TAIL OF A MOEBIUS SUM WITH COPRIMALITY CONDITIONS

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Received: 2/14/17, Accepted: 12/20/17, Published: 1/16/18

Abstract

We show that $\sum_{k>K,(k,q)=1} \mu(k)/k^2 = o(1/K)$ uniformly in q. A more precise bound is given, as well as an extension to similar sums. The precise rate of decay is however, unknown.

1. Introduction and Results

Averages of multiplicative functions have been a major subject of research for many years. The case of non-negative functions has been particularly well studied. When using convolution identities, coprimality conditions naturally enter the scene, where the modulus to which the variable should be coprime, say q^*, can belong to a very wide range. This means that one should use estimates that are uniform with respect to q^*, or at least where the dependence on this q^* is explicitly described. The case of a non-negative function is easier; see for instance [12] or [6, Lemma 2.3]. The more difficult case of oscillating multiplicative functions is the one that we address here. The example $\sum_{n\le x,(n,q^*)=1} \mu(n)/n$ has already been tackled by H. Davenport in [2, Lemma 1], more explicitely by A. Granville and the second author in [3, Lemma 10.2], and expanded by T. Tao in [11]. The more surprising but very interesting example $\sum_{n\le x,(n,q^*)=1} \mu(n)2^{-\omega(n)}/n$ has been investigated in [10, Lemma 4].

Note also that, when aiming at refined estimates for non-negative multiplicative functions, we may have to handle averages of oscillating multiplicative functions

[1]The first author has been partly supported by the Marie Curie IRSES program Moduli, project No 612534

[2]The second author has been partly supported by the Indo-French Centre for the Promotion of Advanced Research – CEFIPRA, project No 5401-1.

twisted by coprimality conditions as is the case in [5]: though we were studying $\sum_{d\le X}\mu^2(d)/\varphi(d)$, we had to prove the following theorem.

Theorem 1. *For any real number $K\ge 14$ and any modulus q^*, we have*

$$\Bigg|\sum_{\substack{k>K,\\(k,q^*)=1}}\frac{\mu(k)}{\varphi(k)k}\Bigg|\le\frac{0.37}{K}.$$

From another angle, one may simply say that with the case of non-negative multiplicative functions being under control, the next step is to remove this non-negativity condition. The proof of Theorem 1 uses some adhoc manipulations and does not disclose whether the sum under examination, when multiplied by K, goes to 0. Our first corollary is the following.

Corollary 1. *We have*

$$\limsup_{K\to\infty} K\max_{q^*}\Bigg|\sum_{\substack{k>K,\\(k,q^*)=1}}\frac{\mu(k)}{\varphi(k)k}\Bigg|=0.$$

The leading idea, which we borrow from [8], is to compare the function $\mu(k)/(\varphi(k)k)$ to a completely multiplicative one, namely $\lambda(k)/k^2$ where we denote by $\lambda(n)$ the Liouville function. Here $\lambda(n)=(-1)^{\Omega(n)}$ is the Liouville function, $\Omega(n)$ denoting the total number of prime factors of the positive integer n, counted with multiplicity. We now present our main theorem.

Theorem 2. *Let $\gamma>1$ and G be two real numbers. There exist two constants $C=C(\gamma,G)$ and $c_0(\gamma,G)>0$ with the following property. For any arithmetic function g such that $\sum_{d\ge1}|g(d)|/d\le G$ and for all $D\ge1$,*

$$\sum_{d\ge D}|g(d)|/d^2\le G/D^\gamma,$$

for any modulus q^* and any $K\ge3$, we have

$$\Bigg|\sum_{\substack{k>K,\\(k,q^*)=1}}\frac{(\lambda\star g)(k)}{k^2}\Bigg|\le C\exp\big(-c_0\sqrt{\log\log K}\big)/K.$$

Corollary 1 is an easy consequence of this result, with $\gamma=2$. We could similarly handle the summand $k\mapsto\mu(k)/k^2$. It is interesting to know how far one can go in the quality of the decreasing rate. The essential case is when $\lambda\star g$ is simply λ. It is not difficult to modify the proof of the above result to infer that, under the Riemann Hypothesis, one has, for any $\varepsilon>0$,

$$\sum_{\substack{k>K,\\(k,q^*)=1}}\frac{\lambda(k)}{k^2}\ll_\varepsilon\frac{1}{K(\log K)^{\frac13-\varepsilon}},\quad\text{(assuming the Riemann Hypothesis).}\qquad(1)$$

On the other side, by considering the modulus $q^* = \prod_{p\le K} p$, we readily check that

$$\limsup_{K\to\infty} \max_{q^*} \Bigg| \sum_{\substack{k>K,\\ (k,q^*)=1}} \frac{\lambda(k)}{k^2} \Bigg| K(\log K) \ge 1. \tag{2}$$

The surprising fact is that we have not been able to improve on this estimate. Even simply getting a larger constant on the right-hand side seems difficult. This $1/(K\log K)$ may be the true order of decay, but we do not venture any conjecture.

An extremely similar proof also proves the next theorem.

