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RELAXED COMPLETE PARTITIONS: AN ERROR-CORRECTING BACHET'S PROBLEM

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Abstract

Motivated by an error-correcting generalization of Bachet's weights problem, we define and classify relaxed complete partitions. We show that these partitions enjoy a succinct description in terms of lattice points in polyhedra. Our main result is an enumeration of the minimal such partitions (those with fewest possible parts) via Brion's formula. This generalizes work of Park on classifying complete partitions and that of Rødseth on enumerating minimal complete partitions.

1. Introduction

Bachet's weights problem asks for the least number of integer weights that can be used on a two-pan scale to weigh any integer between 1 and 40 inclusive. Its unique solution consists of four parts and can be written as the integer partition $40 = 1 + 3 + 9 + 27$. While the problem was popularized by Bachet in 1612 [2, page 143], its noble roots can be traced to Fibonacci's *Liber Abaci* [16, On IIII Weights Weighing Forty Pounds]. Its naming as *Bachet's Weights Problem* occurs in numerous sources, including influential texts from the early twentieth century like Hardy and Wright's *Theory of Numbers* [10, §9.7] and Ball's popular *Mathematical Recreations and Essays* [3, Chapter 1]. Despite these authors' collective awareness that Bachet only recorded the problem – Ball credits Tartaglia for solving the weights problem in 1556 – the attribution to Bachet has endured.

Generalizations of Bachet's problem – replacing 40 with any integer – include MacMahon's perfect and subperfect partitions [11], Brown's complete partitions [8]

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and Park's extension of Brown's work [15]. These generalizations begin with the simple observation that a partition $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ realizes every integer between $-m$ and m on a two-scale pan if and only if $\{\sum_{i=0}^{n} \alpha_i \lambda_i : \alpha_i \in \{0, 1, 2\}\} = \{0, 1, 2, \ldots, 2m\}$; the solution to the original problem now follows from the unique base 3 representation of every integer.

Tanton [19] considered an error-correcting variant of Bachet's problem. Given a fixed but unknown integer weight l, weighing no more than 80 pounds, what is the least number of integer weights that can be used on a two-pan scale to discern l's value? We still need only four parts, the partition $80 = 2 + 6 + 18 + 54$ suffices: if the unknown integer weight l is even then its exact value will be realized by the parts of $2 + 6 + 18 + 54$ with a balanced scale, if the weight l is odd then both $l - 1$ and $l + 1$ can be achieved by the parts of $2 + 6 + 18 + 54$ and the unknown weight will be heavier than the former and lighter than the latter. Equivalently, the parts of $80 = 2 + 6 + 18 + 54$ can be used to weigh every integer between 1 and 80 on a two-scale pan allowing for an error of one. Tanton's variant along with the simple observation common to previous generalizations suggests the following definition.

Definition 1. A partition $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ with the parts in weakly increasing order is an *e-relaxed r-complete partition* (*(e, r)-partition* for short) if no $e + 1$ consecutive integers between 0 and rm are absent from the set $\{\sum_{i=0}^{n} \alpha_i \lambda_i : \alpha_i \in \{0, 1, \ldots, r\}\}$. An (e, r)-partition with the fewest number of parts is called *minimal*.

Park's *r-complete partitions* [15] are the $(0, r)$-partitions. The $(0, 1)$-partitions are the variant of Bachet's problem where weights are allowed on only one of the two pans and were first introduced as Brown's *complete partitions* [8]. MacMahon's *perfect* and *subperfect partitions* [11] are the $(0, 1)$-partitions and $(0, 2)$-partitions respectively, but with additional conditions that generalize the uniqueness of representations in a given base. Tanton's version of Bachet asks for the minimal $(1, 2)$-partitions of 80. The $(0, r)$-complete partitions were enumerated by Park [15]. The minimal partitions were enumerated by O'Shea [13] (a partial enumeration for the $r = 1$ case) and by Rødseth ($r = 1$ [17] and $r \geq 2$ [18]).

We provide a complete classification of minimal (e, r)-partitions as sets of integer points in polyhedra (Theorem 1 and Proposition 1 in Section 2). These sets are enumerated using *Brion's theorem* [7], a formula that encodes lattice points in a given polyhedron by first attaching a generating function to each vertex in the polyhedron and then taking the formal sum of these generating functions. In Section 3, we attain our main result by applying Brion's theorem to $P_n(e, r)$, a transformed version of the polyhedron of the minimal (e, r)-partitions with $n + 1$ parts:

Theorem 2. *If $r \geq 2$, the minimal relaxed complete partitions are enumerated by the generating function* $\sigma_{P_n(e,r)}(x) = \sigma_{\mathcal{K}_{\mu(\emptyset)}}(x) + \sum_{j=1}^{n} \sigma_{\mathcal{K}_{\mu(\{j\})}}(x) + O(x^{(e+1)(r+1)^n})$.

The polyhedron $P_n(e,r)$ is combinatorially isomorphic to the $(n+1)$-cube. Theorem 2 states that we only need enumerate $n+1$ of $P_n(e,r)$'s 2^{n+1} vertex cones – the vertex cone at the origin, indexed by $\mathcal{K}_{\mu(\emptyset)}$, and all but one of the vertex cones of those vertices neighboring the origin, those indexed by $\mathcal{K}_{\mu(\{j\})}$. Each of these vertex cones and their generating functions are described explicitly in Section 3. Theorem 3 addresses the $r=1$ case. Our results agree with those of Park, O'Shea, and Rødseth when $e=0$.

This use of discrete geometric tools to enumerate families of integer partitions described by linear inequalities contributes to an existing literature that is especially inspired by the 1997 papers [5, 6] of Bousquet-Mélou and Eriksson on *lecture hall partitions*. These partitions, arising from Bott's formula for enumerating the lengths of reduced words in the affine Coxeter groups of type C, can be described in terms of lattice points in polyhedra, and they provide a refinement of Euler's classical odd-distinct theorem. Other papers in the same discete geometric spirit include Pak's use of lattice point enumeration when the family of integer partitions are points in a unimodular cone [14], Andrew's proof of the lecture hall theorem [1] utilizing MacMahon's *Partition Analysis* (the Ω operator) [12, Vol. 2, §VIII], and Corteel, Lee, and Savage's broad introduction to discrete geometric methods in integer partitions [9]. What the (e,r)-partitions share with lecture hall partitions is that they both generalize classical integer partition problems by viewing them through a modern geometric lens.

To illustrate our main result, consider the six minimal $(1,2)$-partitions of $m=12$: 1+3+8, 1+4+7, 2+2+8, 2+3+7, 2+4+6 and 2+5+5. These are transformed to six μ lattice points via $\mu_i := (e+1)(r+1)^i - \lambda_i$: $(1,3,10)$, $(1,2,11)$, $(0,4,10)$, $(0,3,11)$, $(0,2,12)$ and $(0,1,13)$ respectively. These six lattice points have coordinate sum equal to $26-12=14$ in the polyhedron $P_2(1,2)$ below.

