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HYPERHARMONIC NUMBERS CAN RARELY BE INTEGERS

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Abstract

We give density results on a conjecture of Mező about hyperharmonic numbers. It is shown with stronger error terms that almost all hyperharmonic numbers are not integers. Other types of density results based on a new sufficient condition about the nonintegerness of hyperharmonic numbers are obtained. Quantitative estimates for the frequency of noninteger hyperharmonic numbers are deduced using the distribution of prime numbers. In particular, assuming the Riemann hypothesis or Cramér's conjecture on the gap between consecutive prime numbers, we come very close to establishing Mező's conjecture.

1. Introduction

This paper is concerned with giving density results on a conjecture of Mező [22] about hyperharmonic numbers. All of our findings are in support of the conjecture and show that it holds for the vast majority of possible cases. The classical harmonic numbers are defined as the partial sums of the harmonic series

$$h_n = \sum_{k=1}^{n} \frac{1}{k}$$

for $n \geq 1$. Theisinger [27] and Kürschák [21] were among the first to show that h_n is never an integer except $h_1 = 1$. This nonintegerness problem was generalized in [14] and [25], where it was proved that multiple harmonic type sums are rarely integers.

In another direction, Ramanujan gave an asymptotic formula for h_n using triangular numbers. However, he didn't explain the structure of the mysterious coefficients in the corresponding asymptotic expansion. Later on, the work of Berndt revealed the essence of the coefficients, see [9, pp. 531–532].

Harmonic numbers enjoy many arithmetical and analytical properties. For recent work on harmonic numbers and their applications to approximation of real numbers via log-sine integrals, we refer to [1, 2, 3]. As an interesting connection to formal logic, Pambuccian [24] verified the nonintegerness of more general sums of fractions in Peano arithmetic without the axiom of induction. Numerators of harmonic type numbers are closely related with Stirling numbers and recent work on such numbers was done by Komatsu and Mező [20, 23]. A fruitful generalization of harmonic numbers was introduced by Conway and Guy [10]. They defined the nth hyperharmonic number of order r by the recursive formula

$$h_n^{(r)} := \sum_{k=1}^{n} h_k^{(r-1)},$$

where initially $h_n^{(1)} = h_n$. Functional equations and special values of zeta functions arising from hyperharmonic and generalized harmonic numbers were studied in [4] and [5]. Congruences and some properties of harmonic type sums were investigated in [11, 16, 17]. Apart from that, polynomials related to hyperharmonic numbers and their closed forms can be found in [13]. Inspired by the nonintegerness of harmonic numbers, Mező [22] made the following conjecture:

"*$h_n^{(r)}$ is never an integer except $h_1^{(r)} = 1$.*"

In [22], he proved the conjecture for $r = 2, 3$. Amrane and Belbachir [7], [8] extended this to $r \leq 25$. They also tabulated other values of n and r such that $h_n^{(r)}$ is not an integer. Although there seems to be no obvious reason for the truth of the conjecture, a significant step towards its resolution was taken in [15]. In Theorem 1 of [15], it was shown that Mező's conjecture holds in almost all cases. Precisely, if

$$S(x) = \left| \{(n, r) \in [0, x] \times [0, x] : h_n^{(r)} \notin \mathbb{Z}\} \right|,$$

then

$$S(x) = x^2 + O\left(x^{\frac{2.475}{1.475}}\right).$$

Theorem 2 of [15] states that $h_n^{(r)}$ is not an integer in the cases when n is even, r is odd and n is a prime power. We will improve both of these theorems here with delicate density results possessing sharper error terms and thereby making further progress towards Mező's conjecture.

Theorem 1. *If $S(x)$ is the number of pairs $(n,r) \in [0,x] \times [0,x]$ such that $h_n^{(r)}$ is not an integer, then for any $A > 0$, we have*

$$S(x) = x^2 + O_A\left(\frac{x^{\frac{80}{59}}}{(\log x)^A}\right),$$

where the implied constant depends only on A. Assuming the Riemann hypothesis, we have

$$S(x) = x^2 + O\left(x^{\frac{10}{9}}(\log x)^{\frac{28}{9}}\right).$$

Finally, assuming Cramér's conjecture, namely that

$$p_{n+1} - p_n = O(\log^2 p_n)$$

for consecutive prime numbers p_n and p_{n+1}, we have

$$S(x) = x^2 + O(x \log^3 x).$$

Note that in Theorem 1 of [15], the error term for $S(x)$ was obtained in the forms

