A METHOD FOR OBTAINING FIBONACCI IDENTITIES

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Abstract
For the Lucas sequence $\{U_k(P,Q)\}$ we discuss the identities such as the well-known Fibonacci identities. We also propose a method for obtaining identities involving recurrence sequences, with the help of which we find an interpolating-type identity for second-order linear recurrence sequences.

1. Introduction

Let $\mathbb{F}$ be an arbitrary field and P, Q be its nonzero elements. The Lucas sequences $\{U_k(P,Q)\}$, $\{V_k(P,Q)\}$ are defined recursively by

$$f_{k+2} = Pf_{k+1} - Qf_k, \tag{1}$$

with the initial values[1] $U_0 = 0,\ U_1 = 1, V_0 = 2, V_1 = P$. The characteristic equation of the recurrence relation (1) is $x^2 - Px + Q = 0$. Its roots are $\alpha = \frac{P+\sqrt{\Delta}}{2}$ and $\bar{\alpha} = \frac{P-\sqrt{\Delta}}{2}$, where $\Delta = P^2 - 4Q$. If $\Delta \neq 0$, the Lucas sequences can be expressed in terms of α and $\bar{\alpha}$ according to Binet formulas

$$U_k = \frac{\alpha^k - \bar{\alpha}^k}{\alpha - \bar{\alpha}},\ \ V_k = \alpha^k + \bar{\alpha}^k,\ \ P = \alpha + \bar{\alpha},\ \ Q = \alpha\bar{\alpha},\ \ \sqrt{\Delta} = \alpha - \bar{\alpha}. \tag{2}$$

We define $U_{-k} = -U_k/Q^k$ and $V_{-k} = V_k/Q^k$ for $k \geq 1$. Then Binet formulas are valid for every $k \in \mathbb{Z}$. More information about the subject can be found in [6, 14, 18].

The sequence of the Fibonacci numbers $\{F_k\}$ is defined by the recurrence relation $F_{k+2} = F_{k+1} + F_k$ ($F_0 = 0$, $F_1 = 1$) [13]. From this definition, it follows that $F_k = U_k(1,-1)$. There are many identities involving the Fibonacci numbers [15, 16]. Some of them are derived from identities involving $\{U_k\}$ if we put $P = 1$, $Q = -1$. In this paper, we generalize some identities for $\{F_k\}$ in terms of $\{U_k\}$. Often such

[1] In this paper instead of $U_k(P,Q)$ and $V_k(P,Q)$ we will write U_k and V_k, if it is not ambiguous.

generalized identities have a form close to the initial one. We also present a method for obtaining identities involving recurrence sequences. To show the efficiency of this method we obtain some identities for the Fibonacci numbers. The most interesting result is presented in Theorem 2.

2. Generalizations of Fibonacci Identities

It is clear that any identity which holds for $\{U_k\}$ can be easy transformed into an identity for $\{F_k\}$. But there exist identities involving $\{F_k\}$ for which analogues in terms of $\{U_k\}$ are unknown. We note that the discussion fits in the literature [5, 10]. For example, Candido's identity [2] was generalized in [3]. This result is as follows:

$(F.1)$ $\quad 2(F_k^4 + F_{k+1}^4 + F_{k+2}^4) = (F_k^2 + F_{k+1}^2 + F_{k+2}^2)^2,$

$(GF.1)$ $\quad 2(Q^4U_k^4 + P^4U_{k+1}^4 + U_{k+2}^4) = (Q^2U_k^2 + P^2U_{k+1}^2 + U_{k+2}^2)^2.$

Catalan's identity and its generalization [15] are:

$(F.2)$ $\quad F_k^2 - F_{k+n}F_{k-n} = (-1)^{k-n}F_n^2,$

$(GF.2)$ $\quad U_k^2 - U_{k+n}U_{k-n} = Q^{k-n}U_n^2.$

2.1. Two Important Identities for $\{U_k(P, Q)\}$

For the Fibonacci numbers the following holds:

$(F.3)$ $\quad F_{2k+1} = F_{k+1}^2 + F_k^2,$

$(F.4)$ $\quad F_{2k} = F_k(2F_{k+1} - F_k).$

The generalizations of $(F.3)$, $(F.4)$ are:

$(GF.3)$ $\quad U_{2k+1} = U_{k+1}^2 - QU_k^2,$

$(GF.4)$ $\quad U_{2k} = U_k(2U_{k+1} - PU_k).$

Lemma 1. *Let $\{U_k\}$ be the Lucas sequence with parameters P, Q in an arbitrary field $\mathbb{F}$. Then $(GF.3)$ and $(GF.4)$ are valid.*

