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SUMS OF CUBES OVER ODD-INDEX FIBONACCI NUMBERS

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Abstract

We study sums of the form $\sum_{k=1}^{n} F_{mk}^3$, $\sum_{k=1}^{n}(-1)^k F_{mk}^3$, $\sum_{k=1}^{n} L_{mk}^3$ and $\sum_{k=1}^{n}(-1)^k L_{mk}^3$ where m is an odd integer. For each class of sums we obtain new closed-form expressions. Our findings complement the results of Clary and Hemenway, and Adegoke.

1. Introduction

Let F_n and L_n $(n \geq 0)$ denote the Fibonacci and Lucas numbers, respectively. Their Binet forms equal

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n, \tag{1.1}$$

where α and β are roots of the quadratic equation $x^2 - x - 1 = 0$.

Clary and Hemenway [2] were the first who obtained remarkable summation formulas for all sums of the form $\sum_{k=1}^{n} F_{mk}^3$, where m is an integer. Two examples of their discoveries are

$$\sum_{k=1}^{n} F_k^3 = \begin{cases} \frac{1}{10} F_v(L_{5v+2} + L_{3v+2} - L_v - L_{v-5}) & \text{if } n \equiv 0 \bmod 4 \\ \frac{1}{10} L_{v-1} L_v F_{v+1}(L_{3v-1} - 2L_{v-2}) & \text{if } n \equiv 1 \bmod 4 \\ \frac{1}{10} L_v(F_{5v+2} + F_{3v+2} - F_v - F_{v-5}) & \text{if } n \equiv 2 \bmod 4 \\ \frac{1}{2} F_{v-1} F_v L_{v+1}(F_{3v-1} - 2F_{v-2}) & \text{if } n \equiv 3 \bmod 4, \end{cases} \tag{1.2}$$

where v is the smallest integer greater than or equal to $n/2$ and

$$\sum_{k=1}^{n} F_{2k}^3 = \begin{cases} \frac{1}{4} F_n^2 L_{n+1}^2 F_{n-1} L_{n+2} & \text{if } n \text{ is even} \\ \frac{1}{4} L_n^2 F_{n+1}^2 L_{n-1} F_{n+2} & \text{if } n \text{ is odd.} \end{cases} \tag{1.3}$$

[1]Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.

If m is an even integer, these sums have been studied further in the two recent papers [1] and [3]. For instance, Adegoke obtained the following generalizations in [1]: if m is even, then

$$F_{3m}\sum_{k=1}^{n}F_{2mk}^{3}=F_{mn}^{2}F_{m(n+1)}^{2}(L_{mn}L_{m(n+1)}+L_{m}), \tag{1.4}$$

and if m is odd, then

$$L_{3m}\sum_{k=1}^{n}F_{2mk}^{3}=\begin{cases}F_{mn}^{2}L_{m(n+1)}^{2}(L_{mn}F_{m(n+1)}+F_{m}) & \text{if } n \text{ is even}\\ L_{mn}^{2}F_{m(n+1)}^{2}(F_{mn}L_{m(n+1)}+F_{m}) & \text{if } n \text{ is odd.}\end{cases} \tag{1.5}$$

He also discovered formulas for the corresponding Lucas sums. Frontczak [3] in turn has also derived expressions for the alternating variants. For instance, he showed that

$$\sum_{k=1}^{n}(-1)^{k}F_{2mk}^{3}=\frac{(-1)^{n}}{5F_{6m}}\Big[F_{3m(n+1)}^{2}-F_{3mn}^{2}\Big]-\frac{3}{5}\frac{(-1)^{n}}{F_{2m}}\Big[F_{m(n+1)}^{2}-F_{mn}^{2}\Big]-\frac{F_{3m}}{5L_{3m}}+\frac{3F_{m}}{5L_{m}}. \tag{1.6}$$

For m being odd the situation turns out to be more nebulous. As Clary and Hemenway mention, the generalization of the formula (1.2) results in another four-part expression. The research community seems not to have analyzed these sums further and, as a matter of fact, no other sum identities are reported. In this note we attempt to fill this gap. We establish simple expressions for the cubic sums when m is odd. We study both the nonalternating and alternating versions. We also present corresponding evaluations for the Lucas cubic sums.