By consulting Figure 1 and Theorem 2, our enumerating function for the lattice

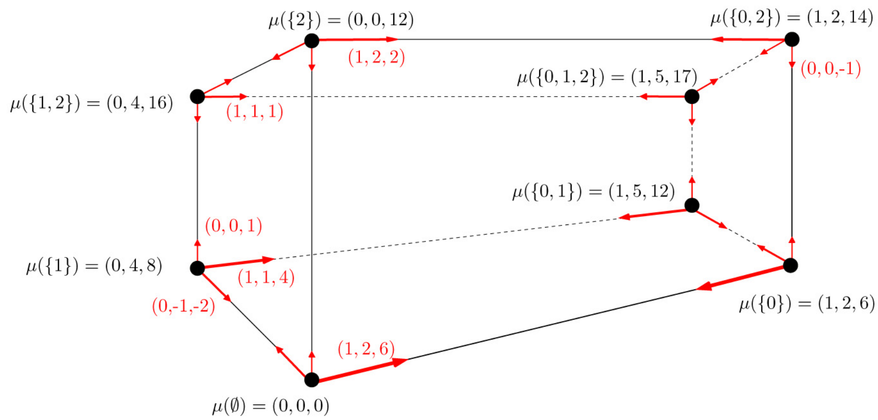

Figure 1: The polyhedron $P_2(1,2)$ with vertices and vertex cones labeled.

points in $P_2(1,2)$ is:

$$\sigma_{P_2(1,2)}(x) = \sigma_{\mathcal{K}_{\mu(\emptyset)}}(x) + \sigma_{\mathcal{K}_{\mu(\{1\})}}(x) + \sigma_{\mathcal{K}_{\mu(\{2\})}}(x) + O(x^{18})$$

$$\begin{aligned}
&= x^{|(0,0,0)|}\frac{1}{1-x^{|(1,2,6)|}}\frac{1}{1-x^{|(0,1,2)|}}\frac{1}{1-x^{|(0,0,1)|}} \\
&\quad + x^{|(0,4,8)|}\frac{1}{1-x^{|(0,-1,-2)|}}\frac{1}{1-x^{|(1,1,4)|}}\frac{1}{1-x^{|(0,0,1)|}} \\
&\quad + x^{|(0,0,12)|}\frac{1}{1-x^{|(0,0,-1)|}}\frac{1}{1-x^{|(1,2,2)|}}\frac{1}{1-x^{|(0,1,1)|}} + O(x^{18})
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(1-x^9)(1-x^3)(1-x^1)} - \frac{x^{15}}{(1-x^3)(1-x^6)(1-x^1)} \\
&\quad - \frac{x^{13}}{(1-x)(1-x^5)(1-x^2)} + O(x^{18}).
\end{aligned}$$

The coefficient of $x^{26-12} = x^{14}$ equals $7 - 0 - 1 = 6$ as expected.

2. Classifying Relaxed Complete Partitions

In this section we classify the (e,r)-partitions as a collection of lattice points in a polyhedron and describe the minimal such partitions. Call $\{\sum_{j=0}^{n}\alpha_j\lambda_j : \alpha_j \in \{0,1,\ldots,r\}\}$ the *r-cover* of $\lambda_0 + \lambda_1 + \cdots + \lambda_n$.

Theorem 1. *A partition* $m = \lambda_0 + \cdots + \lambda_n$ *such that* $\lambda_0 \le e+1$ *is an* (e, r)*-partition if and only if* $\lambda_i \le (e+1) + r\sum_{j=0}^{i-1} \lambda_j$ *for all* $i \le n$.

Proof. If $\lambda_i > (e+1) + r\sum_{j=0}^{i-1} \lambda_j$ then the shifted set $r\sum_{j=0}^{i-1} \lambda_j + \{1, 2, \ldots, e+1\}$ is omitted from the r-cover of $\lambda_0 + \lambda_1 + \cdots + \lambda_n$.

We show necessity by induction on the number of parts in the partition. If $n = 0$ then $\lambda_0 = m \le e+1$ in accordance with our hypothesis. Let $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1}$ be an (e, r)-partition onto which we append any part λ_n with $\lambda_{n-1} \le \lambda_n \le (e+1) + r\sum_{j=0}^{n-1} \lambda_j$. We wish to show that every positive integer $l \le r(\lambda_0 + \lambda_1 + \cdots + \lambda_n)$ is within a distance of at most $e+1$ of some integer in the r-cover of $\lambda_0 + \lambda_1 + \cdots + \lambda_n$.

If $l \le r(\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1})$, then the induction hypothesis readily applies. So we can fix l in the interval $r\sum_{j=0}^{n-1} \lambda_j < l \le r\sum_{j=0}^{n} \lambda_j$. In this case there will always exist an α_n, $1 \le \alpha_n \le r$, such that $(\alpha_n - 1)\lambda_n + r\sum_{j=0}^{n-1} \lambda_j < l \le \alpha_n \lambda_n + r\sum_{j=0}^{n-1} \lambda_j$, or $\alpha_n = \left\lceil \frac{l - r\sum_{j=0}^{n-1} \lambda_j}{\lambda_n} \right\rceil$. Since $l - \alpha_n \lambda_n \le r\sum_{j=0}^{n-1} \lambda_j$, the inductive hypothesis implies that $l - \alpha_n \lambda_n$ is within distance $e+1$ of an integer in the r-cover of $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1}$ and so l must be within distance $e+1$ of an integer in the r-cover of $\lambda_0 + \lambda_1 + \cdots + \lambda_n$. ◻

Park [15] gave a similar proof of the above theorem for $e = 0$. A novel feature of Theorem 1 is that the defining hyperplanes of the polyhedra that cut out the (e, r)-partitions arise from translating by e the defining hyperplanes of the $(0, r)$-partitions. Some simple corollaries include: If $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a $(0, r)$-partition of m then $(e+1)\lambda_0 + (e+1)\lambda_1 + \cdots + (e+1)\lambda_n$ is an (e, r)-partition of $(e+1)m$. The partition $m = 1 + 1 + \cdots + 1$ is always an (e, r)-partition of m. Every (e, r)-partition of m is both an $(e+1, r)$-partition and an $(e, r+1)$-partition of m.

The *minimal* (e, r)-partitions are those (e, r)-partitions with fewest possible parts. Using the inequalities of Theorem 1, a standard inductive argument shows that

$$\lambda_i \le (e+1)(r+1)^i \tag{1}$$

for all (e, r)-partitions $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$. The size of any such partition m cannot exceed $\sum_{i=0}^{n} (e+1)(r+1)^i = \frac{(e+1)}{r}((r+1)^{n+1} - 1)$. That is,

$$m \le \frac{e+1}{r}((r+1)^{n+1} - 1) < \frac{e+1}{r}(r+1)^{n+1}, \quad \text{or} \ \log_{r+1}\left(\frac{rm}{e+1}\right) < n+1.$$

Since $n+1$ is an integer, the integer part of $\log_{r+1}(\frac{rm}{e+1})$ is strictly less than $n+1$, or $\lfloor \log_{r+1}(\frac{rm}{e+1}) \rfloor \le n$, so an (e, r)-partition of m must have at least $\lfloor \log_{r+1}(\frac{rm}{e+1}) \rfloor + 1$ parts. The construction of an (e, r)-partition with this number of necessary parts would ensure that this suffices for the number of parts of a minimal such partition.

Proposition 1. *A minimal* (e, r)*-partition of* m *has* $\lfloor \log_{r+1}(\frac{rm}{e+1}) \rfloor + 1$ *parts.*

Proof. We claim that, for any integer m, the partition whose parts are exactly those in the multiset

$$\{\,(e{+}1),\ (e{+}1)(r{+}1),\ (e{+}1)(r{+}1)^2, \ldots, (e{+}1)(r{+}1)^{n-1},\ m-\frac{(e+1)}{r}((r{+}1)^n-1)\}$$

is an (e,r)-partition of m.

Since $1+(r+1)+(r+1)^2+\cdots+(r+1)^{n-1}$ is a $(0,r)$-partition of $\frac{1}{r}((r+1)^n-1)$, it follows that $(e+1)+(e+1)(r+1)+(e+1)(r+1)^2+\cdots+(e+1)(r+1)^{n-1}$ is an (e,r)-partition of $\frac{(e+1)}{r}((r+1)^n-1)$. For each $0 \le \alpha \le r$, the shifted set

$$\alpha \cdot \left(m - \frac{(e+1)}{r}((r+1)^n-1)\right) + \left\{\sum_{i=0}^{n-1} \alpha_i (e+1)(r+1)^i : \alpha_i \in \{0,1,\ldots,r\}\right\}$$

will not omit any consecutive $(e+1)$ integers and, since $m < \frac{e+1}{r}(r+1)^{n+1}$, the union over $1 \le \alpha \le r$ of these shifted sets ranges from 1 to rm and does not omit any $(e+1)$ consecutive integers. This union is precisely the r-cover of the partition whose parts consist of the elements from the given multiset. $\square$

The above proposition also tells us that *minimality* is preserved by the multiplication of parts by $e+1$. This is consistent with the "doubling of parts" in solving Tanton's variant of Bachet's problem with $80 = 2{+}6{+}18{+}54$ from $40 = 1{+}3{+}9{+}27$.