Proof. Consider the well-known identity $U_{n+m} = U_nU_{m+1} - QU_mU_{n-1}$. If we put $n = k+1$ and $m = k$, then we obtain $(GF.3)$. If we put $n = k$ and $m = k$, then we obtain $U_{2k} = U_k(U_{k+1} - QU_{k-1})$. If we use $QU_{k-1} = PU_k - U_{k+1}$ in the previous equality, then we obtain $(GF.4)$. □

These two identities can be found in the literature: both identities are presented in [17], $(GF.3)$ is in [12, 22]. To show their importance, we note that $(GF.3)$, $(GF.4)$ can be used to calculate the Lucas sequences. This is an alternative to the more known method [11], which uses the properties of both sequences $\{U_k\}$ and $\{V_k\}$.

2.2. Higher Order Fibonacci Identities

The following identities hold:

$(F.5)$ $\quad F_{3k} = F_{k+1}^3 + F_k^3 - F_{k-1}^3,$

$(F.6)$ $\quad F_{4k} = F_{k+1}^4 + 2F_k^4 - F_{k-1}^4 + 4F_k^3 F_{k-1},$

$(F.7)$ $\quad F_{5k} = F_{k+1}^5 + 4F_k^5 - F_{k-1}^5 + 10F_{k+1}F_k^3 F_{k-1}.$

The reader can find $(F.5)$, $(F.6)$ in [1, 4]. The generalizations of the above are:

$(GF.5)$ $\quad U_{3k} = U_{k+1}^3/P + (U_3 - P^2)U_k^3 + Q^3 U_{k-1}^3/P,$

$(GF.6)$ $\quad U_{4k} = U_{k+1}^4/P + (U_4 - P^3)U_k^4 - Q^4 U_{k-1}^4/P + 4Q^2 U_k^3 U_{k-1},$

$(GF.7)$ $\quad U_{5k} = U_{k+1}^5/P + (U_5 - P^4)U_k^5 + Q^5 U_{k-1}^5/P + 10Q^2 U_{k+1} U_k^3 U_{k-1}.$

One way to prove $(GF.5)$-$(GF.7)$ is to use (2). But obtaining such identities by using the Binet formulas is difficult. In the next section we discuss how we obtain similar identities.

3. A New Method for Obtaining Identities for Linear Recurrences

First we consider the matrix method that is often used to prove some identities concerning the generalized Fibonacci and Lucas numbers [12, 17]. We have the following matrix formulas:

$$\begin{pmatrix} U_{k+1} & V_{k+1} \\ U_k & V_k \end{pmatrix} = M^k \begin{pmatrix} 1 & P \\ 0 & 2 \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} P & -Q \\ 1 & 0 \end{pmatrix}, \tag{3}$$

$$\begin{pmatrix} U_k & U_{k+1} \\ V_k & V_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & P \end{pmatrix} R^k, \quad \text{where } R = \begin{pmatrix} 0 & -Q \\ 1 & P \end{pmatrix}. \tag{4}$$

Theorem 1. *Let $\{U_r(P,Q)\}$, $\{V_r(P,Q)\}$ be the Lucas sequences with $P, Q \in \mathbb{F}$. Then there exist the following representations of U_{mk} and V_{mk} via U_k, U_{k+1}:*

$$U_{mk} = \sum_{i=0}^{m} \binom{m}{i} (-1)^{i+1} U_i U_k^i U_{k+1}^{m-i}, \tag{5}$$

$$V_{mk} = \sum_{i=0}^{m} \binom{m}{i} (-1)^{i} V_i U_k^i U_{k+1}^{m-i}, \tag{6}$$

where $m \in \mathbb{Z}^+$, $k \in \mathbb{Z}$ and such that $U_k U_{k+1} \neq 0$. Moreover, there are no other representations of the form $\sum_{i=0}^{m} c_i U_k^i U_{k+1}^{m-i}$, where c_i are coefficients that do not depend on k.