$$O_\epsilon\left(x^{\frac{5}{3}+\epsilon}\right)$$

for all $\epsilon > 0$ and

$$O\left(x^{\frac{3}{2}} \log x\right),$$

assuming the Riemann hypothesis or Cramér's conjecture, respectively. Moreover, we have

$$\frac{80}{59} \approx 1.35 < 1.67 \approx \frac{2.475}{1.475}, \quad \frac{10}{9} \approx 1.11 < 1.66 \approx \frac{5}{3}$$

which represent considerable improvements in the exponents. Finally, assuming Cramér's conjecture, our asymptotic for $S(x)$ comes very close to establishing Mező's conjecture as the number of all lattice points (n,r) in $[0,x] \times [0,x]$ with $n > 1$ is

$$x^2 + O(x).$$

Our next result gives a sufficient condition for $h_n^{(r)} \notin \mathbb{Z}$ which is easy to check numerically. Consequently, this condition leads to another density type outcome not covered by Theorem 1. As usual, we let $\{x\} = x - [x]$ be the fractional part of a real number x.

Theorem 2. *Assume that m is the highest power of a prime number p such that p^m divides at least one of the n consecutive numbers $r, r+1, \ldots, r+n-1$. If p^m divides exactly one of $r, r+1, \ldots, r+n-1$ and the condition*

$$\left\{\frac{r-1}{p^k}\right\} + \left\{\frac{n}{p^k}\right\} < 1 \tag{1}$$

holds for some k with $p^k \le n$, then $h_n^{(r)} \notin \mathbb{Z}$. In particular, if

$$\left\{\frac{r-1}{2^k}\right\} + \left\{\frac{n}{2^k}\right\} < 1 \tag{2}$$

holds for some k with $2^k \le n$, then $h_n^{(r)} \notin \mathbb{Z}$.

Let us remark that if $n \ge 2$ is even, then taking $k = 1$ in (2), we see that $h_n^{(r)} \notin \mathbb{Z}$. If r is odd, then we see again from (2) that $h_n^{(r)} \notin \mathbb{Z}$ for $n \ge 2$. Lastly, if $n = p^s$ is a prime power, where we may assume that p is odd, then taking $k = s$ in (1) and observing that p^m with $m \ge s$ divides exactly one of $r, r+1, \ldots, r+n-1$, we have $h_n^{(r)} \notin \mathbb{Z}$. Therefore, Theorem 2 is an improvement of Theorem 2 of [15], but it gives much more than that as can be realized in the following consequence.

Corollary 1. *Let $k \ge 1, n, a$ be nonnegative integers subject to the conditions $n \equiv a \pmod{2^k}$, $a \in \{0, 1, \ldots, 2^k - 1\}$ and $2^k \le n$. If $r \equiv 1, 2, \ldots, 2^k - a \pmod{2^k}$, then $h_n^{(r)} \notin \mathbb{Z}$. Consequently, for such a fixed n, the set of integers r with $h_n^{(r)} \notin \mathbb{Z}$ contains a set of density*

$$1 - \frac{a}{2^k}.$$

Furthermore, let $k \ge 1, n, r, a$ be nonnegative integers with the conditions $r \equiv a \pmod{2^k}$, $a \in \{1, 2, \ldots, 2^k\}$ and $2^k \le n$. If $n \equiv 0, 1, \ldots, 2^k - a \pmod{2^k}$, then $h_n^{(r)} \notin \mathbb{Z}$.

Corollary 1 signifies the dual role of n and r over arithmetic progressions that leads to the nonintegerness of $h_n^{(r)}$. For a fixed r, we know from Theorem 4 of [15] that $h_n^{(r)} \notin \mathbb{Z}$ whenever $n \ge r/16$. It follows that the density of such n is 1. Although the second part of Corollary 1 falls short of this density, it is still not implied by Theorem 4 of [15] since for $n < r/16$, there are values of n over the indicated progressions such that $h_n^{(r)} \notin \mathbb{Z}$. However, for a fixed n, the frequency of r such that $h_n^{(r)} \notin \mathbb{Z}$ can also be estimated from below in terms of the largest prime number not exceeding n.