2. Preliminary Results

In what follows we will require the following standard facts about Fibonacci and Lucas numbers:

$$\begin{aligned}
F_{-n} &= (-1)^{n+1}F_{n}, &(2.1)\\
F_{n}^{2} &= F_{n-1}F_{n+1}-(-1)^{n}, &(2.2)\\
L_{n}^{2} &= L_{n-1}L_{n+1}+5(-1)^{n}, &(2.3)\\
L_{m+n} &= L_{m+1}F_{n}+L_{m}F_{n-1}, &(2.4)\\
F_{3n} &= F_{n+1}^{3}+F_{n}^{3}-F_{n-1}^{3}, &(2.5)\\
F_{3n} &= 5F_{n}^{3}+3(-1)^{n}F_{n}, &(2.6)\\
5L_{3n} &= L_{n+1}^{3}+L_{n}^{3}-L_{n-1}^{3}. &(2.7)
\end{aligned}$$

These facts are well-known to the Fibonacci community (see, for instance, [4]). Furthermore, we will require the following results, each of which we state as a separate lemma.

Lemma 1. *For every integer $n \geq 1$, it holds that*

$$5L_{3n} = 2L_n^3 + 3L_{n-1}L_nL_{n+1}. \tag{2.8}$$

Proof. Since

$$\begin{aligned} 2L_n^2 + 3L_{n-1}L_{n+1} &= 2\big(L_n^2 + L_{n-1}L_{n+1}\big) + L_{n-1}L_{n+1} \\ &= 5\big(L_{n-1}L_{n+1} + 2(-1)^n\big), \end{aligned}$$

we must show that

$$L_{3n} = L_n\big(L_{n-1}L_{n+1} + 2(-1)^n\big). \tag{2.9}$$

Keeping in mind that $\alpha\beta = -1, \alpha + \beta = 1, \alpha^2 = 1 + \alpha$ and $\beta^2 = 1 + \beta$, the last identity follows from the Binet form for Lucas numbers:

$$\begin{aligned} L_n\big(L_{n-1}L_{n+1} + 2(-1)^n\big) &= (\alpha^n + \beta^n)\big((\alpha^{n-1} + \beta^{n-1})(\alpha^{n+1} + \beta^{n+1}) + 2(\alpha\beta)^n\big) \\ &= (\alpha^n + \beta^n)\big(\alpha^{2n} + \beta^{2n} + \alpha^n\beta^n(\alpha^{-1}\beta + \alpha\beta^{-1}) + 2(\alpha\beta)^n\big) \\ &= (\alpha^n + \beta^n)\big(\alpha^{2n} + \beta^{2n} - \alpha^n\beta^n(\alpha + \beta)\big) \\ &= L_{3n}. \end{aligned}$$

□

Lemma 2. *Let m be an odd integer. Then*

$$\sum_{k=1}^{n} F_{mk} = \frac{F_{m(n+1)} + F_{mn} - F_m}{L_m}, \tag{2.10}$$

$$\sum_{k=1}^{n} (-1)^k F_{mk} = \frac{(-1)^n F_{m(n+1)} + (-1)^{n+1} F_{mn} - F_m}{L_m}, \tag{2.11}$$

$$\sum_{k=1}^{n} L_{mk} = \frac{L_{m(n+1)} + L_{mn} - L_m - 2}{L_m}, \tag{2.12}$$

and

$$\sum_{k=1}^{n} (-1)^k L_{mk} = \frac{(-1)^n L_{m(n+1)} + (-1)^{n+1} L_{mn} - L_m + 2}{L_m}. \tag{2.13}$$

