BINARY REPRESENTATIONS AND THETA FUNCTION

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Abstract

The question to be considered is whether there is a power series, $f(q)$, whose coefficients are ± 1 and for which $\prod_{n\geq 1} f(q^{2n-1}) = \sum_{-\infty}^{\infty} q^{n(3n-1)/2}$. This question will be answered affirmatively, following a study of the binary representation of integers. In addition, related theorems will be developed.

1. Introduction

In [2] and [3], the second author raised questions along the following lines. Given the generating functions for $p(n)$, [1, p.3, Th. 1.1]

$$\begin{aligned}
\sum_{n\geq 0} p(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \\
&= \prod_{n=1}^{\infty} \left(1+q^n+q^{2n}+q^{3n}+\cdots\right), \qquad (1.1)
\end{aligned}$$

could one merely change some of the signs in this last product so that the resulting power series has coefficients of 0, ± 1? The answer is still unknown.

A very natural step in this direction is to consider

$$\begin{aligned}
B(q) &= \prod_{n=0}^{\infty} \left(1-q^{2^n}\right) \\
&= \sum_{n\geq 0} (-1)^{\#_1(n)} q^n \qquad (1.2) \\
&= 1-q-q^2+q^3-q^4+q^5+q^6-q^7-\cdots,
\end{aligned}$$

where $\#_1(n)$ is the number of 1's in the binary representation of n.

Now since every integer uniquely factors into a power of 2 times an odd number, we see that

$$\begin{aligned}\prod_{n=1}^{\infty} B(q^{2n-1}) &= \prod_{n=1}^{\infty}\prod_{m=0}^{\infty}\left(1-q^{(2n-1)2^m}\right)\\ &= \prod_{N=1}^{\infty}\left(1-q^N\right) \qquad (1.3)\\ &= \sum_{-\infty}^{\infty}(-1)^n q^{n(3n-1)/2},\end{aligned}$$

by Euler's pentagonal number theorem [1, p.11, Cor. 1.7].

In light of the success of (1.3), it is natural to ask whether there exists a power series $f(x)$ having ± 1 as coefficients such that

$$\prod_{n=1}^{\infty} f(q^{2n-1}) = \sum_{-\infty}^{\infty} q^{n(3n-1)/2}. \qquad (1.4)$$

Empirically, one finds that

$$\begin{aligned}f(x) &= 1+x+x^2-x^3-x^4-x^5-x^6-x^7 \qquad (1.5)\\ &\quad -x^8+x^9+x^{10}+x^{11}-x^{12}-x^{13}-x^{14}+\cdots.\end{aligned}$$

In Section 3, we show that $f(x)$ exists and determine the sign pattern for the coefficients. In order to achieve this goal we must consider

$$\begin{aligned}B_e(q) &= 1+2\sum_{n=1}^{\infty}(-1)^n q^{b_n}\\ &= 1-2q+2q^3-2q^4+2q^5-2q^7+2q^9\cdots,\end{aligned}$$

where b_n is the nth integer whose binary representation ends in an even number of zeros (cf. [4, sequence A003159]). The explanation of (1.4) and the coefficients of $f(x)$ relies on

Theorem 1. *We have*

$$\prod_{n=1}^{\infty} B_e(q^{2n-1}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}. \qquad (1.6)$$

Section 2 will be devoted to background on $B_e(q)$. Section 3 will be devoted to a proof of Theorem 1, and Section 4 will provide a full explanation of (1.4) and (1.5). In Section 5, we provide an analogous representation of Gauss's triangular number series. We conclude with open questions.

2. Background on Binary Representations

The object of this section is to find a nice closed form representation of $B_e(q)$.

Lemma 2.
$$\sum_{n=1}^{\infty}(-1)^n q^{b_n} = \sum_{n=1}^{\infty}(-1)^{\#_1(b_n)} q^{b_n}.$$

Proof. In light of the fact that $b_1 = 1$ so that $\#_1(b_1) = 1$, we see that all that is required to establish this result is a proof that $\#_1(b_n)$ changes parity as we pass from b_n to b_{n+1}.

There are 3 cases to consider:

1. b_n in binary ends in $2j$ zeros, with $j > 0$.
2. b_n in binary ends in $2k$ ones
3. b_n in binary ends in $2k+1$ ones

In case 1., $b_{n+1} = b_n + 1$ and b_{n+1} has one more 1 than b_n. Hence $\#_1(b_{n+1})$ has opposite parity from $\#_1(b_n)$.

In case 2., we see that in binary
$$b_n = \beta_1 \cdots \beta_s 0 \underbrace{11\ldots1}_{2k \text{ times}},$$
so $b_{n+1} = b_n + 1$. Thus
$$b_{n+1} = \beta_1 \cdots \beta_s 1 \underbrace{00\ldots0}_{2k \text{ times}},$$
and b_{n+1} has $2k-1$ fewer 1's than b_n. Hence $\#_1(b_{n+1})$ has opposite parity from $\#_1(b_n)$.

