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THE LIND-LEHMER CONSTANT FOR 3-GROUPS

Stian Clem[1]
Cornell University, Ithaca, New York
sac369@cornell.edu

Christopher Pinner
Department of Mathematics, Kansas State University, Manhattan, Kansas
pinner@math.ksu.edu

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Abstract

For a finite abelian group the Lind measure of an integer polynomial is a finite analogue of the usual Mahler measure. The determination of the minimal non-trivial measure, the Lind-Lehmer constant for the group, is the counterpart of the famous Lehmer Problem. This corresponds to the minimal non-trivial value taken by the group determinant for integer variables. We establish the Lind-Lehmer constant for some infinite classes of abelian 3-groups, including $G = H \times \mathbb{Z}_{3^t}$ when H is a 3-group of order at most 81 (and H contains at least one component $\mathbb{Z}_3$ or t is sufficiently large). We give many cases where the minimal non-trivial measure equals the trivial bound $|G|-1$, for example $\mathbb{Z}_3^r \times \mathbb{Z}_{3^t}$ when $r \geq 5$ and $t \leq 3 \cdot 2^r$, $\mathbb{Z}_3^r \times \mathbb{Z}_{3^t} \times \mathbb{Z}_{3^s}$ when $t \leq s \leq (2^r - 1)t + 1$, and $\mathbb{Z}_3 \times \mathbb{Z}_{3^t}^r \times \mathbb{Z}_{3^s}$ when $t \leq s \leq (2^r - r)t+1$. Although we have restricted ourselves to 3-groups, these results may be helpful in understanding the general p-group. The unpredictability of the $(p-1)$st roots of unity mod p^k for $p \geq 5$ makes obtaining these kinds of results in the general case much more difficult.

1. Introduction

For a polynomial F in $\mathbb{Z}[x]$, its (logarithmic) Mahler measure $m(F) = \log M(F)$ is defined by

$$\log M(F) = \int_0^1 \log\left|F\left(e^{2\pi i t}\right)\right| dt. \tag{1}$$

Lehmer's problem [5] famously asks if there exists a positive constant $c > 1$ with the property that if $F \in \mathbb{Z}[x]$ then $M(F) = 1$ or $M(F) > c$. This problem remains open.

[1] This was a summer undergraduate research project for the first author

In 2005 Doug Lind [6] viewed the integral over $[0,1)$ as the Haar measure on the group $G=\mathbb{R}/\mathbb{Z}$, and $F\left(e^{2\pi i t}\right)$ as a linear sum of characters on G. This enabled him to generalize the concept of Mahler measure to an arbitrary compact abelian group G with a suitably normalized Haar measure (see also the group generalization of Dasbach and Lalín [1]). For example, for a finite group

$$G=\mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_r}$$

the integral becomes the average of a $\log|f|$ over the elements $(x_1,\ldots,x_r)$ of G, where f is a linear sum

$$f(x_1,\ldots,x_r)=\sum_{(t_1,\ldots,t_r)\in G} a_{t_1,\ldots,t_r}\chi_{t_1,\ldots,t_r}(x_1,\ldots,x_r),$$

of the characters on G

$$\chi_{t_1,\ldots,t_r}(x_1,\ldots,x_r)=\prod_{j=1}^{r} e^{2\pi i x_j t_j/n_j}.$$

Here we write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$, the cyclic group of order n. That is, for an

$$F(x_1,\ldots,x_r)=\sum_{(t_1,\ldots,t_r)\in G} a_{t_1,\ldots,t_r}x_1^{t_1}\cdots x_r^{t_r}$$

in $\mathbb{Z}[x_1,\ldots,x_r]$, modulo the ideal generated by $x_1^{n_1}-1,\ldots,x_r^{n_r}-1$, we can define

$$m_G(F)=\frac{1}{|G|}\log|M_G(F)|,$$

where $M_G(F)$ is the integer

$$M_G(F):=\prod_{j_1=1}^{n_1}\cdots\prod_{j_r=1}^{n_r}F(e^{2\pi i j_1/n_1},\ldots,e^{2\pi i j_r/n_r}). \tag{2}$$

As in Lehmer's problem, for each finite group G one may ask for the smallest non-trivial value of $M_G(F)$

$$\lambda(G):=\min\{|M_G(F)|\;:\;F\in\mathbb{Z}[x_1,\ldots,x_r],\;M_G(F)\neq 0,\pm 1\}.$$

In his thesis, Vipismakul [10] showed the relationship between $M_G(F)$ and the group determinant, that is, the $|G|\times|G|$ determinant

$$D(G)=|\det\left(x_{gh^{-1}}\right)_{g,h\in G}|,$$

with the variables x_g corresponding to the coefficients of F. Thus $\lambda(G)$ is the smallest integer value greater than 1 taken by the group determinant when the variables x_g are all in $\mathbb{Z}$.

The cyclic case $G = \mathbb{Z}_n$ was considered in [6], [4] and [8], and $\lambda(\mathbb{Z}_n)$ has been determined for all n not divisible by $892,371,480 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$. In particular, if p is prime and k is a positive integer, then it is known that

$$\lambda(\mathbb{Z}_{p^k}) = \begin{cases} 3 & \text{if } p = 2, \\ 2 & \text{if } p \geq 3; \end{cases}$$

the extreme values here are achieved by using $F(x) = x^2 + x + 1$ and $x + 1$, respectively.

The case of p-groups

$$G = \mathbb{Z}_{p^{k_1}} \times \cdots \times \mathbb{Z}_{p^{k_r}}, \quad k_1 \leq \cdots \leq k_r, \tag{3}$$

has also received much attention, see [3], [2] and [9]. For example, when p is odd and all the $k_i = 1$ it was shown in [3] that

$$\lambda(\mathbb{Z}_p^r) = B_r(p),$$

where $B_k(p)$ is the smallest non-trivial $(p-1)$st root of unity mod p^k

$$B_k(p) := \min\{a^{p^{k-1}} \bmod p^k \ : \ 1 < a \leq p-1\}.$$

Here we take the least positive residue mod p^k. Additional p-groups were considered in [2]. For example, it was shown that

$$\lambda(\mathbb{Z}_p \times \mathbb{Z}_{p^2}) = \begin{cases} 2^p, & \text{if } p = 3 \text{ or } 5, \\ B_3(p), & \text{if } 7 \leq p < 10^7,\ p \neq 127, \end{cases}$$

along with many other cases of $|H| = p^k$, $k \leq 6$, where

$$\lambda(\mathbb{Z}_p \times H) = \min\{\lambda(H)^p, B_{k+1}(p)\}. \tag{4}$$

The determination of $B_k(p)$ was crucial in these computations. For most p the value $B_k(p)$ can be difficult to predict, but when $p = 3$ the value is very well-behaved:

$$B_k(3) = 3^k - 1,$$

and so we might hope to make extra progress in the special case $p = 3$. For example it was observed in [3] that

$$\lambda(\mathbb{Z}_3^r) = 3^r - 1,$$

and in [7] that

$$\lambda(\mathbb{Z}_3 \times \mathbb{Z}_{3^t}) = 2^3, \quad t \geq 1, \tag{5}$$

and shown in [2] that

$$\lambda(\mathbb{Z}_3^2 \times \mathbb{Z}_{3^t}) = \begin{cases} 2^9, & \text{if } t \geq 4, \\ 3^{t+2} - 1, & \text{if } 1 \leq t \leq 3. \end{cases} \tag{6}$$

Here we are interested in other examples of 3-groups

$$G = \mathbb{Z}_{3^{k_1}} \times \cdots \times \mathbb{Z}_{3^{k_r}}, \quad k_1 \leq \cdots \leq k_r, \tag{7}$$

where we can determine $\lambda(G)$. We note that for any G of the form (7) we have the upper bound

$$\lambda(G) \leq \min\left\{ |G| - 1,\ 2^{|G|/3^{k_r}} \right\}. \tag{8}$$

To see this observe that

$$M_G\left(\pm 1 + m \prod_{j=1}^{r} \frac{(x_j^{3^{k_j}} - 1)}{(x_j - 1)} \right) = m|G| \pm 1 \tag{9}$$

and, since $M_{\mathbb{Z}_{3^k}}(1+x) = 2$,

$$M_G(1 + x_r) = 2^{3^{k_1 + \cdots + k_{r-1}}}. \tag{10}$$

In fact, in all previously determined cases of 3-groups and in all the cases that we will resolve in this paper, (8) is sharp and it is tempting to ask:

Question 1. Do we have equality in (8) for all 3-groups?