Theorem 3. *Let $\gamma > 1$ and G be two real numbers. Let χ be a Dirichlet character. There exist two constants $C = C(\gamma, G, \chi)$ and $c_0(\gamma, G, \chi) > 0$ with the following property. For any arithmetic function g such that $\sum_{d\ge1} |g(d)|/d \le G$ and*

$$\forall D \ge 1, \quad \sum_{d\ge D} |g(d)|/d^2 \le G/D^\gamma,$$

for any modulus q^* and any $K \ge 3$, we have

$$\sum_{\substack{k>K,\\ (k,q^*)=1}} \frac{(\lambda\chi \star g)(k)}{k^2} \le C \exp\bigl(-c_0\sqrt{\log\log K}\bigr)/K.$$

The reader may wonder whether we are now able to improve on Theorem 1, and the answer is no! When following the proof we produce here, we have to rely on an explicit version of Lemma 3. Reading [7] and [9], or [1], it is not difficult to see that we can only hope for an upper bound of the shape $c/(K\log K)$, for a small enough constant, or $c'/(K(\log K)^2)$ for a rather large constant c'. Continuing in this manner, we would reach a bound $\mathcal{O}(1/(K\log\log K))$ in Theorem 2 which is alas numerically of the same strength as $\mathcal{O}(1/K)$ for decent values of K.

2. Lemmas

Lemma 1. *Let $K \ge e^6$ be a real number. Let q be a modulus all whose prime factors are below K. We have*

$$\sum_{\substack{k\le K,\\ (k,q)=1}} 1 \le 7\frac{\varphi(q)}{q}K.$$

This is an easy consequence of the Selberg sieve and is stated as above in [4, Theorem 3.5]. The following is the trivial consequence of the above that we shall use.

Lemma 2. *Let $K' \geq 1$ be a real number. Let q be a modulus. We have*

$$\sum_{\substack{k \leq K', \\ (k,q)=1}} 1 \ll \prod_{\substack{p|q, \\ p \leq K'^2}} \left(1 - \frac{1}{p}\right) K'$$

where the implied constant is independant of q and K'.

This also means that we may restrict the prime p to any subset of the divisors of q that are below K'^2. The right-hand side would at most only increase.

Proof. The lemma is obvious when K' is below e^6. Let us then assume that K' is larger than this bound. We can replace q by $q_1 = \prod_{\substack{p|q, \\ p \leq K'}} p$ and apply Lemma 1. Our next task is to show that one can extend the product

$$\frac{\varphi(q_1)}{q_1} = \prod_{\substack{p|q, \\ p \leq K}} \left(1 - \frac{1}{p}\right)$$

to encompass the primes up to K'^2. Such an extension may modify the $\varphi(q_1)/q_1$ by a factor that is between 1 and

$$\prod_{K < p \leq K'^2} \left(1 - \frac{1}{p}\right),$$

and this last quantity is asymptotically bounded below by $1/2$ by Mertens' Theorem. □

We continue with an easy consequence of the Prime Number Theorem.

Lemma 3. *There exists a positive constant c with the following property. Let $K \geq 1$ be a real number. We have*

$$\sum_{k > K} \frac{\lambda(k)}{k^2} \ll \frac{\exp -c\sqrt{\log(2K)}}{K}.$$

3. Proof of Theorem 2, the Case of the Liouville Function

The modulus q^* being fixed, we define

$$q_1 = \prod_{\substack{p|q^*, \\ p \leq K}} p \tag{3}$$

as well as

$$\rho = \prod_{\substack{p|q^*,\\ p\le K}} \left(1-\frac{1}{p}\right). \tag{4}$$

We next fix the real number K. The proof splits in two cases, according to whether ρ is small or large. Here is our first bound. A summation by parts discloses the estimate

$$\Bigg|\sum_{\substack{k>K,\\ (k,q^*)=1}} \frac{\lambda(k)}{k^2}\Bigg| = \left|2\int_K^\infty \sum_{\substack{K<k\le t,\\ (k,q^*)=1}} \lambda(k)\,\frac{dt}{t^3}\right| \le \left|2\int_K^\infty \sum_{\substack{k\le t,\\ (k,q_1)=1}} 1\,\frac{dt}{t^3}\right|$$

which is $\mathcal{O}(\rho/K)$, by Lemma 2 (with $K' = t$ and restricting in this lemma the product over p to the primes not more than K). This first bound is efficient when ρ is small enough.