In summary, the minimal (e,r)-partitions are realized as $(\lambda_0, \lambda_1 \ldots, \lambda_n) \in \mathbb{Z}^{n+1}$, each lying inside the polyhedron defined by the inequalities $\lambda_{i-1} \le \lambda_i \le (e+1) + r\sum_{j=0}^{i-1} \lambda_j$ for each $i \le n$. The remainder of this article focuses on the geometric character of the partitions starting with a description of Brion's formula [7].

In the interests of brevity, we will give a description of Brion's formula in a manner best suited to our needs. Given an integer vector $\boldsymbol{u}$ define the *primitive part of* $\boldsymbol{u}$ as $\mathrm{p}(\boldsymbol{u}) := \frac{1}{\gcd(\boldsymbol{u})}\boldsymbol{u}$, where the $\gcd(\boldsymbol{u})$ is the greatest common divisor of the entries of $\boldsymbol{u}$. Call $\boldsymbol{u}$ *primitive* if $\gcd(\boldsymbol{u})$ equals 1. Given two integral vectors $\boldsymbol{u}$ and $\boldsymbol{u}'$ write $\mathrm{p}(\boldsymbol{u}'-\boldsymbol{u})$ for $\mathrm{p}_{\boldsymbol{u}}(\boldsymbol{u}')$, the *primitive vector of* $\boldsymbol{u}'$ *relative to* $\boldsymbol{u}$. Note that $\mathrm{p}_{\boldsymbol{u}}(\boldsymbol{u}') = -\mathrm{p}_{\boldsymbol{u}'}(\boldsymbol{u})$.

Given a vertex $\mathbf{v}$ of a polyhedron $\mathcal{P}$, the *vertex cone of* $\mathcal{P}$ *at* $\mathbf{v}$ is the smallest cone with apex $\mathbf{v}$ that contains $\mathcal{P}$. If the polyhedron $\mathcal{P}$ is *integral* (its vertices are lattice points), letting $\mathcal{N}(\mathbf{v})$ denote the neighbors of $\mathbf{v}$ in $\mathcal{P}$, we can write the vertex cone at $\mathbf{v}$ as

$$\mathcal{K}_{\mathbf{v}} := \mathbf{v} + \mathbb{R}_{\ge 0}\{\mathrm{p}_{\mathbf{v}}(\mathbf{v}') \,:\, \mathbf{v}' \in \mathcal{N}(\mathbf{v})\}$$

where the latter component is understood as the non-negative real span of the set of primitive vectors of the neighbors of $\mathbf{v}$ relative to $\mathbf{v}$ itself. The polyhedron $\mathcal{P}$ is said to be *simple* if the generators $\{\mathrm{p}_{\mathbf{v}}(\mathbf{v}') \,:\, \mathbf{v}' \in \mathcal{N}(\mathbf{v})\}$ of each vertex cone $\mathcal{K}_{\mathbf{v}}$ form a linearly independent set.

Assuming that the polyhedron $\mathcal{P}$ is full dimensional, we say that a vertex cone $\mathcal{K}_{\mathbf{v}}$ is *unimodular* if the square matrix with row vectors equal to $\{\mathrm{p}_{\mathbf{v}}(\mathbf{v}') \,:\, \mathbf{v}' \in \mathcal{N}(\mathbf{v})\}$

has determinant ± 1. The vertex cone $\mathcal{K}_{\mathbf{v}}$ being unimodular implies that every lattice point in $\mathcal{K}_{\mathbf{v}}$ can be written uniquely as the apex $\mathbf{v}$ plus a non-negative integer combination of the primitive vectors that generate the cone. Consequently, the set of integer points in a unimodular vertex cone $\mathcal{K}_{\mathbf{v}}$ can be written as a generating function

$$\sigma_{\mathcal{K}_{\mathbf{v}}}(\boldsymbol{z}) := \sum_{\mathbf{m}\in\mathcal{K}_{\mathbf{v}}\cap\mathbb{Z}^{n+1}} \boldsymbol{z}^{\mathbf{m}} = \boldsymbol{z}^{\mathbf{v}} \prod_{\mathbf{v}'\in\mathcal{N}(\mathbf{v})} \frac{1}{1-\mathbf{z}^{\mathrm{p}_{\mathbf{v}}(\mathbf{v}')}},$$

where $\mathbf{u} = (u_0, u_1, \ldots, u_n) \in \mathcal{K}_{\mathbf{v}}$ is encoded as $\boldsymbol{z}^{\boldsymbol{u}} := z_0^{u_0} z_1^{u_1} \cdots z_n^{u_n}$.

The formula of Brion [7], specialized here for a simple, integer polyhedron $\mathcal{P}$ with unimodular vertex cones, states that the lattice points in $\mathcal{P}$ are encoded precisely by the monomials appearing in the sum of the generating functions for the vertex cones:

$$\sigma_{\mathcal{P}}(\boldsymbol{z}) = \sum_{\mathbf{v}\text{ a vertex of }\mathcal{P}} \sigma_{\mathcal{K}_{\mathbf{v}}}(\boldsymbol{z}) = \sum_{\mathbf{v}\text{ a vertex of }\mathcal{P}} \boldsymbol{z}^{\mathbf{v}} \prod_{\mathbf{v}'\in\mathcal{N}(\mathbf{v})} \frac{1}{1-\mathbf{z}^{\mathrm{p}_{\mathbf{v}}(\mathbf{v}')}}. \tag{2}$$

Brion's formula holds in greater generality than presented here. The polyhedron need not be simple, nor integral, nor must all vertex cones be unimodular. These general cases require a more detailed description [4, Chapter 9] which we do not need to call upon here.

To enumerate all lattice points with common 1-norm (denoted by $|\cdot|$), it suffices to set the variables $z_0, z_1, \ldots, z_n$ in (2) to a common variable x to yield

$$\sigma_{\mathcal{P}}(x) = \sum_{\mathbf{v}} \sigma_{\mathcal{K}_{\mathbf{v}}}(x) = \sum_{\mathbf{v}} x^{|\mathbf{v}|} \prod_{\mathbf{v}'\in\mathcal{N}(\mathbf{v})} \frac{1}{1-x^{|\mathrm{p}_{\mathbf{v}}(\mathbf{v}')|}}, \tag{3}$$

where the sum is taken over all $\mathbf{v}$ that are vertices of $\mathcal{P}$. Note that if $|\mathrm{p}_{\mathbf{v}}(\mathbf{v}')| < 0$ then $\dfrac{1}{1-x^{|\mathrm{p}_{\mathbf{v}}(\mathbf{v}')|}} = \dfrac{-x^{|\mathrm{p}_{\mathbf{v}'}(\mathbf{v})|}}{1-x^{|\mathrm{p}_{\mathbf{v}'}(\mathbf{v})|}}$ and

$$\sigma_{\mathcal{K}_{\mathbf{v}}}(x) = x^{|\mathbf{v}|} \times \prod_{\mathbf{v}'\,:\,|\mathrm{p}_{\mathbf{v}}(\mathbf{v}')|<0} \frac{-x^{|\mathrm{p}_{\mathbf{v}'}(\mathbf{v})|}}{1-x^{|\mathrm{p}_{\mathbf{v}'}(\mathbf{v})|}} \times \prod_{\mathbf{v}'\,:\,|\mathrm{p}_{\mathbf{v}}(\mathbf{v}')|>0} \frac{1}{1-x^{|\mathrm{p}_{\mathbf{v}}(\mathbf{v}')|}}, \tag{4}$$

where each product is taken over $\mathbf{v}' \in \mathcal{N}(\mathbf{v})$. Written in this form, every monomial in the vertex cone's generating function $\sigma_{\mathcal{K}_{\mathbf{v}}}(x)$ has degree at least

$$O(\mathcal{K}_{\mathbf{v}}) := |\mathbf{v}| + \sum_{\mathbf{v}'\,:\,|\mathrm{p}_{\mathbf{v}}(\mathbf{v}')|<0} |\mathrm{p}_{\mathbf{v}'}(\mathbf{v})|$$

and we call this quantity the *order* of the vertex cone. We will sometimes write $O(\mathbf{v})$ for the order of the vertex cone with apex $\mathbf{v}$ in $\mathcal{P}$.