We do know that this is true when k_r is large relative to the other k_i; in particular for a 3-group H it was shown in [2, Theorem 2.4] that

$$3^{t+1} \geq 4^{|H|} - 1 \quad \text{implies} \quad \lambda(H \times \mathbb{Z}_{3^t}) = 2^{|H|}. \tag{11}$$

For a general p-group (3) with $p \geq 3$, corresponding to (8) we have

$$\lambda(G) \leq \min\left\{ B_k(p),\ 2^{|G|/p^{k_r}} \right\}, \quad k = k_1 + \cdots + k_r. \tag{12}$$

Again, all known cases have equality in (12), for example when k_r is particularly large compared to the other k_i. Since $B_k(p)^{p^l} \geq B_{k+l}(p)$, always having equality in this bound would be equivalent to always having equality in the bound

$$\lambda(\mathbb{Z}_{p^{k_1}} \times H) \leq \min\{B_k(p),\ \lambda(H)^{p^{k_1}}\}, \quad p^k = |\mathbb{Z}_{p^{k_1}} \times H|.$$

2. The Product of a Small 3-group With a $\mathbb{Z}_{3^t}$

In this section we consider groups of the form $G = H \times \mathbb{Z}_{3^t}$ where H is a 3-group of order at most 81. These are the groups $H = \mathbb{Z}_3$ of order 3, $\mathbb{Z}_3^2$, $\mathbb{Z}_9$ of order 9, $\mathbb{Z}_3^3, \mathbb{Z}_3 \times \mathbb{Z}_9$, $\mathbb{Z}_{27}$ of order 27, and $\mathbb{Z}_3^4$, $\mathbb{Z}_3 \times \mathbb{Z}_{27}$, $\mathbb{Z}_3^2 \times \mathbb{Z}_9$, $\mathbb{Z}_9 \times \mathbb{Z}_9$, $\mathbb{Z}_{81}$ of order 81. With $H = \mathbb{Z}_3$ or $\mathbb{Z}_3^2$ dealt with in (5) and (6), the first case to consider is $H = \mathbb{Z}_9$. We know from (11) that $\lambda(G) = 2^9$ for $t \geq 11$. Although unable to obtain a complete determination we can at least improve this down to $t \geq 5$, a result that we shall need later.

Theorem 1. *For $t \geq 5$*
$$\lambda(\mathbb{Z}_9 \times \mathbb{Z}_{3^t}) = 2^9.$$

We suspect the same is true for $t = 4$, but for $t \leq 4$ our congruences cannot eliminate values that are $\pm 1 \bmod 3^{t+1}$, and the most that we can say is

$$\begin{aligned} \lambda(\mathbb{Z}_9 \times \mathbb{Z}_9) &= 26, 28, 53, 55 \text{ or } 80, \\ \lambda(\mathbb{Z}_9 \times \mathbb{Z}_{27}) &= 80, 82, 161, 163 \text{ or } 242, \\ \lambda(\mathbb{Z}_9 \times \mathbb{Z}_{81}) &= 242, 244, 485, 487 \text{ or } 512. \end{aligned} \tag{13}$$

In line with Question 1 we ask:

Question 2. Is

$$\lambda(\mathbb{Z}_9 \times \mathbb{Z}_{3^t}) = \begin{cases} 2^9, & \text{if } t = 4, \\ 3^{t+2} - 1, & \text{if } 2 \leq t \leq 3? \end{cases}$$

When $|H| = 27$ we know from (11) that $\lambda(G) = 2^{27}$ for $t \geq 34$, and when $H = \mathbb{Z}_3^3$ or $\mathbb{Z}_3 \times \mathbb{Z}_9$ that $\lambda(G) = |G| - 1$ for $2 \leq t \leq 4$ from the Section 6 computations in [2]. When H contains at least one $\mathbb{Z}_3$ we can determine $\lambda(G)$ for all t.

Theorem 2. *For $t \geq 1$*

$$\lambda(\mathbb{Z}_3^3 \times \mathbb{Z}_{3^t}) = \begin{cases} 2^{27}, & \text{if } t \geq 15, \\ 3^{t+3} - 1, & \text{if } 1 \leq t \leq 14. \end{cases} \tag{14}$$

For $t \geq 2$

$$\lambda(\mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_{3^t}) = \begin{cases} 2^{27}, & \text{if } t \geq 15, \\ 3^{t+3} - 1, & \text{if } 2 \leq t \leq 14. \end{cases} \tag{15}$$

Similar to Theorem 1 we are able to evaluate the minimum for the remaining case $H = \mathbb{Z}_{27}$ only for sufficiently large t.

Theorem 3. *For $t \geq 17$*
$$\lambda(\mathbb{Z}_{27} \times \mathbb{Z}_{3^t}) = 2^{27}.$$

For $3 \leq t \leq 16$ the values $\pm 1 \bmod 3^{t+1}$ less that $3^{t+3} - 1$ can again not be eliminated just using our congruences and the most that we can say is that

$$\lambda(\mathbb{Z}_{27} \times \mathbb{Z}_{3^t}) \equiv \pm 1 \bmod 3^{t+1}, \tag{16}$$

for $3 \leq t \leq 14$, and (16) or 2^{27} when $t = 15$ or 16.

Again it seems natural to ask:

Question 3. Is

$$\lambda(\mathbb{Z}_{27} \times \mathbb{Z}_{3^t}) = \begin{cases} 2^{27}, & \text{if } t = 15 \text{ or } 16, \\ 3^{t+3} - 1, & \text{if } 3 \le t \le 14? \end{cases}$$

Finally we try to compute $\lambda(H \times \mathbb{Z}_{3^t})$ for $|H| = 3^4$. From (11) we know that the minimum is 2^{81} for $t \ge 102$. Corresponding to Theorem 2 we get a complete determination when H contains at least one $\mathbb{Z}_3$.

Theorem 4. *For $t \ge 1$ we have*

$$\lambda(\mathbb{Z}_3^4 \times \mathbb{Z}_{3^t}) = \begin{cases} 2^{81}, & \textit{if } t \ge 48, \\ 3^{t+4} - 1, & \textit{if } 1 \le t \le 47, \end{cases}$$

for $t \ge 2$

$$\lambda(\mathbb{Z}_3^2 \times \mathbb{Z}_9 \times \mathbb{Z}_{3^t}) = \begin{cases} 2^{81}, & \textit{if } t \ge 48, \\ 3^{t+4} - 1, & \textit{if } 2 \le t \le 47, \end{cases}$$

and for $t \ge 3$

$$\lambda(\mathbb{Z}_3 \times \mathbb{Z}_{27} \times \mathbb{Z}_{3^t}) = \begin{cases} 2^{81}, & \textit{if } t \ge 48, \\ 3^{t+4} - 1, & \textit{if } 3 \le t \le 47. \end{cases}$$

If $H = \mathbb{Z}_9^2$ or $H = \mathbb{Z}_{3^4}$ then, as with Theorem 1 and Theorem 3, we can only get an evaluation for t sufficiently large, since our congruences cannot rule out $3^{t+3} - 1$ or $3^{t+1} - 1$ respectively.

