SYMMETRIC (NOT COMPLETE INTERSECTION) NUMERICAL SEMIGROUPS GENERATED BY FIVE ELEMENTS

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Abstract
We consider symmetric (not complete intersection) numerical semigroups $S_5$, generated by five elements, and derive inequalities for degrees of syzygies of $S_5$ and find the lower bound $F_5$ for their Frobenius numbers. We study a special case $W_5$ of such semigroups, which satisfy the Watanabe Lemma, and show that the lower bound $F_{5w}$ for the Frobenius number of the semigroup $W_5$ is stronger than $F_5$.

1. Symmetric Numerical Semigroups Generated by Five Integers
Let a numerical semigroup $S_m = \langle d_1, \ldots, d_m \rangle$ be generated by a set of $m$ integers \{d_1, \ldots, d_m\} such that $\gcd(d_1, \ldots, d_m) = 1$, where $d_1$ and $m$ denote a multiplicity and an embedding dimension (\textit{edim}) of $S_m$. There exist $m-1$ polynomial identities [7] for degrees of syzygies associated with the semigroup ring $k[S_m]$. They are a source of various relations for semigroups of a different nature. In the case of complete intersection (CI) semigroups, such a relation for the degrees $e_j$ of the 1st syzygy was found in [7]. The next nontrivial case exhibits a symmetric (not CI) semigroup generated by $m \geq 4$ integers. In [6] such semigroups with $m = 4$ were studied and the lower bound for the Frobenius number $F(S_4)$ was found. In the present paper we deal with a more difficult case of symmetric semigroups $S_5$.

Consider a symmetric numerical semigroup $S_5$, which is not CI, and generated by five elements $d_i$. Its Hilbert series $H(S_5; t)$ with the 1st Betti number $\beta_1$ reads,

$$H(S_5; t) = \frac{Q_5(t)}{\prod_{i=1}^{5} (1 - t^{d_i})},$$

$$Q_5(t) = 1 - \sum_{j=1}^{\beta_1} t^j + \sum_{j=1}^{\beta_1-1} (t^{y_j} + t^{g-y_j}) - \sum_{j=1}^{\beta_1} t^{g-x_j} + t^g, \quad (1)$$

where $x_j, y_j, g \in \mathbb{Z}_{>0}$, $x_j, y_j < g$, and the Frobenius number is defined as follows: $F(S_5) = g - \sigma_1$, where $\sigma_1 = \sum_{j=1}^{5} d_j$. There are two more constraints, $\beta_1 > 4$

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and \(d_1 > 5\). The inequality \(\beta_1 > 4\) holds since the semigroup \(S_5\) is not CI, and the condition \(d_1 > 5\) is necessary since the numerical semigroup \((m, d_2, \ldots, d_m)\) is never symmetric [5].

Polynomial identities for degrees of syzygies for numerical semigroups were derived in [7], Theorem 1. In the case of the symmetric (not CI) semigroup \(S_5\), they read:

\[
\sum_{j=1}^{\beta_1} x_j^r - \sum_{j=1}^{\beta_1-1} [y_j^r + (g - y_j)^r] + \sum_{j=1}^{\beta_1} (g - x_j)^r - g^r = 0, \quad 1 \leq r \leq 3,
\]

\[
\sum_{j=1}^{\beta_1} x_j^4 - \sum_{j=1}^{\beta_1-1} [y_j^4 + (g - y_j)^4] + \sum_{j=1}^{\beta_1} (g - x_j)^4 - g^4 = -24\pi_5, \quad \pi_5 = \prod_j d_j.
\]

Only two of the four identities in (2,3) are not trivial; these are the identities with \(r = 2, 4\),

\[
\sum_{j=1}^{\beta_1} x_j (g - x_j) = \sum_{j=1}^{\beta_1-1} y_j (g - y_j),
\]

\[
\sum_{j=1}^{\beta_1} x_j^2 (g - x_j)^2 + 12\pi_5 = \sum_{j=1}^{\beta_1-1} y_j^2 (g - y_j)^2.
\]

Denote by \(Z_k\) the k-th power symmetric polynomial \(Z_k(z_1, \ldots, z_n) = \sum_{j=1}^n z_j^k\), \(z_j \geq 0\), and recall the Newton-Maclaurin inequality [8],

\[
Z_2^2 \leq nZ_2.
\]

Applying (6) to the right-hand side of equality (5) and substituting into the resulting inequality an equality (4), we obtain

\[
\left( \sum_{j=1}^{\beta_1} x_j (g - x_j) \right)^2 \leq (\beta_1 - 1) \left[ \sum_{j=1}^{\beta_1} x_j^2 (g - x_j)^2 + 12\pi_5 \right].
\]