Theorem 3. *Let $p^{(n)}$ be the largest prime number that is $\le n$. Then the set of integers r with $h_n^{(r)} \notin \mathbb{Z}$ contains a set of density*

$$2 - \frac{n}{p^{(n)}}.$$

Consequently, for all large enough n, the set of integers r with $h_n^{(r)} \notin \mathbb{Z}$ contains a set of density

$$1 - \frac{n^{0.525}}{n - n^{0.525}} = 1 + O\left(\frac{1}{n^{0.475}}\right).$$

Assuming the Riemann hypothesis, this density is

$$1 - \frac{c\sqrt{n}\log n}{n - c\sqrt{n}\log n} = 1 + O\left(\frac{\log n}{\sqrt{n}}\right)$$

for all $n \geq 2$ *and some constant* $c > 0$*. Finally, assuming Cramér's conjecture, this density is*

$$1 - \frac{c \log^2 n}{n - c \log^2 n} = 1 + O\left(\frac{\log^2 n}{n}\right)$$

for all $n \geq 2$ *and some constant* $c > 0$*.*

It is known from [15] that $h_n^{(r)} \notin \mathbb{Z}$ when $1 < n \leq 32$ or when $r \leq 20001$. However, none of the methods used in [15] could show that $h_{33}^{(r)} \notin \mathbb{Z}$ for all r. Nevertheless, our final result gives a lower estimate for the frequency of r such that $h_{33}^{(r)} \notin \mathbb{Z}$ indicating that finding r with $h_{33}^{(r)} \in \mathbb{Z}$ is not very probable.

Corollary 2. *The set of integers* r *such that* $h_{33}^{(r)} \notin \mathbb{Z}$ *contains a set of density*

$$\frac{143}{144} \approx 0.993.$$

2. Proofs

Proof of Theorem 1. Consider a pair $(n, r) \in [0, x] \times [0, x]$, where n and r are positive integers. First, we may assume that $n > M$ for any given constant M as the number of pairs $(n, r) \in [0, x] \times [0, x]$ with $n \leq M$ is $O(x)$ and this will be a negligible error. Next, let us put an upper bound on the size of n in terms of x. As a result of Theorem 10 of [15], we know that if the interval $(n - \Phi(n), n]$ contains a prime number for n large enough, where $\Phi(n)$ is a monotonically increasing function with $\Phi(n) = o(n)$ and

$$r = O\left(\frac{n^2}{\Phi(n)}\right), \tag{3}$$

then $h_n^{(r)} \notin \mathbb{Z}$. As in [15], we may take $\Phi(n) = n^{0.525}$ when n is large enough. It follows from (3) that if

$$r = O\left(n^{1.475}\right), \tag{4}$$

then $h_n^{(r)} \notin \mathbb{Z}$. As $r \leq x$, it is clear from (4) that one may assume

$$n \leq c x^{\frac{1}{1.475}} \tag{5}$$

for some constant $c > 0$, since otherwise $h_n^{(r)}$ is guaranteed to be noninteger. One may fix a large enough n subject to (5). It is known from [10] that

$$h_n^{(r)} = \frac{r(r+1)\ldots(r+n-1)}{n!}\left(\frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{r+n-1}\right). \tag{6}$$

Let $p^{(n)}$ be the largest prime number that is $\leq n$. As n is large enough, Bertrand's postulate gives that $p^{(n)} > n/2$. Therefore, $p^{(n)}$ can divide at most two integers among the n consecutive integers $r, r+1, \ldots, r+n-1$. If $p^{(n)}$ divides exactly one of $r, r+1, \ldots, r+n-1$, then it is easy to see that $h_n^{(r)}$ is not an integer. We may then assume that $r \le x$ is such that exactly two of the n consecutive integers $r, r+1, \ldots, r+n-1$ are divisible by $p^{(n)}$. Note that when n is large enough, we have

$$n - p^{(n)} \le \Phi(n) \tag{7}$$

as soon as the interval $(n-\Phi(n), n]$ contains a prime number. Therefore, if $r \le x$ is such that exactly two of the n consecutive integers $r, r+1, \ldots, r+n-1$ are divisible by $p^{(n)}$, then from (7), there exists an integer k such that

$$|r - kp^{(n)}| = O\left(\Phi(n)\right). \tag{8}$$

Observe that the number of multiples of $p^{(n)}$ that are less than or equal to x is