Theorem 5. *For $t \ge 49$*

$$\lambda(\mathbb{Z}_9^2 \times \mathbb{Z}_{3^t}) = 2^{81}.$$

For $t = 48$

$$\lambda(\mathbb{Z}_9^2 \times \mathbb{Z}_{3^{48}}) = 3^{51} \pm 1, \;\; 2 \cdot 3^{51} \pm 1, \;\; \textit{or} \;\; 2^{81},$$

and for $2 \le t \le 47$

$$\lambda(\mathbb{Z}_9^2 \times \mathbb{Z}_{3^t}) = 3^{t+3} \pm 1, \;\; 2 \cdot 3^{t+3} \pm 1, \;\; \textit{or} \;\; 3^{t+4} - 1.$$

Theorem 6. *For $t \ge 51$*

$$\lambda(\mathbb{Z}_{81} \times \mathbb{Z}_{3^t}) = 2^{81}.$$

For $4 \le t \le 50$ the minimal value is ± 1 mod 3^{t+1}, or 2^{81} when $t \ge 48$.

Following the pattern of Theorem 4 we might expect the minimum to be 2^{81} for $t \ge 48$ in both cases, and so we ask:

Question 4. Is it true that for $t \ge 48$

$$\lambda(\mathbb{Z}_{81} \times \mathbb{Z}_{3^t}) = \lambda(\mathbb{Z}_9^2 \times \mathbb{Z}_{3^t}) = 2^{81}, \tag{17}$$

with $\lambda(\mathbb{Z}_{81} \times \mathbb{Z}_{3^t}) = |G| - 1$ for $4 \le t \le 47$ and $\lambda(\mathbb{Z}_9^2 \times \mathbb{Z}_{3^t}) = |G| - 1$ for $2 \le t \le 47$?

3. Additional 3-groups

The proofs for Section 2 require us to perform some computations. There are other 3-groups where we can obtain the minimum with little extra work. For example, when the trivial bound (9) is optimal we can always add additional $\mathbb{Z}_3$'s.

Theorem 7. *If H is a 3-group with $\lambda(H) = |H| - 1$, or $\lambda(H)^2 \geq 3|H| - 1$, then $G = \mathbb{Z}_3 \times H$ has*

$$\lambda(G) = |G| - 1.$$

In particular, from Theorem 4 we have that for any $r \geq 4$

$$\lambda(\mathbb{Z}_3^r \times \mathbb{Z}_{3^t}) = 3^{r+t} - 1, \quad 1 \leq t \leq 47.$$

In fact, we can improve the 47 to 97 for $r = 5$, 197 for $r = 6$, 398 for $r = 7$, etc.

Corollary 1. *If $r \geq 5$ then*

$$1 \leq t \leq 101 \cdot 2^{r-5} - r + 1 \quad \Rightarrow \quad \lambda(\mathbb{Z}_3^r \times \mathbb{Z}_{3^t}) = 3^{r+t} - 1.$$

More generally, if $|H_1| = 3^3$ and

$$G = \mathbb{Z}_3 \times \mathbb{Z}_{3^{\beta_2}} \times \cdots \times \mathbb{Z}_{3^{\beta_r}} \times H_1 \times \mathbb{Z}_{3^t},$$

where $r \geq 2$ and 3^t is the highest invariant factor of G, then

$$\beta_2 + \cdots + \beta_r + 3 + t \leq 101 \cdot 2^{r-2} \Rightarrow \lambda(G) = |G| - 1.$$

From (11) we know that for large t the minimum is not $|G| - 1$:

$$t \geq \frac{\log(4^{3^r} - 1)}{\log 3} - 1 \quad \Rightarrow \quad \lambda(\mathbb{Z}_3^r \times \mathbb{Z}_{3^t}) = 2^{3^r}, \tag{18}$$

with Question 1 suggesting $\log(2^{3^r} + 1)/\log 3 - r$ as a realistic cutoff for t.

From Lemma 2 below or Theorem 6.1 of [2] we readily obtain

$$\lambda(\mathbb{Z}_3 \times \mathbb{Z}_{3^t} \times \mathbb{Z}_{3^t}) = 3^{2t+1} - 1,$$
$$\lambda(\mathbb{Z}_3 \times \mathbb{Z}_{3^t} \times \mathbb{Z}_{3^{t+1}}) = 3^{2t+2} - 1.$$

We can use Theorem 7 to add $\mathbb{Z}_3$'s or otherwise generalize this:

Corollary 2. *If $r \geq 1$ and*

$$G = \mathbb{Z}_3^r \times \mathbb{Z}_{3^t} \times \mathbb{Z}_{3^s}, \quad t \leq s \leq (2^r - 1)t + 2^{r-1} - r + 1,$$

then $\lambda(G) = |G| - 1$. More generally if $|H_1| = 3^t$ and $|H_2| = 3^s$ and

$$G = \mathbb{Z}_3 \times \mathbb{Z}_{3^{\beta_2}} \times \cdots \times \mathbb{Z}_{3^{\beta_r}} \times H_1 \times H_2$$

with $r \geq 1$ and $1 \leq \beta_2 \leq \cdots \leq \beta_r \leq t \leq s$, then

$$\beta_2 + \cdots + \beta_r + s + t \leq 2^r t + 2^{r-1} \Rightarrow \lambda(G) = |G| - 1.$$

While these give us many examples, there are still plenty of straightforward 3-groups, as in (13), where we cannot determine the minimal measure. It is quite curious that we know that $\lambda(\mathbb{Z}_3 \times \mathbb{Z}_{3^t} \times \mathbb{Z}_{3^t}) = 3^{2t+1} - 1$ for all $t \geq 2$ but cannot determine $\lambda(\mathbb{Z}_{3^t} \times \mathbb{Z}_{3^t})$ for any $t \geq 2$.

4. Some Congruence Conditions on the Measures

We observe that if we obtain G' from G by increasing any of the k_i or adding an additional component $\mathbb{Z}_{3^k}$ then $M_{G'}(F) = M_G(F_1)$ for a related F_1 and $\lambda(G') \geq \lambda(G)$. We also know, see for example [7], that

$$M_G(F) \equiv F(1, \ldots, 1)^{|G|} \bmod 3^r. \tag{19}$$

Writing the measure as a product of norms it is readily seen that $3 \mid M_G(F)$ if and only if $3 \mid F(1, \ldots, 1)$ and all the norms, in which case $3^r|G| \mid M_G(F)$. In particular an extremal measure cannot be divisible by 3. In [2] we obtained some more sophisticated congruences. If H_1 and H_2 are 3-groups with

$$G = H_1 \times H_2,$$

where

$$H_1 = \mathbb{Z}_{3^{\alpha_1}} \times \cdots \times \mathbb{Z}_{3^{\alpha_m}}, \quad H_2 = \mathbb{Z}_{3^{\beta_1}} \times \cdots \times \mathbb{Z}_{3^{\beta_n}}, \tag{20}$$

with $n, m \geq 1$, then $F(x_1, \ldots, x_m, y_1, \ldots, y_n)$ with $3 \nmid M_G(F)$, has

$$M_G(F) = \prod_{j_1=0}^{\alpha_1} \cdots \prod_{j_m=0}^{\alpha_m} N_{j_1,\ldots,j_m}$$

where the $N_{j_1,\ldots,j_m}$ are integers with

$$N_{j_1,\ldots,j_m} \equiv A^{\phi(3^{j_1})\cdots\phi(3^{j_m})} \bmod 3|H_2|,$$

and A is the H_2 measure of $F(1, \ldots, 1, y_1, \ldots, y_n)$. In particular this gives

$$M_G(F) \equiv A^{|H_1|} \bmod 3|H_2|,$$

and hence, by Euler's Theorem,

$$M_G(F)^2 \equiv 1 \bmod 3h, \tag{21}$$

and

$$M_G(F) \equiv \pm 1 \bmod 3h, \tag{22}$$

where

$$h = \min\{|H_1|, |H_2|\}.$$

Notice that from (22) we have

$$\lambda(G) \geq 3\min\{|H_1|, |H_2|\} - 1. \tag{23}$$

Here the $N_{j_1,\ldots,j_m}$ represents the product obtained when the x_i run through the primitive 3^{j_i}th roots of unity and the y_i through all the 3^{β_i}th roots of unity, in particular, by pairing complex conjugates, we know that the $N_{j_1,\ldots,j_m}$ will be positive integers as long as at least one of the $j_i \geq 1$.