3. Enumerating Minimal Relaxed Complete Partitions

We have established that the minimal complete partitions with $n+1$ parts are lattice points in a polyhedron sitting in $\mathbb{R}^{n+1}$. We will show that Brion's formula, with the premises previously stated, applies to this polyhedron and then proceed to compute the generating functions of the polyhedron's vertex cones.

Recall that the (e,r)-partitions with $n+1$ parts are precisely the positive integer points $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ that satisfy the linear inequalities $\lambda_{i-1} \leq \lambda_i$ and ineq_i : $\lambda_i \leq (e+1) + r\sum_{j=0}^{i-1} \lambda_j$, for each $1 \leq i \leq n$. Furthermore, the partitions with size $|\lambda| := \lambda_0 + \lambda_1 + \cdots + \lambda_n$ lying in the interval $[(e+1)\frac{(r+1)^n-1}{r} + 1, (e+1)\frac{(r+1)^{n+1}-1}{r}]$ are the minimal such partitions.

Echoing Rødseth's enumeration of the minimal $(0,r)$partitions [17, 18], consider the following transformation of a partition λ:

$$\mu_i := (e+1)(r+1)^i - \lambda_i$$

for each $i = 0, 1, \ldots, n$. The constraints on λ translate to constraints on μ:

$$0 \leq \mu_0 \leq e \qquad \text{and} \qquad r\sum_{j=0}^{i-1} \mu_j \leq \mu_i \leq (e+1)(r)(r+1)^{i-1} + \mu_{i-1}.$$

There is a one-to-one correspondence between the λ's and the μ's and since $|\mu| = \frac{e+1}{r}((r+1)^{n+1} - 1) - |\lambda|$, the (e,r)-partitions λ with $|\lambda| \in [(e+1)\frac{(r+1)^n-1}{r} + 1, (e+1)\frac{(r+1)^{n+1}-1}{r}]$ correspond to the lattice points $\mu = (\mu_0, \ldots, \mu_n)$ that satisfy the above inequalities with norm constraint $|\mu| \in [0, (e+1)(r+1)^n - 1]$. Hence, it will suffice to find a generating function for the norms of the transformed μ points, $\sum x^{|\mu|} + O(x^{(e+1)(r+1)^n})$.

The lattice points $\mu = (\mu_0, \mu_1, \ldots, \mu_n)$ are contained in the half-spaces $\min^+(0) := \{\mu \in \mathbb{R}^{n+1} : 0 \leq \mu_0\}$, $\max^+(0) := \{\mu \in \mathbb{R}^{n+1} : \mu_0 \leq e\}$, and, for each $i = 1, 2, \ldots, n$,

$$\min{}^+(i) := \{\mu \in \mathbb{R}^{n+1} : r\sum_{j=0}^{i-1} \mu_j \leq \mu_i\} \quad \text{and}$$

$$\max{}^+(i) := \{\mu \in \mathbb{R}^{n+1} : \mu_i \leq (e+1)(r)(r+1)^{i-1} + \mu_{i-1}\}.$$

For every $0 \leq i \leq n$, let $\min(i)$ and $\max(i)$ denote the defining hyperplanes of the above half-spaces, with every inequality replaced by equality in $\min^+(i)$ and $\max^+(i)$ respectively. Define the polyhedron $P_n(e,r)$ as the common intersection of the above μ-half-spaces[2],

$$P_n(e,r) := \bigcap_{i=0}^{n} \min^+(i) \cap \max^+(i).$$

For each $\mathcal{S} \subseteq \{0,1,2,\ldots,n\}$, define the set of points

$$\mu(\mathcal{S}) := \{\mu \in \max(i) : i \in \mathcal{S}\} \cap \{\mu \in \min(i) : i \notin \mathcal{S}\} \subset \mathbb{R}^{n+1}.$$

Proposition 2. *The polyhedron $P_n(e,r)$ is an $(n+1)$-dimensional simple polyhedron. Each $\mu(\mathcal{S})$ is a singleton and the vertices of $P_n(e,r)$ are $\{\mu(\mathcal{S}) : \mathcal{S} \subseteq \{0,1,2,\ldots,n\}\}$. Two vertices $\mu(\mathcal{S})$ and $\mu(\mathcal{T})$ share an edge if and only if their symmetric difference $\mathcal{S} \triangle \mathcal{T}$ is a singleton.*

Proof. It is straightforward to check that there is no point $\mu \in P_n(e,r)$ that simultaneously satisfies both $\max(i)$ and $\min(i)$. Since each $\min/\max(i)$ sequentially introduces a new independent variable μ_i, the normals to the hyperplanes that define $\mu(\mathcal{S})$ are a set of linearly independent vectors and each $\mu(\mathcal{S})$ equals the intersection of $n+1$ affinely independent hyperplanes in $\mathbb{R}^{n+1}$. While no other defining hyperplane of $P_n(e,r)$ can contain $\mu(\mathcal{S})$, it is nonetheless contained in the interior of the other defining half-spaces of $P_n(e,r)$. This implies that each $\mu(\mathcal{S})$ is a singleton and, further, its single element must be a vertex of $P_n(e,r)$. Since the vertex $\mu(\mathcal{S})$ is defined by $n+1$ hyperplanes (and no others), it must have $n+1$ edges in $P_n(e,r)$. That is, $P_n(e,r)$ is an $(n+1)$-dimensional simple polyhedron.

As for the edges, $|\mathcal{S}\triangle\mathcal{T}| = 1$ if and only if $\mu(\mathcal{S})$ and $\mu(\mathcal{T})$ have precisely n of their $n+1$ defining hyperplanes in common. Since $P_n(e,r)$ is simple, this is equivalent to $\mu(S)$ and $\mu(\mathcal{T})$ being neighbors in $P_n(e,r)$. □

Lemma 1. *The vertex cones of $P_n(e,r)$ are unimodular.*

Proof. By Proposition 2, it suffices to study pairs of sets $\mathcal{S}$ and $\mathcal{T}$ where $\mathcal{S}\triangle\mathcal{T} = \{j\}$. For now, let us assume that $\mathcal{T} = \mathcal{S} \cup \{j\}$ and we will show that $\mathbf{p} := \mathrm{p}_{\mu(\mathcal{S})}(\mu(\mathcal{T}))$ equals 0 in entries 0 through $j-1$ and 1 in entry j.

For every $i < j$, since the sets $\mathcal{S}$ and $\mathcal{T}$ agree when restricted to the support of $\{0,1,2,\ldots,j-1\}$, $\mu(\mathcal{S})_j = \mu(\mathcal{T})_i$ and $\mathrm{p}_{\mu(\mathcal{S})}(\mu(\mathcal{T}))_i = 0$. The j-th entry of the difference vector is $\mu(\mathcal{T})_j - \mu(\mathcal{S})_j = r(e+1)(r+1)^{j-1} + \mu(\mathcal{T})_{j-1} - r(\sum_{k=0}^{j-1} \mu(\mathcal{S})_k)$, which is a positive integer – this difference must be positive since it equals $r(\lambda_0 + \cdots + \lambda_{j-1}) - \lambda_{j-1}$ in the original λ-formulation of the (e,r)-partition.

[2]When $e = 0$, $\min^+(0) \cap \max^+(0)$ equals the hyperplane defined by $\mu_0 = 0$. The claims of this section assume that $e \geq 1$ but when $e = 0$ the statements require only minor alteration, if any. For example, the polyhedron $P_n(0,r)$ is of dimension n as opposed to $n+1$.