Proof. By (3), we have

$$M^k = \begin{pmatrix} U_{k+1} & (-PU_{k+1} + V_{k+1})/2 \\ U_k & (-PU_k + V_k)/2 \end{pmatrix}. \tag{7}$$

It is well known that $V_k = PU_k - 2QU_{k-1}$. Then

$$M^k = \begin{pmatrix} U_{k+1} & -QU_k \\ U_k & -QU_{k-1} \end{pmatrix} = \begin{pmatrix} U_{k+1} & -QU_k \\ U_k & U_{k+1} - PU_k \end{pmatrix} \tag{8}$$

since $U_{k+1} = PU_k - QU_{k-1}$ and therefore

$$M^k = U_{k+1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + U_k \begin{pmatrix} 0 & -Q \\ 1 & -P \end{pmatrix} = U_{k+1}I + U_k A, \tag{9}$$

where

$$A = \begin{pmatrix} 0 & -Q \\ 1 & -P \end{pmatrix}. \tag{10}$$

It follows from (4) that if P is replaced by $-P$ in R, then we obtain A. In addition, if we use $V_k = PU_k - 2QU_{k-1}$ in the matrix formula (4), then

$$R^m = \begin{pmatrix} -QU_{m-1}(P,Q) & -QU_m(P,Q) \\ U_m(P,Q) & U_{m+1}(P,Q) \end{pmatrix}. \tag{11}$$

By the Binet formulas, it can be seen that $U_m(-P,Q) = (-1)^{m+1}U_m(P,Q)$. Then

$$A^m = \begin{pmatrix} (-1)^{m+1}QU_{m-1}(P,Q) & (-1)^m QU_m(P,Q) \\ (-1)^{m+1}U_m(P,Q) & (-1)^m U_{m+1}(P,Q) \end{pmatrix}. \tag{12}$$

Thus

$$M^{mk} = (M^k)^m = (U_{k+1}I + U_k A)^m = \sum_{i=0}^{m} \binom{m}{i} U_k^i A^i U_{k+1}^{m-i}. \tag{13}$$

Since

$$M^{mk} = \begin{pmatrix} * & * \\ U_{mk} & * \end{pmatrix} \quad \text{and} \quad A^i = \begin{pmatrix} * & * \\ (-1)^{i+1}U_i & * \end{pmatrix}, \tag{14}$$

the representation (5) follows from (13). Let $\operatorname{tr}(X)$ denote trace of the matrix X. From (8) and (13), it is seen that

$$\begin{aligned} \operatorname{tr}(M^{mk}) = U_{mk+1} - QU_{mk-1} = V_{mk} &= \operatorname{tr}\left(\sum_{i=0}^{m} \binom{m}{i} U_k^i A^i U_{k+1}^{m-i}\right) \\ &= \sum_{i=0}^{m} \binom{m}{i} U_k^i \operatorname{tr}\left(A^i\right) U_{k+1}^{m-i}. \end{aligned} \tag{15}$$

By (12), $\operatorname{tr}\left(A^i\right) = (-1)^i(-QU_{i-1} + U_{i+1}) = (-1)^i V_i$. Using this in (15), we get (6).

Let $a_i(P,Q) = \binom{m}{i}(-1)^{i+1}U_i$, then $U_{mk} = \sum_{i=0}^{m} a_i(P,Q)U_k^i U_{k+1}^{m-i}$. Suppose that there is another representation in the analogical form $U_{mk} = \sum_{i=0}^{m} b_i(P,Q)U_k^i U_{k+1}^{m-i}$. Then $\sum_{i=0}^{m}(b_i - a_i)U_k^i U_{k+1}^{m-i} = 0$. We can take $P = 2, Q = 1$; in this case $U_k = k$. Therefore, we have $\sum_{i=0}^{m}(b_i - a_i)k^i(k+1)^{m-i} \equiv 0 \pmod{k}$, and hence $k \mid (b_0 - a_0)$. This must hold for any $k > 0$, and is possible only if $b_0 = a_0$. By analogy we prove that $b_i = a_i$ for $0 < i \leq m$. So the representation (5) is unique. The uniqueness of the representation (6) can be proved similarly. □

We note that the study fits in the literature. For a generalization of (5) see Remark 4.1 in [20]. When P^2-4Q is not a perfect square (see [19], Corollary 2.10), both identities are proved.

Now we consider another way of obtaining the identity (5). We can use a method similar to the partial fraction decomposition. We begin with the supposed identity $U_{mk}=\sum_{i=0}^{m} a_i(P,Q)U_k^iU_{k+1}^{m-i}$. Then we get the system of $m+1$ equations whose variables and coefficients are functions of P,Q:

$$U_{mk}=\sum_{i=0}^{m} a_i(P,Q)U_k^iU_{k+1}^{m-i} \quad \left(-\lfloor (m-1)/2\rfloor \le k \le \lceil m/2\rceil\right). \tag{16}$$

Since $U_0=0$, $U_1=1$, it is seen that $a_0=0$, $a_m=(-1)^{m+1}U_m$. The remaining coefficients can be found by solving the system.