For a 3-group $G = \mathbb{Z}_{3^l} \times H$ we can write the measure of an F in $\mathbb{Z}[x_1, \ldots, x_r]$ as

$$M = N_0 N_1 \cdots N_l, \quad N_j \equiv N_0^{\phi(3^j)} \bmod 3|H|, \tag{24}$$

where N_0 is the H measure of $F(1, x_2, \ldots, x_r)$ and N_j the H measure of

$$\prod_{\substack{l=1 \\ (3,l)=1}}^{3^j} F(e^{2\pi i l/3^j}, x_2, \ldots, x_r) \in \mathbb{Z}[x_2, \ldots, x_r].$$

Replacing F by $-F$ as necessary we can make $N_0 > 0$ and hence assume all $N_j > 0$.

Lemma 1. *For $G = \mathbb{Z}_{3^l} \times H$, if any of the $N_j = 1$ then $M \equiv \pm 1$ mod $3|H|$.*

Proof. If $N_0 = 1$ then all the $N_j \equiv 1 \bmod 3|H|$ and $M \equiv 1 \bmod 3|H|$. If $N_j = 1$ for some $1 \leq j \leq l$ then $N_i \equiv 1 \bmod 3|H|$ for any $j < i \leq l$, with $N_0^{2\cdot 3^{j-1}} \equiv 1 \bmod 3|H|$ giving $N_0^{3^{j-1}} \equiv \pm 1 \bmod 3|H|$ and

$$M \equiv N_0^{1+\phi(3)+\cdots+\phi(3^{j-1})} = N_0^{3^{j-1}} \equiv \pm 1 \bmod 3|H|.$$

□

Notice that when $l = 1$, as in Theorems 2, 4 and 7, the values $\pm 1 \bmod 3|H|$ do not beat the trivial bound $|G| - 1$, and when $l > 1$ in Theorems 1, 3, 5 and 6 these are exactly the problem cases that we cannot eliminate when t is small. Hence in the proofs in Section 2 we may assume that all the $N_j > 1$ with $3 \nmid N_j$. In particular, since they are H measures, $N_j \geq \lambda(H)$ all j, with $N_j \geq 3^j + 1$ for $j \geq 1$ by Euler's Theorem. We can also assume that we are taking the least residue in (24), in fact that all the $N_j < 3|H|/2$, since $2 \cdot 3|H|/2 > |G| - 1$ when $l = 1$, and $2 \cdot (3+1) \cdots (3^{l-1}+1)3|H|/2 > 3^l|H| > |G| - 1$ when $l > 1$.

When $G = \mathbb{Z}_3 \times H$ is a 3-group we get

$$M_G(F) = AL, \quad L \equiv A^2 \bmod |G|.$$

Here L and A are the H measures of $F(e^{2\pi i/3}, x_2, \ldots, x_r)F(e^{-2\pi i/3}, x_2, \ldots, x_r)$ and $F(1, x_2, \ldots, x_r)$ respectively. Replacing F by $-F$ as necessary we can always assume that we are working with an $A > 0$. We have the following useful lemma:

Lemma 2. *If $H = H_1 \times H_2$ with H_1, H_2 of the form* (20) *with $n, m \geq 1$ and*

$$\lambda(H)\,(3\min\{|H_1|, |H_2|\} + 1) \geq 3|H| - 1,$$

then $G = \mathbb{Z}_3 \times H$ has

$$\lambda(G) = |G| - 1.$$

Proof. As above we can assume that a measure $M_G(F) < |G| - 1$ takes the form $M = AL$ with $L \equiv A^2 \bmod |G|$ and $A, L \geq \lambda(H)$. From (21) we have

$$L \equiv A^2 \equiv 1 \bmod 3h.$$

So $L \geq 3h + 1$ and $M_G(F) = AL \geq \lambda(H)(3\min\{|H_1|, |H_2|\} + 1) \geq |G| - 1$. □

5. Proofs for Section 2

Proof of Theorem 1. Suppose that $G = \mathbb{Z}_9 \times \mathbb{Z}_{3^t}$ with $t \geq 2$. In this case our trivial upper bound (8) takes the form

$$\lambda(G) \leq \min\{2^9, |G| - 1\} = \begin{cases} 2^9, & \text{if } t \geq 4, \\ 3^{t+2} - 1, & \text{if } 2 \leq t \leq 3. \end{cases} \tag{25}$$

Hence to prove Theorem 1 and (13) we just have to show that there are no measures $M > 1$ with $M < 2^9$ for $t \geq 5$, and none less than (25) that are not congruent to $\pm 1 \bmod 3^{t+1}$ for $t \leq 4$. Note that $3^{t+1} - 1 > 2^9$ for $t \geq 5$.

As above we can write

$$M = ABC, \;\; B \equiv A^2 \bmod 3^{t+1}, \; C \equiv A^6 \bmod 3^{t+1},$$

where B, C are positive integers and, replacing F by $-F$ as necessary, A is a positive integer with $3 \nmid A$. Note, since $M \equiv A^9 \bmod 3^{t+1}$, we have $M^2 \equiv A^{\phi(3^3)} \equiv 1 \bmod 3^3$ and $M \equiv \pm 1 \bmod 27$. So there is nothing to show for $t = 2$ and we can assume that $t \geq 3$. From the discussion above we can assume that $A, B, C \geq 2$, with $B \equiv 1 \bmod 3$, $C \equiv 1 \bmod 9$.

`Case 1.` Suppose that $2 \leq A < 3^{(t+1)/6}$. Then A^2 and A^6 are less than 3^{t+1} and $B \geq A^2$, $C \geq A^6$ and $M = ABC \geq A^9 \geq 2^9$.

`Case 2.` Suppose that $3^{(t+1)/6} < A < 3^{(t+1)/2}$. Since $A^2 < 3^{t+1}$ we have $B \geq A^2$, $C \geq 10$. For $t \geq 3$ we have $A \geq 4$

$$M = ABC \geq 10A^3 > 10 \cdot 4^3 > 2^9.$$

`Case 3.` Suppose that $A > 3^{(t+1)/2}$. Then $C \geq 10$, $B \geq 4$ and for $t \geq 4$ we have $A \geq 16$ and $M = ABC \geq 16 \cdot 4 \cdot 10 > 2^9$. For $t = 3$ we have $A \geq 10$ and $M = ABC \geq 10 \cdot 4 \cdot 10 > |G| - 1$.