Proof. The results follow from more general identities from [5]. □

Lemma 3. *Let k and m be integers. Then*

$$F_{k+m}^3 + (-1)^m F_{k-m}^3 = L_m\big(F_m^2 F_{3k} + (-1)^m F_k^3\big), \tag{2.14}$$

and

$$L_{k+m}^3 + (-1)^m L_{k-m}^3 = L_m\big(5F_m^2 L_{3k} + (-1)^m L_k^3\big). \tag{2.15}$$

Proof. The first identity is stated as Problem H-95 and solved in [6]. Therefore we only give a proof for the second identity. We start with

$$L_{k+m} = L_{k+1}F_m + L_k F_{m-1},$$

and

$$L_{k-m} = (-1)^m (L_k F_{m+1} - L_{k+1} F_m).$$

Hence,

$$\begin{aligned}
L_{k+m}^3 + (-1)^m L_{k-m}^3 &= L_{k+m}^3 + (-1)^{3m} L_{k-m}^3 \\
&= \big(L_{k+1}F_m + L_k F_{m-1}\big)^3 + \big(L_k F_{m+1} - L_{k+1}F_m\big)^3 \\
&= \big(L_{k+1}F_m + L_k F_{m-1}\big)^3 - \big(L_{k+1}F_m - L_k F_{m+1}\big)^3 \\
&= L_k^3\big(F_{m+1}^3 + F_{m-1}^3\big) \\
&\quad +3L_{k+1}F_m L_k F_{m-1}\big(L_{k+1}F_m + L_k F_{m-1}\big) \\
&\quad -3L_{k+1}F_m L_k F_{m+1}\big(L_k F_{m+1} - L_{k+1}F_m\big) \\
&= L_k^3\big(F_{m+1} + F_{m-1}\big)\big(F_{m+1}^2 + F_{m-1}^2 - F_{m-1}F_{m+1}\big) \\
&\quad -3L_{k+1}F_m L_k^2\big(F_{m+1}^2 - F_{m-1}^2\big) \\
&\quad +3L_{k+1}^2 F_m^2 L_k\big(F_{m+1} + F_{m-1}\big) \\
&= L_k^3 L_m\big(\big(F_{m+1} - F_{m-1}\big)^2 + F_{m-1}F_{m+1}\big) \\
&\quad -3L_{k+1}F_m L_k^2\big(F_{m+1} + F_{m-1}\big)\big(F_{m+1} - F_{m-1}\big) \\
&\quad +3L_{k+1}^2 F_m^2 L_k L_m \\
&= L_k^3 L_m\big(F_m^2 + F_{m-1}F_{m+1}\big) - 3L_{k+1}F_m^2 L_k^2 L_m \\
&\quad +3L_{k+1}^2 F_m^2 L_k L_m \\
&= L_k^3 L_m\big(2F_m^2 + (-1)^m\big) + 3L_{k+1}L_k L_m F_m^2\big(L_{k+1} - L_k\big) \\
&= L_m F_m^2\big(2L_k^3 + 3L_{k-1}L_k L_{k+1}\big) + (-1)^m L_m L_k^3.
\end{aligned}$$

The statement follows from Lemma 1. □

3. Main Results

Our main results for the Fibonacci numbers are contained in the next theorem.

Theorem 1. *Let m be an odd integer. Then for each $n \geq 1$*

$$\sum_{k=1}^{n} F_{mk}^3 = \frac{F_m^2}{L_{3m}}\Big(F_{3m(n+1)} + F_{3mn} - F_{3m}\Big) - \frac{1}{L_m}\Big(F_{m(n+1)}^3 + F_{mn}^3 - F_m^3\Big), \quad (3.1)$$

and

$$\sum_{k=1}^{n}(-1)^k F_{mk}^3 = \frac{F_m^2}{L_{3m}}\Big((-1)^n F_{3m(n+1)} + (-1)^{n+1} F_{3mn} - F_{3m}\Big) - \frac{1}{L_m}\Big((-1)^n F_{m(n+1)}^3 + (-1)^{n+1} F_{mn}^3 - F_m^3\Big). \quad (3.2)$$