$$\left[\frac{x}{p^{(n)}}\right],$$

and consequently the number of $r \le x$ satisfying (8) is

$$O\left(\frac{x}{p^{(n)}}\Phi(n)\right) = O\left(\frac{x}{n}\Phi(n)\right) \tag{9}$$

when $n \le cx^{\frac{1}{1.475}}$ is large enough so that $(n-\Phi(n), n]$ contains a prime number. One infers from (9) that

$$S(x) = x^2 + O\left(x \sum_{n \le cx^{\frac{1}{1.475}}} \frac{\Phi(n)}{n}\right), \tag{10}$$

where $\Phi(n)$ is subject to the only condition that $(n-\Phi(n), n]$ contains a prime number. Put $X = cx^{\frac{1}{1.475}}$. Jia [19] showed that for all $n \le X$ with at most

$$O_A\left(\frac{X}{(\log X)^A}\right)$$

exceptions, where A is any positive number and the implied constant depends only on A, the interval $(n - n^{1/19}, n]$ contains a prime number (Jia [19] indeed obtained the stronger conclusion that the exponent here can be any number $> 1/20$, but we will not need this in the sequel). Therefore, we may take $\Phi(n) = n^{1/19}$ with the number of exceptional $n \le X$ as above. Let us define a set of good n as follows:

$$\mathcal{G} := \{n \le X \ : \ \Phi(n) = n^{1/19}\}. \tag{11}$$

According to (11), the set of bad n is defined as follows:

$$\mathcal{B} := \{n \le X \ : \ n \notin \mathcal{G}\}. \tag{12}$$

From (11) and (12), we have

$$\sum_{n \le X} \frac{\Phi(n)}{n} = \sum_{n \in \mathcal{G}} \frac{\Phi(n)}{n} + \sum_{n \in \mathcal{B}} \frac{\Phi(n)}{n}. \tag{13}$$

Again using (11), one obtains

$$\sum_{n \in \mathcal{G}} \frac{\Phi(n)}{n} \ll X^{1/19}. \tag{14}$$

We also know that

$$M(X) := \sum_{n \in \mathcal{B}} 1 \ll_A \frac{X}{(\log X)^A}. \tag{15}$$

Taking $\Phi(n) = n^{0.525}$ for the sum over $n \in \mathcal{B}$ and applying partial summation, one gets

$$\sum_{n \in \mathcal{B}} \frac{\Phi(n)}{n} = \sum_{n \in \mathcal{B}} \frac{1}{n^{0.475}} = \frac{M(X)}{X^{0.475}} + 0.475 \int_2^X \frac{M(t)}{t^{1.475}}\ dt + O(1). \tag{16}$$

Using (15), we see that

$$\frac{M(X)}{X^{0.475}} \ll_A \frac{X^{0.525}}{(\log X)^A} \tag{17}$$

and

$$\int_2^X \frac{M(t)}{t^{1.475}}\ dt \ll_A \int_2^X \frac{1}{t^{0.475}(\log t)^A}\ dt. \tag{18}$$

By partial integration, one obtains for $2 \le Y \le X$ that

$$\begin{aligned} \int_Y^X \frac{1}{t^{0.475}(\log t)^A}\ dt &= \frac{X^{0.525}}{0.525(\log X)^A} \\ &\quad + \frac{A}{0.525} \int_Y^X \frac{1}{t^{0.475}(\log t)^{A+1}}\ dt + O_Y(1). \end{aligned} \tag{19}$$

If Y is large enough only in terms of A, then

$$\frac{A}{(0.525)\log t} \le \frac{1}{2} \tag{20}$$

holds for any $t \ge Y$. It is clear from (20) by the monotonicity of integrals that

$$\frac{A}{0.525} \int_Y^X \frac{1}{t^{0.475}(\log t)^{A+1}}\ dt \le \frac{1}{2} \int_Y^X \frac{1}{t^{0.475}(\log t)^A}\ dt \tag{21}$$

when Y is large enough only in terms of A. Assembling (19) and (21), we obtain

$$\begin{aligned} \int_2^X \frac{1}{t^{0.475}(\log t)^A}\ dt &= \int_2^Y \frac{1}{t^{0.475}(\log t)^A}\ dt + \int_Y^X \frac{1}{t^{0.475}(\log t)^A}\ dt \\ &\ll_A \frac{X^{0.525}}{(\log X)^A}. \end{aligned} \tag{22}$$