Notice that $A \equiv \pm 1 \bmod 3^{t+1}$, $B = C = 1$ satisfies our congruences and so the $M \equiv \pm 1 \bmod 3^{t+1}$ will not be eliminated when any of these are less than 2^9. $\square$

Proof of Theorem 2. We take $G = \mathbb{Z}_3 \times H$ where $H = \mathbb{Z}_3^2 \times \mathbb{Z}_{3^t}$ or $\mathbb{Z}_9 \times \mathbb{Z}_{3^t}$. As above our measures can be written

$$M = AL, \quad L \equiv A^2 \bmod |G|.$$

We know that we can achieve 2^{27} and $|G|-1$ so need to show that there are no

$$M < B, \;\; B := \min\{2^{27}, |G|-1\} = \begin{cases} 3^{t+3}-1, & \text{for } 2 \leq t \leq 14, \\ 2^{27}, & \text{for } t \geq 15. \end{cases} \tag{26}$$

As discussed above we can assume that

$$\lambda(H) \leq A, L < |G|. \tag{27}$$

Moreover from (21) and (22) with $H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9$, $H_2 = \mathbb{Z}_{3^t}$ we have

$$A \equiv \pm 1 \bmod 27, \;\; L \equiv 1 \bmod 27.$$

In particular $A \geq 26, L \geq 28$ and $AL > B$ for $t \leq 3$. From (6) and Theorem 1 we can assume that $A, L \geq \lambda(H) \geq 242$ for $t = 4$ and $A, L \geq 2^9$ for $t \geq 5$. These give $AL > B$, and hence nothing to check, for $t \leq 8$.

For $9 \leq t \leq 15$ we found just two cases of $2^9 \leq A < B/2^9$, $A = \pm 1 \bmod 27$, that produced a least residue $L \equiv A^2 \bmod 3^{t+3}$ with $M = AL < B$, namely

$$\begin{aligned} t = 14, \; A = 27836, \; L = 1918, \; M = 53389448, \; B = 129140162, \\ t = 15, \; A = 27836, \; L = 1918, \; M = 53389448, \; B = 134217728. \end{aligned}$$

Fortunately we can eliminate these by showing that $A = 27836$ is not an H measure.

If A is an $H = \mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_{3^t})$ measure we know that

$$A = \mathcal{A}\mathcal{L}$$

where,

$$\mathcal{A} \equiv \pm 1 \bmod 9, \quad \mathcal{L} \equiv \mathcal{A}^2 \bmod 3^{t+2}. \tag{28}$$

If A is a $\mathbb{Z}_9 \times \mathbb{Z}_{3^t}$ measure then

$$A = \mathcal{A}\mathcal{L}\mathcal{T}, \;\; \mathcal{L} \equiv \mathcal{A}^2 \bmod 3^{t+1}, \; \mathcal{T} \equiv \mathcal{A}^6 \bmod 3^{t+1}. \tag{29}$$

Since $A < 3^{t+1}$ we can assume that $\mathcal{L}$ and $\mathcal{T}$ are least residues in the congruences, ruling out $\mathcal{A} = 1$. Also $A^2 \equiv L \not\equiv 1 \bmod 3^{t+1}$ ruling out $\mathcal{A} = A$. Hence one just has to check the proper divisors of $A = 27836$, namely $\mathcal{A} = 2, 4, 6959, 13918$. None of these has $\mathcal{A} \equiv \pm 1 \bmod 9$ as needed for (28) or produces an $\mathcal{A}\mathcal{L}\mathcal{T} = A$ in (29).

Thus our bound B is optimal for $2 \le t \le 15$. For $t = 15$ the minimum is 2^{27} and hence this will be the minimum for all $t \ge 15$, since the minimum does not go down as we increase t and we can always achieve this value.

This is slightly different from the approach used to rule out problem A's in [2]. Since L is the resultant of a polynomial with the third cyclotomic polynomial $\Phi_3(x) = x^2 + x + 1$, we know (see [2, Lemma 4.2]) that if q is a prime with $q^t || L$ then t cannot be smaller than the order of q mod 3. That is, L cannot be divisible by a single power of a prime $q \equiv 2 \bmod 3$, with $q = 2$ ruling out $L = 1918$. □

Proof of Theorem 3. With $G = \mathbb{Z}_{3^3} \times \mathbb{Z}_{3^t}$ we write

$$M = ABCD,\ B \equiv A^2 \bmod 3^{t+1},\ C \equiv A^6 \bmod 3^{t+1},\ D \equiv A^{18} \bmod 3^{t+1}.$$

As above, an $M \not\equiv \pm 1 \bmod 3^{t+1}$ must have $A, B, C > 1$. Notice that $3^{t+1} - 1 > 2^{27}$ when $t \ge 17$. Our achievable upper bound takes the form

$$\mathcal{B} := \min\{2^{27}, |G| - 1\} = \begin{cases} 2^{27}, & \text{if } t \ge 15, \\ 3^{t+3} - 1, & \text{if } 3 \le t \le 14. \end{cases}$$

Note $M \equiv A^{27} \bmod 3^{t+1}$. Hence $M^2 \equiv 1 \bmod 81$ and $M \equiv \pm 1 \bmod 81$ and there is nothing to show when $t = 3$. So assume that $t \ge 4$. By Euler's Theorem we have

$$B \equiv 1 \bmod 3, \quad C \equiv 1 \bmod 9, \quad D \equiv 1 \bmod 27.$$

Hence, to prove Theorem 3 and (16) we need to show that for $A, B, C, D > 1$ we have $M > \mathcal{B}$. Since $2 \cdot 4 \cdot 10 \cdot 28 > 3^7 - 1$, we can assume that $t \ge 5$.

We could at this stage, as in the proof of Theorem 2, simply test all the $2 \le A \le \mathcal{B}/4 \cdot 10 \cdot 28$ to see that they produce no $M = ABCD < \mathcal{B}$ for $t = 17$ and only $M \equiv \pm 1 \bmod 3^{t+1}$ for $t = 5$ to 16. Instead, as in the proof of Theorem 1, we consider different ranges for A.

`Case 1.` Suppose that B and $C \le 3^{(t+1)/3}$. Then $C = B^3$, $D = C^3$ and

$$M = AB^{13} \le 2 \cdot 4^{13} = 2^{27}.$$

This includes $A \le 3^{(t+1)/18}$ where $B = A^2$, $C = A^6$, $D = A^{18}$ and $M = A^{27} \ge 2^{27}$.

`Case 2.` Suppose that $3^{(t+1)/18} < A < 3^{(t+1)/6}$. Then $B = A^2$, $C = A^6$ and

$$M = A \cdot A^2 \cdot A^6 \cdot D = A^9 D.$$

If $t \ge 26$ then $A \ge 7$ and $D \ge 28$ gives

$$M \ge 28 \cdot 7^9 > 2^{27}.$$

Since $28 \cdot 2^9 > 3^8 - 1$ and $28 \cdot 4^9 > 3^{14} - 1$ that leaves $A = 2$ for $6 \le t \le 10$ and $A = 4$ or 5 for $12 \le t \le 25$. Now $2^{18} \equiv 1891 \bmod 3^7$ and $6265 \bmod 3^8$ with

$1891 \cdot 2^9 > 3^9 - 1$ and $6265 \cdot 2^9 > 3^{13} - 1$ so we can ignore $A = 2$. Similarly $4^{18} \equiv 2323 \bmod 3^8$, $5^{18} \equiv 2648 \bmod 3^7$ with $2323 \cdot 4^9 > 2^{27}$, resolving $A = 4$ or 5.

`Case 3.` Suppose that $3^{(t+1)/6} < A < 3^{(t+1)/2}$. Then $B = A^2$, $A \geq 4$ and

$$M = A \cdot A^2 \cdot CD = A^3 CD.$$

For $A \geq 79$ we have

$$M \geq 79^3 \cdot 10 \cdot 28 > 2^{27}.$$

For $4 \leq A \leq 77$, $3 \nmid A$, we have

$$(A^6 \bmod 3^6)(A^{18} \bmod 3^6) \geq 2296, \quad (A^6 \bmod 3^9)(A^{18} \bmod 3^9) \geq 511147.$$

Since $4^3 \cdot 2296 > 3^{10} - 1$ we can assume that $t \geq 8$ and

$$M = A^3 \cdot CD \geq 3^{(t+1)/2} \cdot 511147 > \mathcal{B}.$$

`Case 4.` Suppose that $A > 3^{(t+1)/2}$ with B or $C > 3^{(t+1)/3}$.