In case 3., we see that in binary
$$b_n = \beta_1 \cdots \beta_s 0 \ \underbrace{11\ldots1}_{2k+1 \text{ times}}\ ,$$
so $b_{n+1} = b_n + 2$. Thus
$$b_{n+1} = \beta_1 \cdots \beta_s 1 \underbrace{00\ldots0}_{2k \text{ times}} 1,$$
and b_{n+1} again has $2k-1$ fewer 1's than b_n. Hence $\#_1(b_{n+1})$ has opposite parity from $\#_1(b_n)$. □

Lemma 3.
$$1 + \sum_{n\geq 0} \frac{q^{2^n}}{\prod_{j=0}^{n}\left(1-q^{2^j}\right)} \quad = \quad \prod_{n=0}^{\infty}\frac{1}{1-q^{2^n}}, \tag{2.1}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{2^n}}{\prod_{j=0}^{n}\left(1-q^{2^j}\right)} \quad = \quad q. \tag{2.2}$$

Proof. Equation (2.1) is the generating function for partitions into powers of 2 expressed in two different ways. The nth term of the series generates those partitions whose largest part is 2^n. The infinite product is the standard form for the generating function [1, p.3, Th. 1,1].

As for (2.2), we shall prove

$$\sum_{n=0}^{N} \frac{(-1)^n q^{2^n}}{\prod_{j=0}^{n}\left(1-q^{2^j}\right)} \quad = \quad q + \frac{q^{2^{N+1}}(-1)^N}{\prod_{j=0}^{n}\left(1-q^{2^j}\right)}. \tag{2.3}$$

When $N = 0$, (2.3) reduces to

$$\frac{q}{1-q} = q + \frac{q^2}{1-q},$$

which is immediate.

Assuming true up through a given N,

$$\begin{aligned}
\sum_{n=0}^{N+1} \frac{(-1)^n q^{2^n}}{\prod_{j=0}^{n}\left(1-q^{2^j}\right)} \quad &= \quad q + \frac{q^{2^{N+1}}(-1)^N}{\prod_{j=0}^{N}\left(1-q^{2^j}\right)} + \frac{q^{2^{N+1}}(-1)^{N+1}}{\prod_{j=0}^{N+1}\left(1-q^{2^j}\right)} \\
&= \quad q + \frac{q^{2^{N+1}}(-1)^N\left(\left(1-q^{2^{N+1}}\right)-1\right)}{\prod_{j=0}^{N+1}\left(1-q^{2^j}\right)} \\
&= \quad q + \frac{(-1)^{N+1} q^{2^{N+2}}}{\prod_{j=0}^{N+1}\left(1-q^{2^j}\right)},
\end{aligned}$$

which proves (2.3).

Equation (2.2) now follows by letting $N \to \infty$ in (2.3). $\square$

Theorem 4. $B_e(q) = (1-q)\prod_{j=0}^{\infty}\left(1-q^{2^j}\right)$.

Proof. In light of Lemma 2, it follows that

$$\begin{aligned}
B_e(q) &= 1 - \sum_{n\geq 0} \left(1+(-1)^n\right) q^{2^n} \prod_{j=n+1}^{\infty} \left(1-q^{2^j}\right) \\
&= 1 - \prod_{j=0}^{\infty} \left(1-q^{2^j}\right) \sum_{n\geq 0} \frac{\left(1+(-1)^n\right) q^{2^n}}{\prod_{j=0}^{n} \left(1-q^{2^j}\right)} \\
&= 1 - \prod_{j=0}^{\infty} \left(1-q^{2^j}\right) \left(\frac{1}{\prod_{j=0}^{\infty} \left(1-q^{2^j}\right)} - 1 + q \right) \quad \text{(by Lemma 3)} \\
&= (1-q) \prod_{j=0}^{\infty} \left(1-q^{2^j}\right).
\end{aligned}$$

□

3. Proof of Theorem 1

$$\begin{aligned}
\prod_{n=1}^{\infty} B_e(q^{2n-1}) &= \prod_{n=1}^{\infty} \left\{ \left(1-q^{2n-1}\right) \prod_{j=0}^{\infty} \left(1-q^{(2n-1)2^j}\right) \right\} \\
&= \prod_{n=1}^{\infty} \left(1-q^{2n-1}\right) \prod_{n=1}^{\infty} (1-q^n) \quad \text{(by (1.3))} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1+q^n)} \quad \text{(by [1, p.5, eq (1.2.5)])} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \quad \text{(by [1, p.23, eq. (2.2.12)]),}
\end{aligned}$$

and Theorem 1 is proved.