For $i = j+1, \ldots, n$, recalling that $\mathcal{S}$ and $\mathcal{T}$ agree outside of $\{j\}$, the i-th entry of the difference equals

$$\mu(\mathcal{T})_i - \mu(\mathcal{S})_i = \left\{ \begin{array}{lcl} \mu(\mathcal{T})_{i-1} - \mu(\mathcal{S})_{i-1} & : & \text{for } i \in \mathcal{T} \\ r\Big(\displaystyle\sum_{k=j}^{i-1} \big(\mu(\mathcal{T})_k - \mu(\mathcal{S})_k\big)\Big) & : & \text{for } i \notin \mathcal{T} \end{array} \right\}. \tag{5}$$

Each $\mu(\mathcal{T})_i - \mu(\mathcal{S})_i$ is recursively defined from $\{\mu(\mathcal{T})_k - \mu(\mathcal{S})_k : k = j, \ldots, i-1\}$ and each such difference is a multiple of $\mu(\mathcal{T})_j - \mu(\mathcal{S})_j$. Hence, $\mu(\mathcal{T})_j - \mu(\mathcal{S})_j$ equals $\gcd(\mu(\mathcal{T}) - \mu(\mathcal{S}))$ and so $\mathrm{p}_{\mu(\mathcal{S})}(\mu(\mathcal{T}))_j = 1$.

If $\mathcal{S} = \mathcal{T} \cup \{j\}$ then $\mathrm{p}_{\mu(\mathcal{T})}(\mu(\mathcal{S})) = -\mathrm{p}_{\mu(\mathcal{S})}(\mu(\mathcal{T}))$ and this case follows in the same fashion except for -1 in entry j. Consequently, the primitive vectors of the vertex cone of any given vertex of $P_n(e,r)$ form the rows of a square matrix with determinant ± 1. □

Applying Brion's formula (Equation (4)) requires explicit calculation of the norms of the vertices of $P_n(e,r)$ and their vertex cones. Since we are only interested in the lattice points of $P_n(e,r)$ that have order less than $(e+1)(r+1)^n$, we can eliminate all vertex cones with order greater than or equal to $(e+1)(r+1)^n$.

To calculate $|\mu(\mathcal{S})|$ for each $\mathcal{S} \subseteq \{0,1,2,\ldots,n\}$, a preliminary lemma is required. Writing μ as shorthand for $\mu(\mathcal{S})$, $|\mu|_k^{k'-1}$ for $\mu_k + \mu_{k+1} + \cdots + \mu_{k'-1}$, $|\mu|^{k'-1}$ for $|\mu|_0^{k'-1}$, and $|\mu|_k$ for $|\mu|_k^n$, we make the following claims.

Lemma 2. (a) *If $k, \ldots, k'-1 \notin \mathcal{S}$ then*

$$|\mu|_k^{k'-1} = ((r+1)^{k'-k} - 1)|\mu|^{k-1} \qquad \text{and} \qquad |\mu|^{k'-1} = (r+1)^{k'-k}|\mu|^{k-1}.$$

(b) *If $k \notin \mathcal{S}$ then $\mu_k = \dfrac{r}{r+1}|\mu|^k$.*

(c) *If $0 < k, \ldots, k'-1 \in \mathcal{S}$ then*

$$|\mu|_k^{k'-1} = \frac{(e+1)(r+1)^k}{r}\left[(r+1)^{k'-k} - 1 - (k'-k)\frac{r}{r+1}\right] + (k'-k)\mu_{k-1}.$$

(d) *If $0, \ldots, k'-1 \in \mathcal{S}$ then $|\mu|_0^{k'-1} = \dfrac{(e+1)}{r}\left[(r+1)^{k'} - 1 - k'\dfrac{r}{e+1}\right]$.*

Proof. (a) Since $k \notin \mathcal{S}$, it follows that $\mu_k = r|\mu|^{k-1}$. Similarly, $k+1 \notin \mathcal{S}$ implies $\mu_{k+1} = r|\mu|^k = r(|\mu|^{k-1} + \mu_k) = r(|\mu|^{k-1} + r|\mu|^{k-1}) = r(r+1)|\mu|^{k-1}$. Recursively, $\mu_{k+i} = r(r+1)^i|\mu|^{k-1}$, hence

$$|\mu|_k^{k'-1} = r\sum_{i=0}^{k'-1-k}(r+1)^i|\mu|^{k-1} = ((r+1)^{k'-k} - 1)|\mu|^{k-1}.$$

It follows that $|\mu|^{k'-1} = |\mu|_k^{k'-1} + |\mu|^{k-1}$.

(b) Since $|\mu|^j = \mu_j + |\mu|^{j-1}$ and $j \notin \mathcal{S}$, we have $\frac{1}{r}\mu_j = |\mu|^{j-1}$. That is, $|\mu|^j = (1+\frac{1}{r})\mu_j$.

(c) Since $k \in \mathcal{S}$, we have that $\mu_k = r(e+1)(r+1)^{k-1} + \mu_{k-1}$. Similarly, $k+1 \in \mathcal{S}$ implies $\mu_{k+1} = r(e+1)(r+1)^k + \mu_k = r(e+1)(r+1)^{k-1}[(r+1)+1] + \mu_{k-1}$. Recursively, for every $i = 0, 1, \ldots, (k'-1)-k$, we have

$$\begin{aligned}
\mu_{k+i} &= r(e+1)(r+1)^{k-1}[(r+1)^i + (r+1)^{i-1} + \cdots + (r+1) + 1] + \mu_{k-1} \\
&= (e+1)(r+1)^{k-1}[(r+1)^{i+1} - 1] + \mu_{k-1}
\end{aligned}$$

and so $|\mu|_k^{k'-1} = (e+1)(r+1)^{k-1}\sum_{i=0}^{k'-1-k}[(r+1)^{j+1} - 1] + \sum_{i=0}^{k'-1-k}\mu_{k-1}$, or

$$|\mu|_k^{k'-1} = (e+1)(r+1)^{k-1}\left[\frac{(r+1)}{r}((r+1)^{k'-k} - 1) - (k'-k)\right] + (k'-k)\mu_{k-1}$$

and is easily rewritten in the desired form.

(d) Follows in the same manner as (c), with the initial condition of $\mu_0 = e$. $\square$

With Lemma 2 in hand, we can compute the norm of each vertex of $P_n(e,r)$. If $(\alpha_i, n_i) := \{\alpha_i, \alpha_i + 1, \ldots, \alpha_i + n_i - 1\} \subseteq \mathcal{S}$ then (α_i, n_i) is said to be a *chain in* $\mathcal{S}$; it is said to be *maximal* if no other chain in $\mathcal{S}$ properly contains (α_i, n_i). We can write a set $\mathcal{S}$ in terms of the union of its maximal chains vis-a-vis $\mathcal{S} = \{(\alpha_1, n_1), (\alpha_2, n_2), \ldots, (\alpha_p, n_p)\}$. Since each chain is maximal, it follows that $\alpha_i + n_i < \alpha_{i+1}$ and $|\mathcal{S}| = n_1 + n_2 + \cdots + n_p$. The following result describes the norm of any vertex $\mu(\mathcal{S})$ in terms of the maximal chains in $\mathcal{S}$.

Lemma 3. *For* $\mathcal{S} = \{(\alpha_1, n_1), \ldots, (\alpha_p, n_p)\}$ *with* $\alpha_1 > 0$,

$$|\mu(\mathcal{S})| = \frac{(e+1)(r+1)^{n+1-|\mathcal{S}|}}{r}\left[(r+1)^{|\mathcal{S}|} - \prod_{i=1}^{p}\left(1 + \frac{rn_i}{r+1}\right)\right]. \tag{6}$$

For $\mathcal{S} = \{(0, n_1), \ldots, (\alpha_p, n_p)\}$,

$$|\mu(\mathcal{S})| = \frac{(e+1)(r+1)^{n+1-|\mathcal{S}|}}{r}\left[(r+1)^{|\mathcal{S}|} - \left(1 + \frac{n_1 r}{e+1}\right)\prod_{i=2}^{p}\left(1 + \frac{rn_i}{r+1}\right)\right]. \tag{7}$$

Proof. We prove the $\alpha_1 > 0$ case by induction on the number of maximal chains in $\mathcal{S}$, leaving the reader to confirm the $\alpha_1 = 0$ case in a similar fashion.