Note that for $t \geq 5$ we have $A \geq 28$, so if $B = 4$ then $A \equiv \pm 2 \bmod 3^{t+1}$ and

$$M \geq (3^{t+1} - 2) \cdot 4 \cdot 10 \cdot 28 > 3^{t+3} - 1.$$

So we can assume that $B \geq 7$ and

$$M \geq 3^{(t+1)/2} \cdot 7 \cdot 3^{(t+1)/3} \cdot 28 > \begin{cases} 2^{27}, & \text{if } t \geq 14, \\ 3^{t+3} - 1, & \text{it } t \leq 13. \end{cases}$$

$\square$

Proof of Theorem 4. We suppose that $G = \mathbb{Z}_3 \times H$ where

$$H = \mathbb{Z}_3^3 \times \mathbb{Z}_{3^t}, \quad \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_{3^t}, \quad \text{or} \quad \mathbb{Z}_{27} \times \mathbb{Z}_{3^t}.$$

Our achievable upper bound takes the form

$$B := \max\{2^{81}, 3^{t+4} - 1\} = \begin{cases} 2^{81}, & \text{if } t \geq 48, \\ 3^{t+4} - 1, & \text{if } t \leq 47. \end{cases}$$

We know from Theorem 2 and Theorem 3 that $\lambda(H) = 2^{27}$ for $t \geq 17$, and that $\lambda(H) \geq 3^{t+1} - 1$ for $t \leq 16$. We already know that $\lambda(\mathbb{Z}_3^5) = 3^5 - 1$ so we can assume that $t \geq 2$. We have

$$(3^{t+1} - 1) \cdot 28 \geq 3^{t+4} - 1,$$

and for $t = 17$

$$2^{27} \cdot 82 \geq 3^{t+4} - 1,$$

and by Lemma 2 we have $\lambda(G) = |G| - 1$ for $t \leq 17$. So assume that $t \geq 18$.

As usual we write the measures

$$M = AL, \quad L \equiv A^2 \bmod 3^{t+4}.$$

From the initial discussion we can assume that L is the least residue mod 3^{t+4} and

$$A, L \geq \lambda(H) = 2^{27}.$$

Since $AL \geq 2^{54} > 3^{t+4} - 1$ for $t \leq 30$ we can assume that $t \geq 31$. As A is an H-measure we have $A \equiv \pm 1 \bmod 81$ and $L \equiv 1 \bmod 81$.

If $A < 3^{(t+4)/2}$, then $L = A^2$ and

$$M = A^3 \geq 2^{81}.$$

So we can assume that

$$A > 3^{(t+4)/2}, \quad 2^{27} \leq L < B/A < B/3^{(t+4)/2} < 3^{(t+4)/2}. \tag{30}$$

We could proceed as before to find any pairs A, L with

$$3^{(t+4)/2} < A < (3^{t+4} - 1)/2^{27}, \quad A \equiv \pm 1 \bmod 81, \; L \equiv A^2 \bmod 3^{t+4},$$

that give an $AL < B$. But for large t the A range becomes unmanageable so we work instead with the smaller range of L.

`a) Small` L. Reversing the roles of A and L, for each $2^{27} \leq L \leq 3^{22}$, with $L \equiv 1 \bmod 81$, we found the value A with $A^2 \equiv L \bmod 3^{t+4}$. Using Hensel's Lemma it was straightforward to find a square root of L to successively higher powers; the recursively defined sequence

$$x_4 = 1, \;\; x_{j+1} := x_j + \lambda_j 3^j, \quad \lambda_j := \left(\frac{x_j^2 - L}{3^j} \right) \bmod 3,$$

will have $x_j^2 \equiv L \bmod 3^j$. We can assume that $A < 3^{t+4}/2$ so A will be the smaller of x_{t+4} and $3^{t+4} - x_{t+4}$.

For $t = 31$ all the way up to $t = 48$ we checked to find any $L < 3^{\min\{(t+4)/2, 22\}}$ giving an A with $AL < B$. Only two examples were found:

$t = 36$, $A = 11564355583, L = 437053078, M = 5054237202636634474,$

$t = 46$, $A = 2076248883915523, L = 227356795, M = 472049291869360360028785.$

For $31 \leq t \leq 40$ this checked all L from (30).

`b) Large` L. For $41 \leq t \leq 48$ we have already checked the small $L < 3^{22}$ and are left with

$$A > 3^{(t+4)/2}, \quad L > 3^{22}.$$

For $j^{1/2}3^{(t+4)/2} < A < (j+1)^{1/2}3^{(t+4)/2}$ we plainly have $L = A^2 - j3^{t+4}$, and for each j it is a matter of checking the first few $A > j^{1/2}3^{(t+4)/2}$ satisfying $A \equiv \pm 1 \bmod 81$ until $M = A(A^2 - j3^{t+4})$ exceeds B. We performed this check for $j = 1, \ldots, 2500$, and found no new A, L with $AL < B$.

This just leaves the $A > \sqrt{2501}\, 3^{(t+4)/2}$, $L > 3^{22}$. But these have

$$AL > \sqrt{2501}\, 3^{t/2+24} > \begin{cases} 3^{t+4} - 1, & \text{if } t \leq 47, \\ 2^{81}, & \text{if } t = 48, \end{cases}$$

and we are done.

It remains to rule out the two values encountered in Step a). We show that A is not an H-measure. For $H = \mathbb{Z}_3^3 \times \mathbb{Z}_{3^t}$ or $\mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_{3^t}$ we write $H = \mathbb{Z}_3 \times H_1$, with $H_1 = \mathbb{Z}_3^2 \times \mathbb{Z}_{3^t}$ or $\mathbb{Z}_9 \times \mathbb{Z}_{3^t}$, and observe that

$$A = ab, \quad b \equiv a^2 \bmod 3^{t+3},$$

where, since a is an H_1 measure $a \equiv \pm 1 \bmod 27$ and $b \equiv 1 \bmod 27$.

For $H = \mathbb{Z}_{27} \times \mathbb{Z}_{3^t}$ we have

$$A = abcd, \quad b \equiv a^2 \bmod 3^{t+1}, c \equiv a^6 \equiv (ab)^2 \bmod 3^{t+1}, d \equiv a^{18} \equiv (abc)^2 \bmod 3^{t+1},$$

where the most we can say is $b \equiv 1 \bmod 3$, $c \equiv 1 \bmod 9$ and $d \equiv 1 \bmod 27$.

Notice that an a, b, c or $d = 1$ would lead to an $A \equiv \pm 1 \bmod 3^{t+1}$ and in both cases $A < 3^{t+1} - 1$. So we can assume that $a, b, c, d > 1$.

Hence it is enough to check whether the few, if any, proper divisors ℓ of A with $\ell \equiv 1 \bmod 27$, have $(A/\ell)^2 \equiv \ell \mod 3^{t+1}$, with ℓ playing the role of b in the first case or d in the second case. No such ℓ were found. Alternatively we could eliminate these L using [2, Lemma 4.2] with $q = 2$ or 5.