Example 1. $U_{3k}=a_0U_{k+1}^3+a_1U_{k+1}^2U_k+a_2U_{k+1}U_k^2+a_3U_k^3$. The system (16) for this case is:

$$\begin{cases} U_{-3}=a_0U_0^3+a_1U_0^2U_{-1}+a_2U_0U_{-1}^2+a_3U_{-1}^3,\\ U_0=a_0U_1^3+a_1U_1^2U_0+a_2U_1U_0^2+a_3U_0^3,\\ U_3=a_0U_2^3+a_1U_2^2U_1+a_2U_2U_1^2+a_3U_1^3,\\ U_6=a_0U_3^3+a_1U_3^2U_2+a_2U_3U_2^2+a_3U_2^3. \end{cases} \tag{17}$$

Using $U_{-3}=-(P^2-Q)/Q^3$, $U_{-1}=-1/Q$, $U_0=0$, $U_1=1$, $U_3=P^2-Q$, $U_6=P^5-4P^3Q+3PQ^2$, we get the solution $a_0=0$, $a_1=3$, $a_2=-3P$, $a_3=P^2-Q$. So $U_{3k}=3U_{k+1}^2U_k-3PU_{k+1}U_k^2+(P^2-Q)U_k^3$. This is consistent with (5).

Example 2. Consider the Fibonacci Pythagorean triples identity [9].

$(F.8)\qquad (F_{k-1}F_{k+2})^2+(2F_kF_{k+1})^2=F_{2k+1}^2.$

Suppose that there exists an identity of the form

$$c_1(U_{k-1}U_{k+2})^2+c_2(U_kU_{k+1})^2+c_3U_{2k+1}^2=0. \tag{18}$$

To get the system for the variables c_i we put $k=-1,0,1$. Then

$$\begin{cases} c_1P^2/Q^4+c_3/Q^2=0,\\ c_1P^2/Q^2+c_3=0,\\ c_2P^2+c_3(P^2-Q)^2=0. \end{cases} \tag{19}$$

If the system has no solutions (the rank is 3), then we can state that the identity of the form (18) does not exist. In this case we may modify it by adding new terms and obtain a new system. In fact, the rank is 2. We put $c_3=-1$, then the solution is $c_1=Q^2/P^2, c_2=(P^2-Q)^2/P^2, c_3=-1$. But when we use the Binet formulas to verify the resulting formula we see that it is not valid. Moreover, if we get the system which corresponds $k=0,1,2$, then the determinant of the system matrix is $-2P^4\left(P^2-2Q\right)\left(P^2-Q\right)\left(P^2+Q\right)/Q$. Thus, the solution exists if one of the following holds: $P = 0, P^2 = 2Q, P^2 = \pm Q$. But if we add $c_4U_{k-1}U_kU_{k+1}U_{k+2}$ to the left side of (18), then we find the generalization of $(F.8)$ as follows:

$(GF.8)$ $\quad (QU_{k-1}U_{k+2})^2 + ((P^2 - Q)U_kU_{k+1})^2 = (PU_{2k+1})^2 + 2Q(P^2+Q)U_{k-1}U_kU_{k+1}U_{k+2}.$

Remark. In general, we cannot assert that the method leads to the final result which holds for all k since a supposed formula similar to (18) is checked by a system only for some values of k. So we need to use the Binet formulas to prove that the final result is valid for all k. In Example 1 this check is not necessary, since by Theorem 1 we know that the identity which involves U_{mk}, U_k, U_{k+1} exists. But we obtained the unique solution using the method.