Proof. The key ingredients in our proofs are the following telescoping sum relations: Let $f(k)$ be a real sequence and m, n and j be positive integers. Then

$$\sum_{k=1}^{n}\Big[f(m(k+j)) - f(m(k-j))\Big] = \sum_{k=n+1-j}^{n+j} f(mk) - \sum_{k=1-j}^{j} f(mk), \quad (3.3)$$

and

$$\sum_{k=1}^{n}(-1)^{k-1}\Big[f(m(k+j)) - f(m(k-j))\Big] = \sum_{k=n+1-j}^{n+j} (-1)^{k+j-1} f(mk) - \sum_{k=1-j}^{j} (-1)^{k+j-1} f(mk). \quad (3.4)$$

These sum relations may be proved straightforwardly by shifting the summation index and are also utilized in [3].

To prove Theorem 1 we start with Equation (2.14). We replace k by mk and use the fact that m is odd to get

$$F_{m(k+1)}^3 - F_{m(k-1)}^3 = L_m F_m^2 F_{3mk} - L_m F_{mk}^3. \quad (3.5)$$

Set $f(k) = F_k^3$ and $j = 1$ in (3.3) to obtain

$$F_{m(n+1)}^3 + F_{mn}^3 - F_m^3 = L_m F_m^2 \sum_{k=1}^{n} F_{3mk} - L_m \sum_{k=1}^{n} F_{mk}^3. \quad (3.6)$$

The first identity follows from (2.10) upon replacing m by $3m$ and rearranging terms. The second identity follows from (3.4) and (2.11):

$$(-1)^n F_{m(n+1)}^3 + (-1)^{n+1} F_{mn}^3 - F_m^3 = L_m F_m^2 \sum_{k=1}^{n}(-1)^k F_{3mk} - L_m \sum_{k=1}^{n}(-1)^k F_{mk}^3. \quad (3.7)$$

□

Equivalent versions of our discoveries are presented in the next two Propositions:

Proposition 1. *Let m be an odd integer. Then*

$$\sum_{k=1}^{n} F_{mk}^3 = \frac{F_{m(n+1)}^3 + F_{mn}^3 + 2F_m^3 + 3(-1)^{n+1}F_m^2\big(F_{m(n+1)} - F_{mn}\big)}{L_m\big(5F_m^2 - 1\big)}, \tag{3.8}$$

and

$$\sum_{k=1}^{n}(-1)^k F_{mk}^3 = \frac{(-1)^n\big(F_{m(n+1)}^3 - F_{mn}^3\big) + 2F_m^3 - 3F_m^2\big(F_{m(n+1)} + F_{mn}\big)}{L_m\big(5F_m^2 - 1\big)}. \tag{3.9}$$

Proof. Using (2.5) we can write (3.6) as

$$F_{m(n+1)}^3 + F_{mn}^3 - F_m^3 = L_m F_m^2 \sum_{k=1}^{n}\Big(F_{mk+1}^3 - F_{mk-1}^3\Big) + \Big(L_m F_m^2 - L_m\Big)\sum_{k=1}^{n} F_{mk}^3. \tag{3.10}$$

Now,

$$\begin{aligned}
F_{mk+1}^3 &= \Big(F_{mk} + F_{mk-1}\Big)^3 \\
&= F_{mk}^3 + F_{mk-1}^3 + 3F_{mk}F_{mk-1}F_{mk+1} \\
&= F_{mk}^3 + F_{mk-1}^3 + 3F_{mk}\Big(F_{mk}^2 + (-1)^{mk}\Big) \\
&= 4F_{mk}^3 + F_{mk-1}^3 + 3(-1)^k F_{mk},
\end{aligned}$$

and the first formula follows from (2.11). The second statement is proved similarly:

$$\begin{aligned}
(-1)^n F_{m(n+1)}^3 + (-1)^{n+1}F_{mn}^3 - F_m^3 &= L_m F_m^2 \sum_{k=1}^{n}(-1)^k\Big(F_{mk+1}^3 - F_{mk-1}^3\Big) \\
&\quad + \Big(L_m F_m^2 - L_m\Big)\sum_{k=1}^{n}(-1)^k F_{mk}^3.
\end{aligned}$$