This gives the result for $t \leq 48$. Since the minimal measure cannot go down for higher t, and 2^{81} is always achievable, we get $\lambda(G) = 2^{81}$ for all $t \geq 48$. □

Proof of Theorem 5. We write $G = \mathbb{Z}_9 \times H$ where $H = \mathbb{Z}_9 \times \mathbb{Z}_{3^t}$. Our achievable upper bound (8) takes the form

$$\mathcal{B} = \max\{2^{81}, 3^{t+4} - 1\} = \begin{cases} 2^{81}, & \text{if } t \geq 48, \\ 3^{t+4} - 1, & \text{if } 2 \leq t \leq 47. \end{cases} \tag{31}$$

As usual we write

$$M = ABC, \quad B \equiv A^2 \bmod 3^{t+3}, \ C \equiv A^6 \equiv B^3 \equiv (AB)^2 \bmod 3^{t+3}.$$

As observed above either $M \equiv \pm 1 \bmod 3^{t+3}$, or we can assume that

$$\lambda(H) \leq A, B, C < 3^{t+3},$$

with $3^{t+3}-1>2^{81}$ for $t\geq 49$. Since A is an H measure we have

$$A\equiv \pm 1 \bmod 27, \;\; B\equiv 1 \bmod 27, \;\; C\equiv 1 \bmod 81.$$

For $t\leq 4$ we have
$$M\geq \lambda(H)^3\geq (3^{t+1}-1)^3>3^{t+4}-1,$$
and for $5\leq t\leq 13$
$$M\geq \lambda(H)^3=2^{27}>3^{t+4}-1,$$
so we can assume that $t\geq 14$.

Observe that if we have an H-measure then we can similarly write it in the form

$$m=abc, \; b\equiv a^2 \bmod 3^{t+1}, \;\; c\equiv a^6\equiv (ab)^2 \bmod 3^{t+1}.$$

In particular if $m<3^{(t+1)/2}$ then we must have $a,ab<3^{(t+1)/2}$. Since $3^{t+1}+1>m$ we know that b, c must be the least residues and $b=a^2$, $c=(ab)^2$ and $m=a^9$.

We consider four ranges for A and B.

`Case 1.` Suppose that $A<3^{(t+3)/6}$.

Then A^2, $A^6<3^{t+3}$ and

$$M=A\cdot A^2\cdot A^6=A^9\geq \lambda(H)^9\geq 2^{81}.$$

`Case 2.` Suppose that $3^{(t+3)/6}<A<3^{(t+3)/2}$.

Then $B=A^2$ and $M=A^3C$. We can assume that $82A^3<\mathcal{B}<3^{t+4}$ and $A<3^{(t+1)/2}$ and hence $A=a^9$ and $M=a^{27}C$ for some a with

$$3^{(t+3)/54}<a<(\mathcal{B}/82)^{1/27}$$

where $(2^{81}/82)^{1/27}<7$. That is we just have to check $a=2$ for $14\leq t\leq 31$, $a=4$ for $35\leq t\leq 49$, and $a=5$ for $40\leq t\leq 49$ and test that in these cases

$$a^{27}\left(a^{54} \bmod 3^{t+3}\right)>\mathcal{B}.$$

`Case 3.` Suppose that $A>3^{(t+3)/2}$ and $B<3^{(t+3)/3}$.

Since $B^3<3^{t+3}$ we have $C=B^3$ and $M=AB^4$. Since $B<3^{(t+3)/3}<3^{(t+1)/2}$ we have $B=b^9$ for some b. Hence $M=Ab^{36}$. Since $A\geq \lambda(H)\geq 2^9$, clearly we just have to check $b<(\mathcal{B}/2^9)^{1/36}\leq (2^{81}/2^9)^{1/36}=4$ which leaves only $b=2$. But $B=2^9\not\equiv 1 \bmod 27$, ruling $b=2$ out as well.

`Case 4.` Suppose that $A>3^{(t+3)/2}$ and $B>3^{(t+3)/3}$.

Then $AB>3^{\frac{5}{6}(t+3)}$ and $C<\mathcal{B}/AB<3^{t+4}/3^{\frac{5}{6}(t+3)}=3^{(t+9)/6}<3^{(t+1)/2}$. So we know that $C=c^9$ for some $c<3^{(t+9)/54}$ with $3^{58/54}<4$, and we are just left with $c=2$. But $C=2^9\not\equiv 1 \bmod 81$ so this cannot occur. $\square$

Proof of Theorem 6. For $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^t}$ we again have the achievable upper bound (31) and measures take the form $M = ABCDE$ with

$$B \equiv A^2 \bmod 3^{t+1}, \quad C \equiv A^6 \equiv B^3 \equiv (AB)^2 \bmod 3^{t+1},$$
$$D \equiv A^{18} \equiv C^3 \equiv (ABC)^2 \bmod 3^{t+1}, \quad E \equiv A^{54} \equiv D^3 \equiv (ABCD)^2 \bmod 3^{t+1}.$$

Setting the $M \equiv \pm 1 \bmod 3^{t+1}$ aside, we can assume that $2 \leq A, B, C, D, E < 3^{t+1}$, with $B \equiv 1 \bmod 3$, $C \equiv 1 \bmod 9$, $D \equiv 1 \bmod 27$ and $E \equiv 1 \bmod 81$.

Hence any $M < \mathcal{B}$ must come from $2 \leq A \leq \mathcal{B}/(4 \cdot 10 \cdot 28 \cdot 82)$. In particular we can assume that $t \geq 8$, otherwise there are no A to test.

For $8 \leq t \leq 51$ we checked all $2 \leq A \leq \min\{12029, \mathcal{B}/91840\}$ and found no $M < \mathcal{B}$.

Case 1. Suppose that $A < 3^{(t+1)/2}$.

Then $B = A^2$ and $M = A^3CDE$.

(i) Suppose that $C < 3^{(t+1)/9}$. Then $D = C^3$, $E = C^9$ and $M = A^3C^{13}$.

If $A < 3^{(t+1)/6}$ then $C = A^6$ and $M = A^{81} \geq 2^{81}$.

If $A > 3^{(t+1)/6}$ then

$$M > 3^{(t+1)/2}\, 10^{13} = 3^{t+4}3^{-(t+7)/2}10^{13},$$

with this greater than 2^{81} for $t \geq 48$ and $3^{t+4} - 1$ for $t \leq 47$.

(ii) Suppose that $3^{(t+1)/9} < C < 3^{(t+1)/3}$. Then $D = C^3$ and $M = A^3C^4E$.

For $t \geq 48$ we have $C \geq 397$ and for $A \geq 10589$

$$M = A^3C^4E \geq 10589^3\, 397^4\, 82 > 2^{81},$$

while for $t \leq 47$ and $A \geq 12030$

$$M = A^3C^4E \geq A^3 3^{4(t+1)/9} 82 = 3^{t+4}A^3 3^{-(5t+32)/9}82 > 3^{t+4} - 1.$$

(iii) Suppose that $C > 3^{(t+1)/3}$ and $D < 3^{(t+1)/3}$. Then $E = D^3$ and $M = A^3CD^4$.

If $A < 3^{(t+1)/6}$ then $C = A^6$ and for $A \geq 117$ we have

$$M = A^9D^4 \geq 117^9\, 28^4 > 2^{81}.$$

If $A > 3^{(t+1)/6}$ then for $t \leq 51$

$$M = A^3CD^4 > 3^{(t+1)/2}3^{(t+1)/3}28^4 = 3^{t+4}3^{-(t+19)/6}28^4 > 3^{t+4} - 1.$$

(iv) Suppose that $C > 3^{(t+1)/3}$ and $D > 3^{(t+1)/3}$.

If $A < 3^{(t+1)/6}$ then $C = A^6$ and for $A \geq 44$ we have

$$M = A^9DE \geq 44^9 3^{(t+1)/3}82 = 3^{t+4}44^9 3^{-(2t+11)/3}82$$

greater than 2^{81} for $t \geq 48$ and $3^{t+4}-1$ for $t \leq 7$.

If $A > 3^{(t+1)/6}$ then

$$M = A^3CDE > 3^{(t+1)/2}3^{(t+1)/3}3^{(t+1)/3}82 = 3^{7(t+1)/6}82 > 3^{t+4}-1. \tag{32}$$

Case 2. Suppose that $A > 3^{(t+1)/2}$.