Therefore, (18) and (22) give

$$\int_2^X \frac{M(t)}{t^{1.475}}\, dt \ll_A \frac{X^{0.525}}{(\log X)^A}. \tag{23}$$

Gathering (16), (17) and (23), we have

$$\sum_{n\in\mathcal{B}} \frac{\Phi(n)}{n} \ll_A \frac{X^{0.525}}{(\log X)^A}. \tag{24}$$

Next (13), (14) and (24) yield that

$$\sum_{n\le X} \frac{\Phi(n)}{n} \ll_A \frac{X^{0.525}}{(\log X)^A} \ll_A \frac{x^{\frac{0.525}{1.475}}}{(\log x)^A}. \tag{25}$$

Feeding (25) into (10),

$$S(x) = x^2 + O_A\left(\frac{x^{\frac{2}{1.475}}}{(\log x)^A}\right)$$

follows. This completes the proof of the first part of Theorem 1.

For the second part let us assume the Riemann hypothesis. An interesting feature of the argument is the usage of different consequences of the Riemann hypothesis. First, Cramér [12] showed, assuming the Riemann hypothesis, that the interval $(n - c\sqrt{n}\log n, n]$ contains a prime number for $n \ge 2$ and some constant $c > 0$ (in the same paper of Cramér, one can also find a probabilistic model on which Cramér based his conjecture $p_{n+1} - p_n = O(\log^2 p_n)$ for the gaps between consecutive prime numbers). For extensions of Cramér's results to prime ideals in number fields, we refer to [6] and [18]. Therefore, we may take $\Phi(n) = c\sqrt{n}\log n$ and it follows from (3) that if

$$r = O\left(\frac{n^{3/2}}{\log n}\right), \tag{26}$$

then $h_n^{(r)}$ is guaranteed to be noninteger. As $r \le x$, we see from (26) that

$$n \le X := cx^{2/3}(\log x)^{2/3} \tag{27}$$

can be assumed for some constant $c > 0$ as otherwise $h_n^{(r)}$ is not an integer. Using (27) and arguing as above, we arrive at the formula

$$S(x) = x^2 + O\left(x\sum_{n\le X}\frac{\Phi(n)}{n}\right), \tag{28}$$

where $\Phi(n)$ is subject to the only condition that $(n-\Phi(n), n]$ contains a prime number. Our definition of a set of good n is as follows:

$$\mathcal{G} := \{n \leq X \ : \ \Phi(n) = n^{\theta}\}, \tag{29}$$

where $0 < \theta < 1$ is to be determined later. As a result of (29), the set of bad n becomes

$$\mathcal{B} := \{n \leq X \ : \ n \notin \mathcal{G}\}. \tag{30}$$

Using (28), (29) and (30), we may now express

$$S(x) = x^2 + O\left(x \sum_{n \in \mathcal{G}} \frac{\Phi(n)}{n} + x \sum_{n \in \mathcal{B}} \frac{\Phi(n)}{n}\right). \tag{31}$$

As a second consequence of the Riemann hypothesis, let us recall a result of Selberg [26] who proved that

$$\sum_{\substack{p_n \leq x \\ p_{n+1} - p_n \geq y}} 1 = O\left(\frac{x \log^2 x}{y^2}\right). \tag{32}$$

Notice that (32) is very useful in putting a severe limitation on the number of exceptionally large gaps between consecutive prime numbers. Let us decompose the range $[2, x]$ of summation in (32) to dyadic intervals of the form

$$\left[\frac{x}{2^{k+1}}, \frac{x}{2^k}\right]$$

for $k \geq 0$. Applying (32) to each of these intervals, one obtains

$$\sum_{\substack{\frac{x}{2^{k+1}} \leq p_n \leq \frac{x}{2^k} \\ p_{n+1} - p_n \geq p_n^{\theta}}} 1 \leq \sum_{\substack{\frac{x}{2^{k+1}} \leq p_n \leq \frac{x}{2^k} \\ p_{n+1} - p_n \geq \frac{x^{\theta}}{2^{(k+1)\theta}}}} 1 \ll 2^{2(k+1)\theta - k} x^{1-2\theta} \log^2 x. \tag{33}$$