For the base case of $\mathcal{S} = \{(\alpha_1, n_1)\}$, setting $k = \alpha_1 + n_1$ and $k' - 1 = n$ in Lemma 2(a), the vertex μ has norm $|\mu| = (r+1)^{n+1-\alpha_1-n_1}|\mu|_{\alpha_1}^{\alpha_1+n_1-1}$. The quantity

$|\mu|_{\alpha_1}^{\alpha_1+n_1-1}$ is computed by setting $k=\alpha_1$ and $k'=\alpha_1+n_1$ in Lemma 2(c):

$$\begin{aligned}|\mu| &= (r+1)^{n+1-\alpha_1-n_1}|\mu|_{\alpha_1}^{\alpha_1+n_1-1}\\ &= \frac{(e+1)(r+1)^{(n+1-\alpha_1-n_1)+\alpha_1}}{r}\left[(r+1)^{n_1}-1-(n_1)\frac{r}{r+1}\right]+(n_1)0.\end{aligned}$$

Next, assume the claim is true for any set $\mathcal{S}'$ with at most $p-1$ chains. Let $\mathcal{S}=\{(\alpha_1,n_1),\ldots,(\alpha_p,n_p)\}$ be a subset of $\{0,1,2,\ldots,n\}$ with p chains. Since $\alpha_p+n_p,\ldots,n\notin\mathcal{S}$, it follows from Lemma 2(a), that

$$\begin{aligned}|\mu(\mathcal{S})| \;=\; |\mu|^{\alpha_p+n_p-1}+|\mu|_{\alpha_p+n_p}^{n} &= (r+1)^{n+1-(\alpha_p+n_p)}|\mu|^{\alpha_p+n_p-1}\\ &= (r+1)^{n+1-(\alpha_p+n_p)}(|\mu|^{\alpha_p-1}+|\mu|_{\alpha_p}^{\alpha_p+n_p-1}).\end{aligned}$$

Lemma 2(c) with $k=\alpha_p$ and $k'=\alpha_p+n_p$ applies to $|\mu|_{\alpha_p}^{\alpha_p+n_p-1}$ to yield

$$\begin{aligned}|\mu|_{\alpha_p}^{\alpha_p+n_p-1} \;=\; &\frac{(e+1)(r+1)^{\alpha_p+n_p}}{r} \;-\; \frac{(e+1)(r+1)^{\alpha_p}}{r}\\ &-\; n_p(e+1)(r+1)^{\alpha_p-1} \;+\; n_p\frac{r}{r+1}|\mu|^{\alpha_p-1}.\end{aligned}$$

It follows that $|\mu|^{\alpha_p+n_p-1}$ equals

$$\begin{aligned}|\mu|^{\alpha_p-1}+|\mu|_{\alpha_p}^{\alpha_p+n_p-1} \quad = \quad &\tfrac{(e+1)(r+1)^{\alpha_p+n_p}}{r} \;-\; \tfrac{(e+1)(r+1)^{\alpha_p}}{r}\\ &-n_p(e+1)(r+1)^{\alpha_p-1} \;+\; (1+n_p\tfrac{r}{r+1})|\mu|^{\alpha_p-1}.\end{aligned} \tag{8}$$

Setting $\mathcal{S}'=\{(\alpha_1,n_1),\ldots,(\alpha_{p-1},n_{p-1})\}$ as a subset of $\{0,1,2,\ldots,\alpha_p-1\}$, the inductive hypothesis allows for

$$|\mu|^{\alpha_p-1}=\frac{(e+1)(r+1)^{\alpha_p}}{r}-\frac{(e+1)(r+1)^{\alpha_p-|\mathcal{S}'|}}{r}\prod_{i=1}^{p-1}\left(1+\frac{rn_i}{r+1}\right). \tag{9}$$

Substituting the expression for $|\mu|^{\alpha_p-1}$ in (9) into the last term of (8) gives $|\mu|^{\alpha_p-1}+|\mu|_{\alpha_p}^{\alpha_p+n_p-1}$

$$\begin{aligned}&= \frac{(e+1)(r+1)^{\alpha_p+n_p}}{r} \;-\; \frac{(e+1)(r+1)^{\alpha_p}}{r}\\ &\quad -\; n_p(e+1)(r+1)^{\alpha_p-1} \;+\; \left(1+n_p\frac{r}{r+1}\right)\frac{(e+1)(r+1)^{\alpha_p}}{r}\\ &\quad -\; \left(1+n_p\frac{r}{r+1}\right)\frac{(e+1)(r+1)^{\alpha_p-|\mathcal{S}'|}}{r}\prod_{i=1}^{p-1}\left(1+\frac{rn_i}{r+1}\right).\end{aligned}$$

The second and third terms cancel with the fourth so

$$\begin{aligned}|\mu|^{\alpha_p+n_p-1} &= \frac{(e+1)(r+1)^{\alpha_p+n_p}}{r} \\ &\quad - \frac{(e+1)(r+1)^{\alpha_p-|\mathcal{S}'|}}{r}\left(1+n_p\frac{r}{r+1}\right)\prod_{i=1}^{p-1}\left(1+\frac{rn_i}{r+1}\right),\end{aligned}$$

and that last product equals $\prod_{i=1}^{p}\left(1+\frac{rn_i}{r+1}\right)$. Finally,

$$\begin{aligned}|\mu| &= (r+1)^{n+1-\alpha_p-n_p}(|\mu|^{\alpha_p-1}+|\mu|_{\alpha_p}^{\alpha_p+n_p-1}) \\ &= \frac{(e+1)(r+1)^{n+1}}{r} - \frac{(e+1)(r+1)^{n+1-|\mathcal{S}'|-n_p}}{r}\prod_{i=1}^{p}\left(1+\frac{rn_i}{r+1}\right)\end{aligned}$$

as claimed. The case where $\alpha_1 = 0$ is proved in the same manner. □

Given an arbitrary $\mathcal{S} = \{(\alpha_1, n_1), \ldots, (\alpha_p, n_p)\}$ and a choice of $\alpha_1 = 0$ or $\alpha_1 > 0$, the previous lemma makes no reference to the values of the remaining α_i's so we can explicitly calculate the norms of all vertices of $P_n(e, r)$.

Corollary 1. (a) *If* $\mathcal{S} = \{j\}$ *then*

$$|\mu| = \left\{\begin{array}{rcl} (e)(r+1)^n & : & j=0 \\ r(e+1)(r+1)^{n-1} & : & j>0 \end{array}\right\}.$$

(b) *If* $\mathcal{S} = \{j_1, j_2\}$ *then*

$$|\mu| = \left\{\begin{array}{rcl} (e+1)(r+1)^n + [e-1]\,(r+1)^{n-1} & : & 0=j_1=j_2-1 \\ (e+1)(r+1)^n + [r(e-1)-1]\,(r+1)^{n-2} & : & 0=j_1<j_2-1 \\ (e+1)(r+1)^n + [r-1]\,(e+1)(r+1)^{n-2} & : & 0<j_1=j_2-1 \\ (e+1)(r+1)^n + [r(r-1)-1]\,(e+1)(r+1)^{n-3} & : & 0<j_1<j_2-1 \end{array}\right\}.$$

(c) *If* $\mathcal{S} = \{j_1, j_2, j_3\}$ *with* $r = 1$ *then*

$$|\mu| = \left\{\begin{array}{lcl} (e+1)2^n + (7e-2)2^{n-4} & : & 0=j_1<j_2-1<j_3-2 \\ (e+1)2^n + (5e-1)2^{n-3} & : & 0=j_1=j_2-1<j_3-2 \\ (e+1)2^n + 7(e+1)2^{n-5} & : & 0<j_1<j_2-1<j_3-2 \end{array}\right\}.$$