4. The Determination of $f(x)$

We define

$$f(x) = \frac{B_e(x^3)}{1-x} \tag{4.1}$$

$$
\begin{aligned}
&= \frac{1+2\sum_{n=1}^{\infty}(-1)^n x^{3b_n}}{1-x} \\
&= \frac{1+\sum_{n=1}^{\infty}(-1)^n x^{3b_n} - \sum_{n=0}^{\infty}(-1)^n x^{3b_{n+1}}}{1-x} \\
&= \frac{1-x^3}{1-x} + \sum_{n=1}^{\infty}(-1)^n \frac{\left(x^{3b_n} - x^{3b_{n+1}}\right)}{1-x} \\
&= 1+x+x^2+\sum_{n=1}^{\infty}(-1)^n \left(x^{3b_n} + x^{3b_n+1} + \cdots + x^{3b_{n+1}-1}\right),
\end{aligned}
$$

and we see that $f(x)$ is a power series with ± 1 as the coefficients.

Theorem 5. *Equation* (1.4) *holds for the* $f(x)$ *given in* (4.1)*.*

Proof.

$$
\begin{aligned}
\prod_{n=1}^{\infty} f(q^{2n-1}) &= \prod_{n=1}^{\infty} \frac{B_e(q^{3(2n-1)})}{1-q^{2n-1}} \\
&= \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} \cdot \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} \quad \text{(by Theorem 1)} \\
&= \prod_{n=1}^{\infty} (1+q^n) \prod_{n=1}^{\infty} \frac{\left(1-q^{3n}\right)}{\left(1+q^{3n}\right)} \\
&\qquad \text{(by [1, p.5, eq. (1.2.5)] and [1, p.23, eq. (2.2.12)])} \\
&= \prod_{n=1}^{\infty} \left(1-q^{3n}\right)\left(1+q^{3n-1}\right)\left(1+q^{3n-2}\right) \\
&= \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} \\
&\qquad \text{(by [1, p.21, eq. (2.2.10), } q \to q^{\frac{3}{2}},\ z = q^{-\frac{1}{2}}\text{])}
\end{aligned}
$$

and (1.4) is established for the $f(x)$ given by (4.1). □

5. Gauss's Triangular Number Series

This paper would be incomplete without a result relating binary partitions to Gauss's famous series:

$$
\psi(q) = \sum_{n\geq 0} q^{n(n+1)/2}. \tag{5.1}
$$

Here the relevant binary series is

$$\begin{aligned} B_g(q) &= \prod_{j\geq 0}\left(1+(-1)^j q^{2^j}\right) \qquad (5.2)\\ &= 1+q-q^2-q^3+q^4+q^5-q^6-q^7-q^8-q^9+q^{10}+\dots \end{aligned}$$

Again we see that we have a series where all the coefficients are ± 1.

Theorem 6.

$$\prod_{n\geq 1} B_g(q^{2n-1})B_g(q^{4n-2}) = \psi(q). \qquad (5.3)$$

Proof. We begin by noting that

$$\begin{aligned} B_g(q)B_g(q^2) &= \prod_{j\geq 0}\left(1+(-1)^j q^{2^j}\right)\left(1+(-1)^j q^{2^{j+1}}\right) \qquad (5.4)\\ &= (1+q)\prod_{j\geq 1}\left(1+(-1)^j q^{2^j}\right)\left(1-(-1)^j q^{2^j}\right)\\ &= (1+q)\prod_{j\geq 1}\left(1-q^{2^{j+1}}\right)\\ &= \frac{(1+q)}{(1-q)(1-q^2)}\prod_{j\geq 0}\left(1-q^{2^j}\right)\\ &= \frac{1}{(1-q)^2}\prod_{j\geq 0}\left(1-q^{2^j}\right). \end{aligned}$$

Hence

$$\begin{aligned} &\prod_{n\geq 1} B_g(q^{2n-1})B_g(q^{4n-2})\\ &= \left(\prod_{n\geq 1}\frac{1}{(1-q^{2n-1})^2}\right)\left(\prod_{j\geq 0}\prod_{n\geq 1}\left(1-q^{(2n-1)2^j}\right)\right)\\ &= \prod_{n\geq 1}\frac{(1-q^n)(1+q^n)}{(1-q^{2n-1})} \quad \text{by [1, p.5, eq. (1.2.5)]}\\ &= \prod_{n\geq 1}\frac{(1-q^{2n})}{(1-q^{2n-1})}\\ &= \sum_{n\geq 0} q^{n(n+1)/2} \quad \text{by [1, p. 23, eq. (2.2.13)]} \end{aligned}$$

□

6. Conclusion

In light of the important role played by the sequence b_n, one might as well ask what happens if we consider the sequences, a_n, the nth integer whose binary representation ends in an odd number of zeros [4, seq. A036554]. However, nothing new arises because $a_n = 2b_n$.

Apart from the classical theta series given by the right-hand sides of (1.3), (1.4), (1.6), and (5.1) , it would be interesting to examine other classical theta series. In light of the fact that binary representation played a crucial role in all our results, it would be interesting to see if there are similar theorems related to other bases apart from 2.

References

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