The result follows essentially from (2.10). □

Proposition 2. *Let m be an odd integer. Then*

$$\sum_{k=1}^{n} F_{mk}^3 = \frac{F_{3m(n+1)} + F_{3mn} + 10F_m^3}{5L_m\big(5F_m^2 - 1\big)} + \frac{3(-1)^{n+1}}{5L_m}\big(F_{m(n+1)} - F_{mn}\big), \tag{3.11}$$

and

$$\sum_{k=1}^{n}(-1)^k F_{mk}^3 = \frac{(-1)^n\big(F_{3m(n+1)} - F_{3mn}\big) + 10F_m^3}{5L_m\big(5F_m^2 - 1\big)} - \frac{3}{5L_m}\big(F_{m(n+1)} + F_{mn}\big). \tag{3.12}$$

Proof. Since m is odd we immediately get from (2.6)

$$F_{m(n+1)}^3 = \frac{1}{5}\big(F_{3m(n+1)} - 3(-1)^{n+1}F_{m(n+1)}\big),$$

and

$$F_{mn}^3 = \frac{1}{5}\big(F_{3mn} - 3(-1)^n F_{mn}\big).$$

Inserting these relations into the above equation and simplifying gives the statement. The alternating sum is proved in the same manner. □

We conclude this section with two explicit examples for $m = 1$ and $m = 3$ using the results from the last Proposition:

$$\sum_{k=1}^{n} F_k^3 = \frac{1}{20}\big(F_{3n+3} + F_{3n} + 10\big) + \frac{3(-1)^{n+1}}{5}F_{n-1}, \tag{3.13}$$

$$\sum_{k=1}^{n} F_{3k}^3 = \frac{1}{380}\big(F_{9n+9} + F_{9n} + 80\big) + \frac{3(-1)^{n+1}}{20}\big(F_{3n+3} - F_{3n}\big), \tag{3.14}$$

$$\sum_{k=1}^{n}(-1)^k F_k^3 = \frac{1}{20}\big((-1)^n\big(F_{3n+3} - F_{3n}\big) + 10\big) - \frac{3}{5}F_{n+2}, \tag{3.15}$$

$$\sum_{k=1}^{n}(-1)^k F_{3k}^3 = \frac{1}{380}\big((-1)^n\big(F_{9n+9} - F_{9n}\big) + 80\big) - \frac{3}{20}\big(F_{3n+3} + F_{3n}\big). \tag{3.16}$$

4. Analogue Results for Lucas Numbers

Next we state the analogue discoveries for the Lucas sequence. Since the proofs are very similar to that ones given in the last section, we only sketch them.

Theorem 2. *Let m be an odd integer. Then for each $n \geq 1$*

$$\sum_{k=1}^{n} L_{mk}^3 = \frac{5F_m^2}{L_{3m}}\Big(L_{3m(n+1)}+L_{3mn}-L_{3m}-2\Big)-\frac{1}{L_m}\Big(L_{m(n+1)}^3+L_{mn}^3-L_m^3-8\Big), \tag{4.1}$$

and

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k L_{mk}^3 \;&=\; \frac{5F_m^2}{L_{3m}}\Big((-1)^n L_{3m(n+1)} + (-1)^{n+1}L_{3mn} - L_{3m} + 2\Big)\\ &\quad -\frac{1}{L_m}\Big((-1)^n L_{m(n+1)}^3 + (-1)^{n+1}L_{mn}^3 - L_m^3 + 8\Big). \end{aligned}\tag{4.2}$$

Proof. Use (2.15) in combination with (3.3) and (3.4) with $f(k) = L_k^3$. The first-order sums are evaluated according to (2.12) and (2.13). □