Proof. Part (b) when $0 < j_1 < j_2 - 1$ is a special case of Lemma 3 with $\mathcal{S} = \{(\alpha_1, n_1), (\alpha_2, n_2)\}$, $\alpha_1 > 0$ and $n_1 = n_2 = 1$:

$$\begin{aligned}|\mu| &= \frac{(e+1)(r+1)^{n+1-2}}{r}\left[(r+1)^2 - \left(1+\frac{r}{(r+1)}\right)^2\right] \\ &= (e+1)(r+1)^{n-3}\left[(r+1)^3 + r(r-1) - 1\right] \\ &= (e+1)(r+1)^n + [r(r-1)-1](e+1)(r+1)^{n-3}.\end{aligned}$$

The other claims follows in a similar fashion from Lemma 3. □

With the norms of the vertices $\mu(\mathcal{S})$ of $P_n(e,r)$ with $|\mathcal{S}| \leq 2$ computed, we turn to the primitive vectors associated to those vertices. A corollary of Equation (5) (with the greatest common divisor term removed) is that we can describe primitive vectors recursively. If $\mathcal{T} = \mathcal{S} \cup \{j\}$ then

$$\mathrm{p}_{\mu(\mathcal{S})}(\mu(\mathcal{T}))_i = \left\{ \begin{array}{rcl} 0 & : & i < j \\ 1 & : & i = j \\ \mathrm{p}_{\mu(\mathcal{S})}(\mu(\mathcal{T}))_{i-1} & : & i > j,\, i \in \mathcal{T} \\ r\sum_{k=j}^{i-1} \mathrm{p}_{\mu(\mathcal{S})}(\mu(\mathcal{T}))_k & : & i > j,\, i \notin \mathcal{T} \end{array} \right\}. \tag{10}$$

This allows us to explicitly compute the norms of the primitive vectors.

Corollary 2. *When $|\mathcal{S}| \leq 2$, the norms of the primitive vectors associated with the vertex $\mu(\mathcal{S})$ are as follows:*

(a) $$|\mathrm{p}_{\mu(\{\emptyset\})}(\mu(\{j\}))| = (r+1)^{n-j} \quad \text{for each} \quad j = 0, 1, \ldots, n. \tag{11}$$

(b) *If $\mu(\mathcal{S}) = \{k\}$ and $\mathcal{T} = \{j_1, j_2\}$, where $j_1 < j_2$:*

$$|\mathrm{p}_{\mu(\{\mathcal{S}\})}(\mu(\mathcal{T}))| = \left\{ \begin{array}{rcl} (r+1)^{n-j_2} & : & k = j_1 \\ 2(r+1)^{n-j_1-1} & : & k = j_2 \text{ and } j_1 = j_2 - 1 \\ (2r+1)(r+1)^{n-j_1-2} & : & k = j_2 \text{ and } j_1 < j_2 - 1 \end{array} \right\}. \tag{12}$$

(c) *If $\mathcal{S} = \{j_1, j_2\}$ with $j_1 < j_2 - 1$ and $\mathcal{T} = \{j_1, j_2, k\}$ then*

$$|\mathrm{p}_{\mu(\mathcal{S})}(\mu(\mathcal{T}))| = \left\{ \begin{array}{rcl} (2r+1)^2(r+1)^{n-k-4} & : & k < j_1 - 1 \\ 2(2r+1)(r+1)^{n-k-3} & : & k = j_1 - 1 \\ (2r+1)(r+1)^{n-k-2} & : & k < j_2 - 1 \\ 2(r+1)^{n-k-1} & : & k = j_2 - 1 \\ (r+1)^{n-k} & : & k > j_2 \end{array} \right\}. \tag{13}$$

Proof. In each case, the vectors $\mu(\mathcal{T}) - \mu(\mathcal{S})$ can be constructed recursively from Equation (10), and the claims follow from the geometric series $1 + r(1 + (r+1) + \cdots + (r+1)^{k-1}) = (r+1)^k$. For example,

$$\mathrm{p}_{\mu(\emptyset)}(\mu(\{j\}))_i = \left\{ \begin{array}{rcl} 0 & : & i < j \\ 1 & : & i = j \\ r(r+1)^{i-j-1} & : & i > j \end{array} \right\}$$

and so $|\mathrm{p}_{\mu(\emptyset)}(\mu(\{j\}))| = 1 + r(1 + \ldots + (r+1)^{n-j-1}) = (r+1)^{n-j}$. $\square$

We are now on the cusp of proving our main result, the enumeration of the minimal relaxed complete partitions. Recall that the *order* of the vertex cone at $\mu(\mathcal{S})$ is $O(\mu(\mathcal{S})) = |\mu(\mathcal{S})| + \sum_{\mathcal{T}} |\mathrm{p}_{\mu(\mathcal{T})}(\mu(\mathcal{S}))|$, where the sum is taken over each $\mathcal{T}$ with $\mathcal{T} = \mathcal{S} \setminus \{j\}$, and that we are only interested in those vertex cones with order less than $(e+1)(r+1)^n$.

Lemma 4. *If $\mathcal{S} \subseteq \mathcal{T}$ then $|\mu(\mathcal{S})| \leq |\mu(\mathcal{T})|$.*

Proof. As noted in Lemma 1, each $\mu(\mathcal{T})_j - \mu(\mathcal{S})_j \geq 0$, with strict inequality when $j \in \mathcal{T} \setminus \mathcal{S}$. □

Lemma 5. *If $0 \in \mathcal{S}$ and $|\mathcal{S}| \leq 2$ then $O(\mu) \geq (e+1)(r+1)^n$.*

Proof. All claims regarding $|\mu|$ and the norms of primitive vectors can be read directly from Corollaries 1 and 2 respectively.

For $\mathcal{S} = \{0\}$, $|\mu| = e(r+1)^n$ and $|\mathrm{p}_{\mu(\emptyset)}(\mu(\{0\}))| = (r+1)^n$, which implies $O(\mu(\{0\})) \geq |\mu(\{0\})| + |\mathrm{p}_{\mu(\emptyset)}(\mu(\{0\}))| = e(r+1)^n + (r+1)^n = (e+1)(r+1)^n$.

For $\mathcal{S} = \{0,1\}$, $|\mu| = (e+1)(r+1)^n + [e-1](r+1)^{n-1}$ and $|\mathrm{p}_{\mu(\{0\})}(\mu(\{0,1\}))| = (r+1)^{n-1}$, which implies $O(\mu) \geq (e+1)(r+1)^n + [e-1](r+1)^{n-1} + (r+1)^{n-1} \geq (e+1)(r+1)^n$.

For $\mathcal{S} = \{0, j_2\}$ with $j_2 > 1$, $|\mu| = (e+1)(r+1)^n - [r(e-1)-1](r+1)^{n-2}$ and $|\mathrm{p}_{\mu(\{j_2\})}(\mu(\{0,j_2\}))| = (2r+1)(r+1)^{n-0-2}$, which implies $O(\mu) \geq (e+1)(r+1)^n + [r(e-1)-1](r+1)^{n-2} + (2r+1)(r+1)^{n-2} \geq (e+1)(r+1)^n$. □

Theorem 2. *If $r \geq 2$ then $\sigma_{P_n(e,r)}(x) = \sigma_{\mathcal{K}_{\mu(\emptyset)}} + \sum_{j=1}^{n} \sigma_{\mathcal{K}_{\mu(\{j\})}} + O(x^{(e+1)(r+1)^n})$.*

Proof. We claim that $O(\mu) \geq (e+1)(r+1)^n$ whenever (i) $|\mathcal{S}| \geq 2$, or (ii) $0 \in \mathcal{S}$.

(i): If $\mathcal{S} = \{j_1, j_2\}$ with $0 < j_1 < j_2$ then, by Corollary 1(b), $|\mu(\mathcal{S})| \geq (e+1)(r+1)^n$. Any set $\mathcal{T}$ of size 3 or greater contains a set of the form $\mathcal{S} = \{j_1, j_2\}$ and, by Lemma 4, $|\mu(\mathcal{T})| \geq |\mu(\mathcal{S})| \geq (e+1)(r+1)^n$.