On the other hand, according to (6), another inequality holds:

\[
\left( \sum_{j=1}^{\beta_1} x_j (g - x_j) \right)^2 \leq \beta_1 \sum_{j=1}^{\beta_1} x_j^2 (g - x_j)^2.
\]

**Proposition 1.** Let a symmetric (not CI) semigroup \(S_5\) be given with the Hilbert series \(H(S_5; z)\) according to (1). Then the following inequality holds:

\[
g \geq g_5, \quad g_5 = \lambda(\beta_1) \sqrt[5]{\pi_5}, \quad \lambda(\beta_1) = \frac{3}{4} \sqrt[4]{\frac{3\beta_1 - 1}{\beta_1}}.
\]
Proof. Inequality (8) holds always, while inequality (7) is not valid for every set \( \{x_1, \ldots, x_{\beta_1}, g\} \). In order to make both inequalities consistent, we have to find a range for \( g \) where both inequalities (7) and (8) are satisfied. As we will see in (12), this requirement also constrains the admissible values of \( x_j \). To provide both inequalities to be correct, it is enough to require

\[
\frac{1}{\beta_1 - 1} \left( \sum_{j=1}^{\beta_1} x_j (g - x_j) \right)^2 - 12 \pi_5 \geq \frac{1}{\beta_1} \left( \sum_{j=1}^{\beta_1} x_j (g - x_j) \right)^2. \tag{10}
\]

Making use of the notation in (6), rewrite inequality (10) for \( X_1 = \sum_{j=1}^{\beta_1} x_j \) and \( X_2 = \sum_{j=1}^{\beta_1} x_j^2 \),

\[
g X_1 - C \geq X_2, \quad C = \sqrt{12 \beta_1 (\beta_1 - 1) \pi_5},
\]

and combine it with inequality (6), namely, \( X_2 \geq \beta_1^{-1} X_1^2 \). Thus, we obtain

\[
X_1^2 - \beta_1 g X_1 + \beta_1 C \leq 0. \tag{11}
\]

Represent the last inequality (11) in the following way,

\[
\left( \frac{\beta_1 g}{2} \right)^2 \geq \beta_1 C + \left( X_1 - \frac{\beta_1 g}{2} \right)^2,
\]

and obtain immediately the lower bound \( g_5 \) in accordance with (9).

The lower bound for the Frobenius number is given by \( F_5 = g_5 - \sigma_1 \). Inequality (11) constrains the degrees \( x_j \) of the 1st syzygy for the symmetric (not CI) semigroup \( S_5 \),

\[
\frac{\beta_1 g}{2} \left( 1 - \sqrt{1 - \frac{g_5^2}{g^2}} \right) \leq X_1 \leq \frac{\beta_1 g}{2} \left( 1 + \sqrt{1 - \frac{g_5^2}{g^2}} \right). \tag{12}
\]

Below we present twelve symmetric (not CI) semigroups \( S_5 \) with different Betti’s numbers \( \beta_1 = 7, 8, 9, 13 \) and give a comparative Table 1 for their largest degree \( g \) of syzygies and its lower bound \( g_5 \):

\begin{align*}
\beta_1 &= 7, \quad A_1^7 = (6, 10, 14, 15, 19), \quad A_2^7 = (6, 10, 14, 17, 21), \quad A_3^7 = (9, 10, 11, 13, 17), \\
\beta_1 &= 8, \quad A_1^8 = (6, 10, 14, 19, 23), \quad A_2^8 = (8, 10, 13, 14, 19), \quad A_3^8 = (8, 9, 12, 13, 19), \\
\beta_1 &= 9, \quad A_1^9 = (7, 12, 13, 18, 23), \quad A_2^9 = (9, 12, 13, 14, 19), \quad A_3^9 = (8, 11, 12, 15, 25), \\
\beta_1 &= 13, \quad A_1^{13} = (19, 23, 29, 31, 37), \quad A_2^{13} = (19, 27, 28, 31, 32), \quad A_3^{13} = (23, 28, 32, 45, 54).
\end{align*}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
$S_5$ & $A_1$ & $A_2$ & $A_3$ & $A_4$ & $A_5$ & $A_6$ & $A_7$ & $A_8$ & $A_9$ & $A_{10}$ & $A_{11}$ \\
\hline
\hline
$g$ & 87 & 93 & 85 & 99 & 89 & 84 & 102 & 96 & 100 & 240 & 236 & 331 \\
$g_5$ & 79.2 & 83.8 & 77.5 & 88.6 & 82.6 & 77.4 & 93.7 & 89.4 & 90.7 & 225.3 & 224.2 & 306.9 \\
\hline
\end{tabular}

Table 1. The largest degree $g$ of syzygies and its lower bound $g_5$ for symmetric (not CI) semigroups $S_5$ with different Betti’s numbers $\beta_1 = 7, 8, 9, 13$.