Proposition 3. *Let m be an odd integer. Then*

$$\sum_{k=1}^{n} L_{mk}^3 = \frac{L_{m(n+1)}^3 + L_{mn}^3 + 15F_m^2(-1)^n\big(L_{m(n+1)} - L_{mn}\big) - 15F_m^2(L_m-2) - L_m^3 - 8}{L_m\big(5F_m^2-1\big)}, \tag{4.3}$$

and

$$\sum_{k=1}^{n}(-1)^k L_{mk}^3 = \frac{(-1)^n\big(L_{m(n+1)}^3 - L_{mn}^3\big) + 15F_m^2\big(L_{m(n+1)} + L_{mn}\big) - 15F_m^2(L_m+2) - L_m^3 + 8}{L_m\big(5F_m^2-1\big)}. \tag{4.4}$$

Proof. Use (2.7) and

$$L_{mk+1}^3 = 4L_{mk}^3 + L_{mk-1}^3 - 15(-1)^k L_{mk}. \tag{4.5}$$

The remaining first-order sums are evaluated again according to (2.12) and (2.13). □

Proposition 4. *Let m be an odd integer. Then*

$$\sum_{k=1}^{n} L_{mk}^3 = \frac{L_{3m(n+1)} + L_{3mn} - \big(L_{3m}+2\big)}{L_m\big(5F_m^2-1\big)} + \frac{3}{L_m}\big((-1)^n\big(L_{m(n+1)} - L_{mn}\big) - L_m + 2\big), \tag{4.6}$$

and

$$\sum_{k=1}^{n}(-1)^k L_{mk}^3 = \frac{(-1)^n\big(L_{3m(n+1)} - L_{3mn}\big) - \big(L_{3m}-2\big)}{L_m\big(5F_m^2-1\big)} + \frac{3}{L_m}\big(L_{m(n+1)} + L_{mn} - L_m - 2\big). \tag{4.7}$$

Proof. The formulas are a consequence of further modifications of (4.3) and (4.4) using the identity $L_{3n} = L_n^3 - 3(-1)^n L_n$, which can be derived from (2.3) and (2.8). □

For $m = 1$ and $m = 3$ the explicit examples are:

$$\sum_{k=1}^{n} L_k^3 = \frac{1}{4}\big(L_{3n+3} + L_{3n} - 6\big) + 3\big((-1)^n L_{n-1} + 1\big), \tag{4.8}$$

$$\sum_{k=1}^{n} L_{3k}^3 = \frac{1}{76}\big(L_{9n+9} + L_{9n} - 78\big) + \frac{3}{4}\big((-1)^n\big(L_{3n+3} - L_{3n}\big) - 2\big), \tag{4.9}$$

$$\sum_{k=1}^{n}(-1)^k L_k^3 = \frac{1}{4}\big((-1)^n\big(L_{3n+3} - L_{3n}\big) - 2\big) + 3\big(L_{n+2} - 3\big), \tag{4.10}$$

and

$$\sum_{k=1}^{n}(-1)^k L_{3k}^3 = \frac{1}{76}\big((-1)^n\big(L_{9n+9} - L_{9n}\big) - 74\big) + \frac{3}{4}\big(L_{3n+3} + L_{3n} - 6\big). \tag{4.11}$$

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References

[1] K. Adegoke, Factored closed-form expressions for the sums of cubes of Fibonacci and Lucas numbers, Preprint available via arXiv, June 2017.

[2] S. Clary and P.D. Hemenway, On sums of cubes of Fibonacci numbers, *Appl. of Fibonacci Numbers* **5** (1993), 123-136.

[3] R. Frontczak, Sums of powers of Fibonacci and Lucas numbers: A new bottom-up approach, Preprint, Under Review, October 2017.

[4] R.P. Grimaldi, *Fibonacci and Catalan Numbers: An Introduction*, John Wiley & Sons, New York, 2012.

[5] J. Seibert and P. Trojovský, On sums of certain products of Lucas numbers, *Fibonacci Quart.* **44 (2)** (2006), 172-180.

[6] Advanced Problems and Solutions, Problem H-95: Another Identity by J. A. Hunter, *Fibonacci Quart.* **6 (2)** (1968), 142-150.