Summing (33) over all possible values of k and noting that the series

$$\sum_{k=0}^{\infty} 2^{(2\theta-1)k}$$

is convergent when $0 < \theta < 1/2$, we deduce that $\Phi(n) = n^{\theta}$ for all $n \leq X$ with $O\left(X^{1-2\theta} \log^2 X\right)$ exceptions. It follows that

$$\sum_{n \in \mathcal{G}} \frac{\Phi(n)}{n} \ll X^{\theta}. \tag{34}$$

We also know that

$$M(X) := \sum_{n \in \mathcal{B}} 1 \ll X^{1-2\theta} \log^2 X. \tag{35}$$

Of course for $n \in \mathcal{B}$, we may take $\Phi(n) = c\sqrt{n}\log n$ and this gives

$$\sum_{n\in\mathcal{B}} \frac{\Phi(n)}{n} \ll \sum_{n\in\mathcal{B}} \frac{\log n}{n^{1/2}}. \tag{36}$$

Using (35) and applying partial summation, one obtains

$$\sum_{n\in\mathcal{B}} \frac{\log n}{n^{1/2}} \ll X^{\frac{1}{2}-2\theta}\log^3 X + \int_2^X t^{-\frac{1}{2}-2\theta}\log^3 t\, dt. \tag{37}$$

Clearly, we have

$$\int_2^X t^{-\frac{1}{2}-2\theta}\log^3 t\, dt \ll X^{\frac{1}{2}-2\theta}\log^3 X. \tag{38}$$

Assembling (36)–(38), one gets the estimate

$$\sum_{n\in\mathcal{B}} \frac{\Phi(n)}{n} \ll X^{\frac{1}{2}-2\theta}\log^3 X. \tag{39}$$

To optimize the exponent, let us take $\theta = 1/6$ so that from (27), (34) and (39),

$$\sum_{n\in\mathcal{G}} \frac{\Phi(n)}{n} + \sum_{n\in\mathcal{B}} \frac{\Phi(n)}{n} \ll X^{1/6}\log^3 X \ll x^{1/9}(\log x)^{28/9} \tag{40}$$

follows. Lastly, feeding (40) into (31), we complete the proof of the second part of Theorem 1 assuming the Riemann hypothesis.

Let us now assume Cramér's conjecture so that $p_{n+1} - p_n = O(\log^2 p_n)$ holds for all consecutive prime numbers p_n and p_{n+1}. This shows that it is possible to take $\Phi(n) = c\log^2 n$ for some constant $c > 0$. Similarly as above, we get from (3) that if

$$r = O\left(\frac{n^2}{\log^2 n}\right), \tag{41}$$

then $h_n^{(r)}$ is not an integer. As $r \le x$, from (41),

$$n \le X := c\sqrt{x}\log x \tag{42}$$

follows for some constant $c > 0$ as otherwise $h_n^{(r)}$ is not an integer. Using (42), we have

$$S(x) = x^2 + O\left(x\sum_{n\le X}\frac{\Phi(n)}{n}\right). \tag{43}$$

Just noting that

$$\sum_{n\le X}\frac{\Phi(n)}{n} \ll \sum_{n\le X}\frac{\log^2 n}{n} \ll \log^3 X \ll \log^3 x \tag{44}$$

and combining (43) and (44), one completes the proof of Theorem 1. $\square$

Proof of Theorem 2. Let m be the highest power of a prime number p such that p^m divides at least one of $r, r+1, \dots, r+n-1$. Assume that u satisfies $p^u \le n < p^{u+1}$. Since

$$\left[\frac{r+n-1}{p^u}\right] - \left[\frac{r-1}{p^u}\right] \ge \left[\frac{n}{p^u}\right] \ge 1,$$

we see that at least one of $r, r+1, \dots, r+n-1$ is divisible by p^u. This forces $u \le m$. From (6), $h_n^{(r)}$ can be written in the form

$$h_n^{(r)} = \sum_{j=r}^{r+n-1} x_j, \tag{45}$$

where each

$$x_j = \frac{r(r+1)\dots(r+n-1)}{jn!}$$

is a rational number. We define the exact power of p dividing x_j as the exact power of p dividing $r(r+1)\dots(r+n-1)$ minus the exact power of p dividing $jn!$. Let $\nu(x_j)$ be the exact power of p dividing x_j. As p^m divides exactly one of $r, r+1, \dots, r+n-1$ by our assumption, we observe that there is a unique j_0 such that