The semigroups $A_{13}^i$, $i = 1, 2, 3$, were studied by H. Bresinsky [3]; the other semigroups were generated numerically with the help of the package “NumericalSgps” [4] by M. Delgado.

Comparing $F_5$ with the two known lower bounds of the Frobenius numbers $F_{CI5}$ and $F_{NS5}$ for the symmetric CI [7] and nonsymmetric [10] semigroups generated by five elements, respectively, we have

$$F_{CI5} = 4\sqrt{5} - \sigma_1, \quad F_{NS5} = \sqrt{24\pi_5} - \sigma_1, \quad F_{NS5} < F_5 < F_{CI5}. $$

2. Symmetric Numerical (Not CI) Semigroups $S_5$ With $W$ Property

Watanabe [11] gave a construction of the numerical semigroup $S_m$ generated by $m$ elements starting with a semigroup $S_{m-1}$ generated by $m-1$ elements and proved the following lemma.

**Lemma 1.** ([11]). Let a numerical semigroup $S_{m-1} = \langle \delta_1, \ldots, \delta_{m-1} \rangle$ be given and $a \in \mathbb{Z}_{>0}$, $a > 1$, $d_m > m$, such that $\gcd(a, d_m) = 1$, $d_m \in S_{m-1}$. Consider a numerical semigroup $S_m = \langle a\delta_1, \ldots, a\delta_{m-1}, d_m \rangle$ and denote it by $S_m = \langle aS_{m-1}, d_m \rangle$. Then $S_m$ is symmetric if and only if $S_{m-1}$ is symmetric, and $S_m$ is symmetric CI if and only if $S_{m-1}$ is symmetric CI.

For our purpose the following Corollary of Lemma 1 is important.

**Corollary 1.** Let a numerical semigroup $S_{m-1} = \langle \delta_1, \ldots, \delta_{m-1} \rangle$ be given and $a \in \mathbb{Z}_{>0}$, $a > 1$, $d_m > m$, such that $\gcd(a, d_m) = 1$, $d_m \in S_{m-1}$. Consider a semigroup $S_m = \langle aS_{m-1}, d_m \rangle$. Then $S_m$ is symmetric (not CI) if and only if $S_{m-1}$ is symmetric (not CI).

To utilize this construction we define the following property.

**Definition 1.** A symmetric (not CI) semigroup $S_m$ has the property $W$ if there exists another symmetric (not CI) semigroup $S_{m-1}$ giving rise to $S_m$ by the construction, described in Corollary 1.

Note that in Definition 1 the semigroup $S_{m-1}$ does not necessarily possess the property $W$. The symbol $W$ stands for Kei-ichi Watanabe. The next Proposition distinguishes the minimal $\text{edim}$ of symmetric CI and symmetric (not CI) numerical semigroups with the property $W$. 
Proposition 2. A minimal edim of symmetric (not CI) semigroup $S_m$ with the property $W$ is $m = 5$.

Proof. All numerical semigroups of $edim = 2$ are symmetric CI, and all symmetric numerical semigroups of $edim = 3$ are CI [9]. Therefore, according to Definition 1, a minimal $edim$ of symmetric CI semigroups with the property $W$ is $edim = 3$. A minimal $edim$ of numerical semigroups, which are symmetric (not CI), is $edim = 4$. Therefore, according to Definition 1 and Corollary 1, a minimal $edim$ of symmetric (not CI) semigroups with the property $W$ is $edim = 5$.  

According to Proposition 2, let us choose a symmetric (not CI) semigroup of $edim = 5$ with the property $W$ and denote it by $W_5$ in order to distinguish it from other symmetric (not CI) semigroups $S_5$ (irrespective of the $W$ property). Then the following containment holds: $\{W_5\} \subset \{S_5\}$. A minimal free resolution associated with semigroups $W_5$ was described recently ([1], section 4), where the degrees of all syzygies were also derived ([1], Corollary 12), e.g., its 1st Betti number is $\beta_1 = 6$.