Example 3. We want to get an identity of the form

$$c_1U_{k+1}^2 + c_2U_k^2 + c_3U_{k-1}^2 = 0. \tag{20}$$

We put $k = -1, 0, 1$ and obtain the following system

$$\begin{cases} c_2/Q^2 + c_3P^2/Q^4 = 0, \\ c_1 + c_3/Q^2 = 0, \\ c_1P^2 + c_2 = 0. \end{cases} \tag{21}$$

Since the determinant of the system matrix is $2P^2/Q^4$, we conclude that for a nonzero P there is no identity which contains only squares of three consecutive terms of $\{U_k\}$. But if we try to find an identity of the form

$$c_1U_{k+1}^2 + c_2U_k^2 + c_3U_{k-1}^2 + c_4U_{k-2}^2 = 0, \tag{22}$$

then we obtain

$(GF.9)$ $\quad U_{k+1}^2 - Q^3U_{k-2}^2 = (P^2 - Q)(U_k^2 - QU_{k-1}^2).$

If $P = 1$ and $Q = -1$, then we get the identity for the Fibonacci numbers:

$(F.9)$ $\quad F_{k+1}^2 + F_{k-2}^2 = 2(F_k^2 + F_{k-1}^2).$

As shown the method is good not only for generalizing Fibonacci identities, but also for finding new identities. Below we present some results that we obtained using this method. Some of them are well-known.

$(F.10)$ $\quad F_{2k} = F_{k+1}^2 - F_{k-1}^2,$

$(GF.10)$ $\quad PU_{2k} = U_{k+1}^2 - Q^2U_{k-1}^2,$

$(F.11)$ $\quad F_{k+1}^2 = 4F_kF_{k-1} + F_{k-2}^2,$

$(GF.11)$ $\quad U_{k+1}^2 = 2P(P^2 - Q)U_kU_{k-1} + (Q^2 - P^4)U_{k-1}^2 + P^2Q^2U_{k-2}^2,$

$(F.12)$ $\quad F_{k+2}^2 - F_{k-2}^2 = 3(F_{k+1}^2 - F_{k-1}^2),$

$(GF.12)$ $\quad U_{k+2}^2 - Q^4U_{k-2}^2 = (P^2 - 2Q)(U_{k+1}^2 - Q^2U_{k-1}^2).$

Note that $(GF.9)$ and $(GF.12)$ have similar forms. To generalize $(GF.12)$ consider $c_1U_{k+m}^2 + c_2U_{k+l}^2 + c_3U_{k-l}^2 + c_4U_{k-m}^2 = 0$. With the help of the method we get the following system:

$$\begin{cases} c_1U_{m-1}^2 + c_2U_{l-1}^2 + c_3U_{l+1}^2/Q^{2(l+1)} + c_4U_{m+1}^2/Q^{2(m+1)} = 0, \\ c_1U_m^2 + c_2U_l^2 + c_3U_l^2/Q^{2l} + c_4U_m^2/Q^m = 0, \\ c_1U_{m+1}^2 + c_2U_{l+1}^2 + c_3U_{l-1}^2/Q^{2(l-1)} + c_4U_{m-1}^2/Q^{2(m-1)} = 0. \end{cases} \tag{23}$$

The rank is 2. If we put $c_1 = U_{l+1}^2 - Q^2U_{l-1}^2$, then $c_2 = -(U_{m+1}^2 - Q^2U_{m-1}^2)$, $c_3 = Q^{2l}(U_{m+1}^2 - Q^2U_{m-1}^2)$, $c_4 = -Q^{2m}(U_{l+1}^2 - Q^2U_{l-1}^2)$. Using $(GF.10)$, we obtain

$(GF.13)$ $\quad U_{2l}(U_{k+m}^2 - Q^{2m}U_{k-m}^2) = U_{2m}(U_{k+l}^2 - Q^{2l}U_{k-l}^2),$

$(F.13)$ $\quad F_{2l}(F_{k+m}^2 - F_{k-m}^2) = F_{2m}(F_{k+l}^2 - F_{k-l}^2).$

Another way to prove $(GF.13)$ is to use Catalan's identity $(GF.2)$. It follows that $U_{k+m}^2 - Q^{2m}U_{k-m}^2 = U_{2k}U_{2m}$, $U_{k+l}^2 - Q^{2l}U_{k-l}^2 = U_{2k}U_{2l}$, which completes the proof.

To generalize $(GF.9)$ we consider the most general formula which involves only four squares of sequence terms: $U_k^2 + c_1U_{k+m}^2 + c_2U_{k+l}^2 + c_3U_{k+n}^2 = 0$. Using the method, we get

$(GF.14)$ $\quad U_k^2 = U_{k+m}^2U_lU_s/(Q^{2m}U_{l-m}U_{s-m}) + U_{k+l}^2U_sU_m/(Q^{2l}U_{s-l}U_{m-l}) + U_{k+s}^2U_mU_l/(Q^{2s}U_{m-s}U_{l-s}),$