$$\nu(x_{j_0}) < \nu(x_j)$$

holds for all $j \neq j_0$. Assume for a moment that $\nu(x_{j_0}) < 0$. Then using (45), $h_n^{(r)}$ can be written in the form

$$h_n^{(r)} = \frac{a_1}{p^{s_1}b_1} + \frac{a_2}{p^{s_2}b_2} + \dots + \frac{a_n}{p^{s_n}b_n},$$

where a_j, b_j are not divisible by p, $s_1 \ge 1$ and $\max(s_2, \dots, s_n) < s_1$ (indeed we have $s_1 = -\nu(x_{j_0})$). If $h_n^{(r)}$ happens to be an integer, then we see from the above formula that

$$p^{s_1}\left(\prod_{j=1}^{n} b_j\right) h_n^{(r)} = a_1\left(\prod_{j=2}^{n} b_j\right) + N, \tag{46}$$

where N is divisible by p. Clearly, (46) leads to a contradiction as

$$a_1\left(\prod_{j=2}^{n} b_j\right)$$

is not divisible by p. Therefore,

$$\nu(x_{j_0}) < 0 \tag{47}$$

is a sufficient condition for the nonintegerness of $h_n^{(r)}$. In order to focus on the condition (47), note that the exact power of p dividing $n!$ is

$$\sum_{k=1}^{u}\left[\frac{n}{p^k}\right].$$

The exact power of p dividing $r(r+1)\dots(r+n-1)$ is

$$\sum_{k=1}^{m}\left(\left[\frac{r+n-1}{p^k}\right]-\left[\frac{r-1}{p^k}\right]\right). \tag{48}$$

Observe that if $u<k\le m$, then exactly one of $r, r+1, \dots, r+n-1$ is divisible by p^k and

$$\left[\frac{r+n-1}{p^k}\right]-\left[\frac{r-1}{p^k}\right]=1$$

holds for such k. Taking this into account, (48) gives that the exact power of p dividing $r(r+1)\dots(r+n-1)$ is

$$m-u+\sum_{k=1}^{u}\left(\left[\frac{r+n-1}{p^k}\right]-\left[\frac{r-1}{p^k}\right]\right). \tag{49}$$

Using (49), one may deduce that

$$\nu(x_{j_0})=-u+\sum_{k=1}^{u}\left(\left[\frac{r+n-1}{p^k}\right]-\left[\frac{r-1}{p^k}\right]-\left[\frac{n}{p^k}\right]\right). \tag{50}$$

Clearly,

$$\left[\frac{r+n-1}{p^k}\right]-\left[\frac{r-1}{p^k}\right]-\left[\frac{n}{p^k}\right]=0 \text{ or } 1$$

for all k. Since by our assumption,

$$\left\{\frac{r-1}{p^k}\right\}+\left\{\frac{n}{p^k}\right\}<1$$

holds for some $k\le u$, we see that

$$\left[\frac{r+n-1}{p^k}\right]-\left[\frac{r-1}{p^k}\right]-\left[\frac{n}{p^k}\right]=0$$

for at least one value of $k\le u$. Therefore (50) implies (47). In particular, when $p=2$, the divisibility of exactly one of $r, r+1, \dots, r+n-1$ by 2^m (which is easy to show) comes for free from a simple but useful observation of Kürschák [21]. This completes the proof of Theorem 2. $\square$

Proof of Corollary 1. Assume that $n\equiv a \pmod{2^k}$, $a\in\{0,1,\dots,2^k-1\}$ and $2^k\le n$. It follows that

$$\left\{\frac{n}{2^k}\right\}=\frac{a}{2^k}. \tag{51}$$

Similarly, if $r\equiv 1,2,\dots 2^k-a \pmod{2^k}$, then

$$\left\{\frac{r-1}{2^k}\right\}\le\frac{2^k-a-1}{2^k}. \tag{52}$$

Combining (51) and (52), one obtains that

$$\left\{\frac{r-1}{2^k}\right\}+\left\{\frac{n}{2^k}\right\}\le\frac{2^k-1}{2^k}. \tag{53}$$