Lemma 2. Let two symmetric (not CI) semigroups $W_5 = \langle aS_4, d_5 \rangle$ and $S_4 = \langle \delta_1, \delta_2, \delta_3, \delta_4 \rangle$ be given and $\gcd(a, d_5) = 1$, $d_5 \in S_4$. Let the lower bound $F_{5w}$ of the Frobenius number $F(W_5)$ of the semigroup $W_5$ be represented as, $F_{5w} = g_{5w} - (a \sum_{j=1}^{4} \delta_j + d_5)$. Then

$$g_{5w} = a \left( \sqrt[4]{25\pi_4(\delta)} + d_5 \right), \quad \pi_4(\delta) = \prod_{j=1}^{4} \delta_j. \quad (13)$$

Proof. Consider a symmetric (not CI) numerical semigroup $S_4$ generated by four integers (without the $W$ property), and apply the recent result [6] on the lower bound $F_4$ of its Frobenius number, $F(S_4)$, to obtain

$$F(S_4) \geq F_4, \quad F_4 = h_4 - \sum_{j=1}^{4} \delta_j, \quad h_4 = \sqrt[4]{25\pi_4(\delta)}. \quad (14)$$

The following relationship between the Frobenius numbers $F(W_5)$ and $F(S_4)$ was derived in [2]:

$$F(W_5) = aF(S_4) + (a - 1)d_5. \quad (15)$$

Substituting two representations $F(S_4) = h - \sum_{j=1}^{4} \delta_j$ and $F(W_5) = g - a \sum_{j=1}^{4} \delta_j - d_5$ into equality (15), we obtain

$$g - a \sum_{j=1}^{4} \delta_j - d_5 = ah - a \sum_{j=1}^{4} \delta_j + (a - 1)d_5 \quad \rightarrow \quad g = a(h + d_5). \quad (16)$$

Comparing the last equality in (16) with the lower bound of $h$ in (14), we arrive at (13).  

\[ \square \]
Proposition 3. Let two symmetric (not CI) semigroups $W_5 = \langle aS_4, d_5 \rangle$ and $S_4 = \langle \delta_1, \delta_2, \delta_3, \delta_4 \rangle$, where $\gcd(a, d_5) = 1$, $d_5 \in S_4$ be given with the Hilbert series $H(W_5; z)$ according to (1). Then
\[ g_{5w} > g_5. \] (17)

Proof. Keeping in mind that $b_1 = 6$ (see [1], Corollary 12), we determine $g_5$ for the semigroup $W_5$ according to (9),
\[ g_5 = a\lambda(6)\sqrt[3]{\pi_4(\delta)d_5}, \quad \lambda(6) = 4\sqrt[3]{5/8} \simeq 3.556, \]
and calculate the following ratio:
\[ \rho = \frac{g_{5w}}{g_5} = \frac{1}{\lambda(6)} \frac{\sqrt[3]{25\pi_4(\delta)}}{\sqrt[3]{\pi_4(\delta)d_5}} = \frac{1}{\lambda(6)} \left( \sqrt[3]{25\eta} + \frac{1}{\eta} \right), \quad \eta = \sqrt[3]{\pi_4(\delta)d_5}. \]
The function $\rho(\eta)$ is positive for $\eta > 0$ and has an absolute minimum
\[ \rho(\eta_m) \simeq 1.1033, \quad \eta_m \simeq 1.01943, \quad \pi_4(\delta) \simeq 1.08d_5^3. \]
In other words, we arrive at $\rho(\eta_m) > 1$, which proves the Proposition. \hfill \Box

We present ten symmetric (not CI) semigroups $A_j^6$ with the $W$ property and the Betti number $b_1 = 6$. All semigroups $A_j^6$ are built according to Lemma 1 and based on symmetric (not CI) semigroups $S_4$, generated by four integers [7]. We give a comparative Table 2 for their $g$, lower bounds $g_{5w}$ and $g_5$, and the parameter $\eta$.

\[ A_j^6 = \langle 14, 15, 16, 18, 26 \rangle, \quad A_2^6 = \langle 20, 21, 24, 27, 39 \rangle, \quad A_3^6 = \langle 10, 11, 12, 14, 16 \rangle, \]
\[ A_4^6 = \langle 14, 16, 17, 18, 26 \rangle, \quad A_5^6 = \langle 16, 21, 26, 30, 34 \rangle, \quad A_6^6 = \langle 23, 24, 39, 45, 51 \rangle, \]
\[ A_j^6 = \langle 35, 40, 41, 45, 65 \rangle, \quad A_8^6 = \langle 302, 305, 308, 314, 316 \rangle, \]
\[ A_9^6 = \langle 302, 308, 314, 315, 316 \rangle, \quad A_{10}^6 = \langle 453, 462, 469, 471, 474 \rangle. \]

<table>
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<th>$W_5$</th>
<th>$A_1^6$</th>
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<th>$A_3^6$</th>
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Table 2. The largest degree $g$ of syzygies, its lower bounds $g_5$ and $g_{5w}$ and the parameter $\eta$ for symmetric (not CI) semigroups $W_5$ with the Betti number $b_1 = 6$.

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