$(F.14)$ $\quad F_k^2 = F_{k+m}^2F_lF_s/(F_{l-m}F_{s-m}) + F_{k+l}^2F_sF_m/(F_{s-l}F_{m-l}) + F_{k+s}^2F_mF_l/(F_{m-s}F_{l-s}).$

Here we mean that l, m, s are distinct integers. The analog for cubes is

$(GF.15)$ $\quad U_k^3 = U_{k+m}^3U_lU_pU_s/(Q^{3m}U_{l-m}U_{p-m}U_{s-m}) + U_{k+l}^3U_mU_pU_s/(Q^{3l}U_{m-l}U_{p-l}U_{s-l}) + U_{k+p}^3U_lU_mU_s/(Q^{3p}U_{l-p}U_{m-p}U_{s-p}) + U_{k+s}^3U_lU_mU_p/(Q^{3s}U_{l-s}U_{m-s}U_{p-s}).$

Similar identities on sums of powers of the terms of the Lucas sequences can be found in [7, 21].

Theorem 2. *Let $\{U_k\}$ be the Lucas sequence with parameters P, Q in an arbitrary field $\mathbb{F}$. Let d_i $(0 \le i \le n)$ be distinct integers. Then the following holds:*

$$U_{k+x}^n = \sum_{i=0}^{n} U_{k+d_i}^n \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{U_{x-d_j}}{U_{d_i-d_j}}. \tag{24}$$

Proof. We denote $(\alpha/\bar{\alpha})^x = y$. Then

$$U_{k+x}^n = \bar{\alpha}^{xn}\left(\frac{\alpha^{k+x}/\bar{\alpha}^x - \bar{\alpha}^k}{\alpha - \bar{\alpha}}\right)^n = \bar{\alpha}^{xn}\left(\frac{\alpha^k y - \bar{\alpha}^k}{\alpha - \bar{\alpha}}\right)^n. \tag{25}$$

We see that U^n_{k+x} is the product of $\bar{\alpha}^{xn}$ and the polynomial in y of degree n with coefficients in $\mathbb{F}$. Since $U_{x-d_j} = \bar{\alpha}^x(\alpha^{-d_j}y - \bar{\alpha}^{-d_j})/(\alpha - \bar{\alpha})$, the right side of (24) has the same structure as U^n_{k+x}. Since $U_0 = 0$, it easy to see that (24) is valid for $x = d_i$ $(0 \leq i \leq n)$. Thus, the polynomials of degree n on the left and right sides of (24) are equal, since they are equal for $n + 1$ different values of the variable. □

Note that (24) is related to Lagrange interpolation. If we put $x = 0$, then after simple transformations we obtain an identity that generalizes (GF.14) and (GF.15):

$$U^n_k = \sum_{i=0}^{n} \frac{U^n_{k+d_i}}{Q^{nd_i}} \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{U_{d_j}}{U_{d_j-d_i}}. \tag{26}$$

As an example, we give some Fibonacci identities that can be obtained from (24):

$$F^3_{k+2} + F^3_{k-2} = 3\left(F^3_{k+1} - F^3_{k-1}\right) + 6F^3_k, \tag{27}$$

$$F^4_{k+3} - F^4_{k-3} = 4(F^4_{k+2} - F^4_{k-2}) + 20(F^4_{k+1} - F^4_{k-1}), \tag{28}$$

$$F^5_{k+3} - F^5_{k-3} = 8(F^5_{k+2} + F^5_{k-2}) + 40(F^5_{k+1} - F^5_{k-1}) - 60F^5_k. \tag{29}$$

Corollary 1. *Let p_0, p_1 be nonzeros in $\mathbb{F}$, and let the sequence $\{W_k(a_0, a_1; p_0, p_1)\}$ be defined by the relation $W_{k+2} = p_0W_{k+1} + p_1W_k$, with $W_0 = a_0, W_1 = a_1$, where $a_0, a_1 \in \mathbb{F}$. Let s be an integer such that $W_s = 0$, and d_i $(0 \leq i \leq n)$ be integers such that $W_{d_i-d_j+s} \neq 0$ for $i \neq j$. Then the following holds*

$$W^n_{k+x} = \sum_{i=0}^{n} W^n_{k+d_i} \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{W_{x-d_j+s}}{W_{d_i-d_j+s}}. \tag{30}$$

Proof. It is known [8] that $W_k(a_0, a_1; p_0, p_1) = a_1U_k(p_0, -p_1) + a_0p_1U_{k-1}(p_0, -p_1)$. The rest of the proof is analogous to the proof of Theorem 2. □

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