As a result of (53), one verifies condition (2) of Theorem 2 and the consequence $h_n^{(r)}\notin\mathbb{Z}$. Note that for a fixed $n\equiv a \pmod{2^k}$, r runs over 2^k-a distinct residue classes modulo 2^k so that the set of r with $h_n^{(r)}\notin\mathbb{Z}$ contains a set of density $1-a/2^k$. Next assume that $r\equiv a \pmod{2^k}$, $a\in\{1,2,\dots,2^k\}$ and $2^k\le n$. If $n\equiv 0,1,\dots,2^k-a \pmod{2^k}$, then

$$\left\{\frac{n}{2^k}\right\}\le\frac{2^k-a}{2^k}. \tag{54}$$

Moreover, we have

$$\left\{\frac{r-1}{2^k}\right\}=\frac{a-1}{2^k}. \tag{55}$$

Thus, (53) holds again by (54) and (55), and $h_n^{(r)}\notin\mathbb{Z}$ follows from (2) of Theorem 2. This completes the proof. □

Proof of Theorem 3. Let $p:=p^{(n)}$ be the largest prime number $\le n$. Again we know from (6) that if p divides exactly one of $r,r+1,\dots,r+n-1$, then $h_n^{(r)}$ is not an integer. Note that exactly one of the p consecutive integers $r,r+1,\dots,r+p-1$ is divisible by p, say $r+p-1-k$ is divisible by p with $0\le k\le p-1$. Therefore, if the next integer divisible by p, namely $r+2p-1-k$ exceeds $r+n-1$, then $h_n^{(r)}$ is not an integer. This forces

$$0\le k\le\min\{p-1,2p-n-1\}=2p-n-1. \tag{56}$$

Consequently, from (56), we deduce that as r runs over $2p-n$ distinct residue classes modulo p, then $h_n^{(r)}$ is not an integer. Hence the set of r with $h_n^{(r)}\notin\mathbb{Z}$ contains a set of density

$$2-\frac{n}{p}.$$

As it was argued in [15],

$$p\ge n-n^{0.525}$$

for all large enough n. It follows that the above density is at least

$$2-\frac{n}{n-n^{0.525}}=1-\frac{n^{0.525}}{n-n^{0.525}}$$

for all large enough n. The remaining assertions follow similarly as

$$p\ge n-c\sqrt{n}\log n$$

for some $c > 0$ and all $n \geq 2$ under the Riemann hypothesis, and

$$p \geq n - c\log^2 n$$

for some $c > 0$ and all $n \geq 2$ under Cramér's conjecture. $\square$

Proof of Corollary 2. As $33 \equiv 1 \pmod{2^5}$, we may use Corollary 1 to see that $h_n^{(r)} \notin \mathbb{Z}$ unless $r \equiv 0 \pmod{2^5}$. Thus we may assume that $r = 32k$ for $k \geq 1$. Let m be the highest power of 3 such that 3^m divides at least one of the 33 consecutive integers $r, r+1, \ldots, r+32$. If $m \geq 4$, then clearly 3^m divides exactly one of $r, r+1, \ldots, r+32$. If $m = 3$ (note that in any case, we have $m \geq 3$), then let us assume that r is subject to the condition

$$\left\{\frac{r-1}{3^3}\right\} + \left\{\frac{33}{3^3}\right\} < 1. \tag{57}$$

Using (57), the number of integers among $r, r+1, \ldots, r+32$ that are divisible by 3^3 is

$$\left[\frac{r+32}{3^3}\right] - \left[\frac{r-1}{3^3}\right] = \left[\frac{33}{3^3}\right] = 1. \tag{58}$$

As a result of (58), exactly one of $r, r+1, \ldots, r+32$ is divisible by 3^3. We conclude by Theorem 2 that $h_n^{(r)} \notin \mathbb{Z}$ when $r = 32k$ is subject to the condition

$$\left\{\frac{32k-1}{3^3}\right\} < \frac{21}{27}. \tag{59}$$

We observe that (59) is equivalent to

$$5k \equiv 1, 2, \ldots, 21 \pmod{27}. \tag{60}$$

One obtains from (60) that $h_n^{(r)} \notin \mathbb{Z}$ when $r = 32k$ and k runs over 21 distinct residue classes modulo 27. It follows that the set of integers r such that $h_n^{(r)} \in \mathbb{Z}$ is contained in a set of density

$$\frac{1}{32} \times \frac{2}{9} = \frac{1}{144}.$$

The proof is now complete. $\square$

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