Let us proceed to get a second bound. We handle the coprimality condition by employing the Moebius function:

$$\begin{aligned}\sum_{\substack{k>K,\\ (k,q^*)=1}} \frac{\lambda(k)}{k^2} &= \sum_{d|q^*} \mu(d) \sum_{d|k>K} \frac{\lambda(k)}{k^2}\\ &= \sum_{d|q^*} \frac{\mu^2(d)}{d^2} \sum_{\ell>K/d} \frac{\lambda(\ell)}{\ell^2}.\end{aligned}$$

Please note that, in the preceding lines, we used in an essential manner the complete multiplicativity of the λ-function. We split the above sum over d according to whether d is larger or smaller than some bound $D < K/2$. We next employ Lemma 3 for the summations over ℓ attached to $d \le D$, and further notice that

$$\sum_{\substack{d|q^*,\\ d\le D}} \frac{\mu^2(d)}{d} \le \prod_{\substack{p|q^*,\\ p\le D}} \frac{1}{1-\frac{1}{p}} \le \prod_{\substack{p|q^*,\\ p\le K}} \frac{1}{1-\frac{1}{p}} \prod_{\substack{p|q^*,\\ D<p\le K}} \left(1-\frac{1}{p}\right) \le 1/\rho.$$

This gives rise to the first contribution:

$$\sum_{d|q^*} \frac{\mu^2(d)}{d^2}\,\frac{\exp -c\sqrt{\log(K/D)}}{(K/d)} \ll \frac{\exp -c\sqrt{\log(K/D)}}{\rho K}. \tag{5}$$

When now d is larger than D, we only use the fact that the partial sums $\sum_{\ell>K/d} \frac{\lambda(\ell)}{\ell^2}$ are bounded (in fact at least by $\pi^2/6$) and are left with finding a bound for

$$\sum_{\substack{d|q^*,\\ d>D}} \frac{\mu^2(d)}{d^2}.$$

Let
$$m = \prod_{\substack{p\le D,\\ p\nmid q^*}} p.$$
An integer d that divides q^* is prime to m, and thus
$$\sum_{\substack{d\mid q^*,\\ d>D}} \frac{\mu^2(d)}{d^2} \le \sum_{\substack{d>D,\\ (d,m)=1}} \frac{1}{d^2}.$$
It is better to disclose at this level our choice for D, namely
$$D = K/\sqrt{\log K}. \tag{6}$$
By employing Lemma 2 again and a summation by parts as above, we show that
$$\sum_{\substack{d\mid q^*,\\ d>D}} \frac{\mu^2(d)}{d^2} \ll \frac{\varphi(m)}{mD} \ll \frac{1}{D}\frac{\prod_{p\le D}(1-p^{-1})}{\prod_{p\le D, p\mid q^*}(1-p^{-1})} \ll \frac{1}{\rho D\log D}. \tag{7}$$
Combining (5) together with (7) we reach
$$\sum_{\substack{k>K,\\ (k,q^*)=1}} \frac{\lambda(k)}{k^2} \ll \frac{\exp -c\sqrt{\log\log K}}{\rho K} + \frac{1}{\rho K\sqrt{\log K}} \ll \frac{\exp -c\sqrt{\log\log K}}{\rho K}. \tag{8}$$
We can put our two bounds in a single expression as follows:
$$\sum_{\substack{k>K,\\ (k,q^*)=1}} \frac{\lambda(k)}{k^2} \ll \min\left(\rho, \frac{\exp -c\sqrt{\log\log K}}{\rho}\right)/K.$$
The theorem for the λ-function is a straightfoward consequence of this bound, with $c_0 = c/2$, i.e.,
$$\sum_{\substack{k>K,\\ (k,q^*)=1}} \frac{\lambda(k)}{k^2} \ll \frac{\exp -c_0\sqrt{\log\log K}}{K}. \tag{9}$$

4. Proof of Theorem 2, Generic Case

We now have (9) at our disposal, and want to apply the "deformation" g to λ. We write
$$\sum_{\substack{k>K,\\ (k,q^*)=1}} \frac{(\lambda\star g)(k)}{k^2} = \sum_{\substack{\delta\ge 1,\\ (\delta,q^*)=1}} \frac{g(\delta)}{\delta^2} \sum_{\substack{k>K/\delta,\\ (k,q^*)=1}} \frac{\lambda(k)}{k^2}.$$

We separate the sum over δ according to whether $\delta \leq K^{\frac{1+\gamma}{2\gamma}}$ or not. In the first case, we employ (9) and get

$$\Bigl|\sum_{\substack{\delta\geq 1,\\ (\delta,q^*)=1,\\ d\leq K^{\frac{1+\gamma}{2\gamma}}}}\frac{g(\delta)}{\delta^2}\sum_{\substack{k>K/\delta,\\ (k,q^*)=1}}\frac{\lambda(k)}{k^2}\Bigr| \leq C\sum_{\substack{\delta\geq 1,\\ d\leq K^{\frac{1+\gamma}{2\gamma}}}}\frac{|g(\delta)|}{\delta}\frac{\exp -c_0\sqrt{\log\log\sqrt{K}}}{K}.$$

In the second case, we simply write

$$\Bigl|\sum_{\substack{\delta\geq 1,\\ (\delta,q^*)=1,\\ d> K^{\frac{1+\gamma}{2\gamma}}}}\frac{g(\delta)}{\delta^2}\sum_{\substack{k>K/\delta,\\ (k,q^*)=1}}\frac{\lambda(k)}{k^2}\Bigr| \leq \frac{\pi^2}{6}G/K^{\frac{1+\gamma}{2}}.$$

The theorem follows readily.

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