(ii): If $0 \in \mathcal{S}$ with $|\mathcal{S}| \geq 3$ then $\mathcal{S}$ would contain a set of the form $\{j_1, j_2\}$ with $j_1 > 0$ so, by case (i) and Lemma 4, $|\mu| \geq (e+1)(r+1)^n$. By Lemma 5, if $|\mathcal{S}| = 1$ or 2 then $O(\mu) \geq (e+1)(r+1)^n$. Hence $O(\mu(\mathcal{S})) \geq (e+1)(r+1)^n$ for every set $\mathcal{S}$ that contains 0. □

The $r = 1$ case follows in much the same way but there are extra terms in the generating fuction.

Theorem 3. *We have*

$$\sigma_{P_n(e,1)}(x) = \sigma_{\mathcal{K}_{\mu(\emptyset)}} + \sum_{j=1}^{n} \sigma_{\mathcal{K}_{\mu(\{j\})}} + \sum_{j_1=1}^{n} \sum_{j_2=j_1+2}^{n} \sigma_{\mathcal{K}_{\mu(\{j_1,j_2\})}} + O(x^{(e+1)2^n}).$$

Proof. We claim that $O(\mu) \geq (e+1)2^n$ whenever (i) $\mathcal{S} = \{j, j+1\}$, (ii) $|\mathcal{S}| \geq 3$, or (iii) $0 \in \mathcal{S}$.

(i): If $\mathcal{S} = \{j_1, j_2\}$ with $0 < j_1 = j_2 - 1$ then, by Corollary 1(b), $|\mu| \geq (e+1)2^n$. Applying Lemma 4, any $\mathcal{T}$ with two consecutive non-zero elements has $|\mu(\mathcal{T})| \geq$

$(e+1)2^n$. In contrast to $r \geq 2$, when $r = 1$ and $\mathcal{S} = \{j_1, j_2\}$ with $0 < j_1 < j_2 - 1$, $|\mu| = (e+1)2^n + [2(2-1)-1]\,(e+1)2^{n-3} = (e+1)2^n - (e+1)2^{n-3} < (e+1)2^n$. Using Corollary 2, one can check that $O(\mu) < (e+1)2^n$, hence $\mathcal{S} = \{j_1, j_2\}$ with $0 < j_1 < j_2 - 1$ appears as the index for the double summation in the $r = 1$ generating function.

(ii): If $|\mathcal{S}| \geq 3$ and $\mathcal{S}$ contains two consecutive elements then, by (i) and Lemma 4, $|\mu| \geq (e+1)2^n$. Alternatively, if $\mathcal{S}$ does not contain two such elements (i.e., $\mathcal{S} = \{j_1, j_2, j_3\}$ with $0 < j_1 < j_2 - 1 < j_3 - 2$) then, by Corollary 1(c), $|\mu(\mathcal{S})| \geq |\mu(\{j_1, j_2, j_3\})| \geq (e+1)2^n$.

(iii): Assume $0 \in \mathcal{S}$. If $|\mathcal{S}| \leq 2$ the claim follows directly from Lemma 5. If $|\mathcal{S}| \geq 4$ then $\mathcal{S}$ must contain three non-zero elements and case (ii) combined with Lemma 4 ensures that $|\mu(\mathcal{S})| \geq (e+1)2^n$. All that remains are the three element sets $\mathcal{S} = \{0, j_2, j_3\}$ with $j_2 < j_3 - 1$.

If $\mathcal{S} = \{0, 1, j_3\}$ then $|\mu| = (e+1)2^n + [5e-1]2^{n-3}$ and $|\mathrm{p}_{\mu(\{1,j_3\})}(\mu(\mathcal{S}))| = 2(3)2^{n-3}$, so $O(\mu) \geq (e+1)2^n + [5e-1+6]2^{n-3} = (e+1)2^n + 5(e+1)2^{n-3} \geq (e+1)2^n$. Otherwise, $\mathcal{S} = \{0, j_2, j_3\}$ with $j_2 > 1$ then $|\mu(\{0, j_1, j_2\})| = (e+1)2^n + (7e-2)2^{n-4}$ and $|\mathrm{p}_{\mu(\{j_2,j_3\})}(\mu(\mathcal{S}))| = 3^2 2^{n-4}$, so $O(\mu) \geq (e+1)2^n + 7(e+1)2^{n-4} \geq (e+1)2^n$. $\square$

We close with explicit descriptions of the above generating functions. Reading directly from Equation (4) and using Corollaries 1 and 2 we have:

$$\sigma_{\mathcal{K}_{\mu(\emptyset)}} = x^0 \prod_{j=0}^{n} \frac{1}{1 - x^{|\mathrm{p}_{\mu(\emptyset)}(\mu(\{j\}))|}} = \prod_{j=0}^{n} \frac{1}{1 - x^{(r+1)^j}}$$

and

$$\begin{aligned}
\sigma_{\mathcal{K}_{\mu(\{j_1\})}} &= x^{r(e+1)(r+1)^{n-1}} \frac{-x^{|\mathrm{p}_{\mu(\emptyset)}(\mu(\{j_1\})|}}{1 - x^{|\mathrm{p}_{\mu(\emptyset)}(\mu(\{j_1\})|}} \prod_{k \neq j_1} \frac{1}{1 - x^{|\mathrm{p}_{\mu(\{j_1\})}(\mu(\{j_1,k\})|}} \\
&= x^{r(e+1)(r+1)^{n-1}} \frac{-x^{(r+1)^{n-j_1}}}{1 - x^{(r+1)^{n-j_1}}} \frac{1}{1 - x^{2(r+1)^{n-j_1}}} \\
&\quad \times \prod_{k=j_1+1}^{n} \frac{1}{1 - x^{(r+1)^{n-k}}} \prod_{k=0}^{j_1-2} \frac{1}{1 - x^{(2r+1)(r+1)^{n-k-2}}} \\
= -&\frac{x^{r(e+1)(r+1)^{n-1} + (r+1)^{n-j_1}}}{(1 - x^{(r+1)^{n-j_1}})(1 - x^{2(r+1)^{n-j_1}})} \times \prod_{t=0}^{n-j_1-1} \frac{1}{1 - x^{(r+1)^t}} \prod_{t=n-j_1}^{n-2} \frac{1}{1 - x^{(2r+1)(r+1)^t}}.
\end{aligned}$$

For $\mathcal{S} = \{j_1, j_2\}$ with $0 < j_1 < j_2 - 1$ and $r = 1$,

$$\begin{aligned}\sigma_{\mathcal{K}_{\mu(\{j_1,j_2\})}} &= x^{(e+1)7\cdot 2^{n-3}} \prod_{k\in\{j_1,j_2\}} \frac{-x^{|\mathrm{p}_{\mu(\{k\})}(\mu(\{j_1,j_2\})|}}{1-x^{|\mathrm{p}_{\mu(\{k\})}(\mu(\{j_1,j_2\})|}} \\ &\quad \times \prod_{k\notin\{j_1,j_2\}} \frac{1}{1-x^{|\mathrm{p}_{\mu(\{j_1,j_2\})}(\mu(\{j_1,j_2,k\})|}} \\ &= x^{(e+1)7\cdot 2^{n-3}} x^{2^{n-j_2}} x^{3\cdot 2^{n-j_1-2}} \prod_{t=0}^{n-j_2+1} \frac{1}{1-x^{2^t}} \\ &\quad \times \prod_{t=n-j_2}^{n-j_1-1} \frac{1}{1-x^{3\cdot 2^t}} \prod_{t=n-j_1-2}^{n-4} \frac{1}{1-x^{9\cdot 2^t}}.\end{aligned}$$

Setting $(e, r) = (1, 2)$ and $n = 2$ we recover the enumerating function $\sigma_{P_2(1,2)}(x)$ described in the introduction. Setting $e = 0$ provides Rødseth's enumerating functions [17, 18] for the minimal complete partitions.

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