If at least two of B, C, D are greater than $3^{(t+1)/3}$ then as in (32)

$$M = A(BCD)E \geq 3^{(t+1)/2}3^{2(t+1)/3}82 > 3^{t+4}.$$

If all B, C, D are less than $3^{(t+1)/3}$ then $C = B^3, D = C^3, E = D^3$ and

$$M = AB^{40} \geq 2 \cdot 4^{40} = 2^{81}.$$

So we can suppose that exactly one of B, C, D, is greater than $3^{(t+1)/3}$. If it is B we get $D = C^3$, $E = D^3$ and for $t \leq 51$

$$M = ABC^{13} \geq 3^{(t+1)/2}3^{(t+1)/3}10^{13} = 3^{t+4}3^{-(t+19)/6}10^{13} > 3^{t+4}-1.$$

If it is C then $C = B^3$, $E = D^3$ and for $t \leq 51$

$$M = AB^4D^4 \geq 3^{(t+1)/2}3^{4(t+1)/9}28^4 = 3^{t+4}3^{-(t+55)/18}28^4 > 3^{t+4}-1.$$

If it is D then $C = B^3, D = B^9$ and for $t \leq 51$

$$M = AB^{13}E > 3^{(t+1)/2}3^{13(t+1)/27}82 = 3^{t+4}3^{-(t+163)/54}82 > 3^{t+4}-1.$$

Since we have checked numerically the $A \leq 12029$ we have proved the claim for all $t \leq 51$, with the minimum value 2^{81} when $t = 51$. Since the value cannot go down and we can achieve 2^{81}, this must be the minimum for all $t \geq 51$. □

6. Proofs for Section 3

Proof of Theorem 7. As discussed in Section 4, when $G = \mathbb{Z}_3 \times H$ we can assume that any measure $1 < M_G(F) < |G|-1$ takes the form

$$M_G(F) = AL, \quad L \equiv A^2 \bmod |G|, \quad 3 \nmid A, \quad A, L \geq \lambda(H).$$

Since $L \equiv A^2 \equiv 1 \bmod 3$ we have $L \geq 4$ and if $\lambda(H) = |H|-1$

$$M_G(F) = AL \geq 4(|H|-1) \geq |G|-1.$$

If $\lambda(H)^2 \geq |G|-1$ then plainly$M_G(F) = AL \geq \lambda(H)^2 \geq |G|-1$. □

Notice that if we have $\lambda(H)^2 \geq 3m|H| \pm 1$ for some integer $m \geq 1$ we can further say that the only measures up $m|G| \pm 1$ are the $j|G| \pm 1$ with $j \leq m$, all achievable by (9), or multiples of 3, though as discussed in Section 4 these are usually large enough to be ignored, for example if $3 \mid M$ then $3^{(1+\alpha_1)\cdots(1+\alpha_r)} \mid M$.

Proof of Corollary 1. We write $G = \mathbb{Z}_3 \times H$ with $H = \mathbb{Z}_{3^{\beta_2}} \times \cdots \times \mathbb{Z}_{3^{\beta_r}} \times H_1 \times \mathbb{Z}_{3^t}$ and proceed by induction on r, noting that $(3^m - 1)^2 > 3^{2m-1}$ and $t \geq \beta_2$.

For $r = 2$ we use Theorem 4. For $t \leq 47$

$$\lambda(H)^2 \geq \lambda(\mathbb{Z}_3 \times H_1 \times \mathbb{Z}_{3^t})^2 = (3^{4+t} - 1)^2 > 3^{2t+7} > 3^{4+\beta_2+t} - 1 = 3|H| - 1,$$

while for $48 \leq t \leq 98 - \beta_2$

$$\lambda(H)^2 \geq \lambda(\mathbb{Z}_3 \times H_1 \times \mathbb{Z}_{3^t})^2 = 2^{162} > 3^{102} - 1 \geq 3^{4+\beta_2+t} - 1 = 3|H| - 1.$$

Suppose $r \geq 3$. If $t \leq 101 \cdot 2^{r-3} - \beta_3 - \cdots - \beta_r - 3$ then by the inductive assumption

$$\begin{aligned}\lambda(H)^2 &\geq \lambda(\mathbb{Z}_3 \times \mathbb{Z}_{3^{\beta_3}} \times \cdots \times \mathbb{Z}_{3^{\beta_r}} \times H_1 \times \mathbb{Z}_{3^t})^2 = (3^{4+\beta_3+\cdots+\beta_r+t} - 1)^2 \\ &> 3^{7+2\beta_3+\cdots+2\beta_r+2t} > 3^{4+\beta_2+\cdots+\beta_r+t} - 1 = 3|H| - 1,\end{aligned}$$

while if $101 \cdot 2^{r-3} - \beta_3 - \cdots - \beta_r - 3 < t \leq 101 \cdot 2^{r-2} - \beta_2 - \cdots - \beta_r - 3$ then

$$\begin{aligned}\lambda(H)^2 &\geq \lambda(\mathbb{Z}_3 \times \mathbb{Z}_{3^{\beta_3}} \times \cdots \times \mathbb{Z}_{3^{\beta_r}} \times H_1 \times \mathbb{Z}_{3^{101\cdot 2^{r-3}-\beta_3-\cdots-\beta_r-3}})^2 \\ &= (3^{1+101\cdot 2^{r-3}} - 1)^2 \\ &> 3^{1+101\cdot 2^{r-2}} > 3^{4+\beta_2+\cdots+\beta_r+t} - 1 = 3|H| - 1.\end{aligned}$$

□

Proof of Corollary 2. We write $G = \mathbb{Z}_3 \times H$ with $H = \mathbb{Z}_{3^{\beta_2}} \times \cdots \times \mathbb{Z}_{3^{\beta_r}} \times H_1 \times H_2$ and proceed by induction on r.

For $r = 1$ we have $H = H_1 \times H_2$ and $s = t$ or $t + 1$. By (23)

$$\lambda(H)\,(3\min\{|H_1|, |H_2|\} + 1) \geq (3^{t+1} - 1)(3^{t+1} + 1) \geq 3^{s+t+1} - 1,$$

and $\lambda(G) = |G| - 1$ from Lemma 2. So assume that $r \geq 2$.

If $t \leq s \leq (2^{r-1} - 1)t + 2^{r-2} - \beta_3 - \cdots - \beta_r$ then by the inductive assumption

$$\begin{aligned}\lambda(H)^2 &\geq \lambda(\mathbb{Z}_3 \times \mathbb{Z}_{3^{\beta_3}} \times \cdots \times \mathbb{Z}_{3^{\beta_r}} \times H_1 \times H_2)^2 = (3^{1+\beta_3+\cdots+\beta_r+t+s} - 1)^2 \\ &> 3^{1+2\beta_3+\cdots+2\beta_r+2s+2t} > 3^{1+\beta_2+\cdots+\beta_r+s+t} - 1 = 3|H| - 1.\end{aligned}$$

If $(2^{r-1} - 1)t + 2^{r-2} - \beta_3 - \cdots - \beta_r < s \leq (2^r - 1)t + 2^{r-1} - \beta_2 - \cdots - \beta_r$ we take H_3 to be a subgroup of H_2 of order $(2^{r-1} - 1)t + 2^{r-2} - \beta_3 - \cdots - \beta_r$, and

$$\begin{aligned}\lambda(H)^2 &\geq \lambda(\mathbb{Z}_3 \times \mathbb{Z}_{3^{\beta_3}} \times \cdots \times \mathbb{Z}_{3^{\beta_r}} \times H_1 \times H_3)^2 = (3^{1+2^{r-1}t+2^{r-2}} - 1)^2 \\ &> 3^{1+2^r t+2^{r-1}} > 3^{1+\beta_2+\cdots+\beta_r+s+t} - 1 = 3|H| - 1.\end{aligned}$$

□

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