#A44 INTEGERS 18 (2018)

SYMMETRIC (NOT COMPLETE INTERSECTION) NUMERICAL SEMIGROUPS GENERATED BY FIVE ELEMENTS

Leonid G. Fel[1]
Department of Civil Engineering, Technion, Haifa, Israel
lfel@technion.ac.il

Received: 7/12/17, Revised: 1/30/18, Accepted: 5/5/18, Published: 5/18/18

Abstract

We consider symmetric (not complete intersection) numerical semigroups S_5, generated by five elements, and derive inequalities for degrees of syzygies of S_5 and find the lower bound F_5 for their Frobenius numbers. We study a special case W_5 of such semigroups, which satisfy the Watanabe Lemma, and show that the lower bound F_{5w} for the Frobenius number of the semigroup W_5 is stronger than F_5.

1. Symmetric Numerical Semigroups Generated by Five Integers

Let a numerical semigroup $\mathsf{S}_m = \langle d_1, \ldots, d_m \rangle$ be generated by a set of m integers $\{d_1, \ldots, d_m\}$ such that $\gcd(d_1, \ldots, d_m) = 1$, where d_1 and m denote a multiplicity and an embedding dimension (*edim*) of S_m. There exist $m-1$ polynomial identities [7] for degrees of syzygies associated with the semigroup ring $k[\mathsf{S}_m]$. They are a source of various relations for semigroups of a different nature. In the case of complete intersection (CI) semigroups, such a relation for the degrees e_j of the 1st syzygy was found in [7]. The next nontrivial case exhibits a symmetric (not CI) semigroup generated by $m \geq 4$ integers. In [6] such semigroups with $m = 4$ were studied and the lower bound for the Frobenius number $F(\mathsf{S}_4)$ was found. In the present paper we deal with a more difficult case of symmetric semigroups S_5.

Consider a symmetric numerical semigroup S_5, which is not CI, and generated by five elements d_i. Its Hilbert series $H(\mathsf{S}_5; t)$ with the 1st Betti number β_1 reads,

$$H(\mathsf{S}_5; t) = \frac{Q_5(t)}{\prod_{i=1}^{5}(1 - t^{d_i})},$$
$$Q_5(t) = 1 - \sum_{j=1}^{\beta_1} t^{x_j} + \sum_{j=1}^{\beta_1 - 1}\left(t^{y_j} + t^{g - y_j}\right) - \sum_{j=1}^{\beta_1} t^{g - x_j} + t^g, \tag{1}$$

where $x_j, y_j, g \in \mathbb{Z}_{>0}$, $x_j, y_j < g$, and the Frobenius number is defined as follows: $F(\mathsf{S}_5) = g - \sigma_1$, where $\sigma_1 = \sum_{j=1}^{5} d_j$. There are two more constraints, $\beta_1 > 4$

[1]The research was partly supported by the Kamea Fellowship.

and $d_1 > 5$. The inequality $\beta_1 > 4$ holds since the semigroup S_5 is not CI, and the condition $d_1 > 5$ is necessary since the numerical semigroup $\langle m, d_2, \ldots, d_m \rangle$ is never symmetric [5].

Polynomial identities for degrees of syzygies for numerical semigroups were derived in [7], Theorem 1. In the case of the symmetric (not CI) semigroup S_5, they read:

$$\sum_{j=1}^{\beta_1} x_j^r - \sum_{j=1}^{\beta_1-1} \left[y_j^r + (g-y_j)^r\right] + \sum_{j=1}^{\beta_1} (g-x_j)^r - g^r = 0, \quad 1 \le r \le 3, \tag{2}$$

$$\sum_{j=1}^{\beta_1} x_j^4 - \sum_{j=1}^{\beta_1-1} \left[y_j^4 + (g-y_j)^4\right] + \sum_{j=1}^{\beta_1} (g-x_j)^4 - g^4 = -24\pi_5, \quad \pi_5 = \prod_j^5 d_j. \tag{3}$$

Only two of the four identities in (2,3) are not trivial; these are the identities with $r = 2, 4$,

$$\sum_{j=1}^{\beta_1} x_j(g-x_j) = \sum_{j=1}^{\beta_1-1} y_j(g-y_j), \tag{4}$$

$$\sum_{j=1}^{\beta_1} x_j^2(g-x_j)^2 + 12\pi_5 = \sum_{j=1}^{\beta_1-1} y_j^2(g-y_j)^2. \tag{5}$$

Denote by Z_k the k-th power symmetric polynomial $Z_k(z_1, \ldots, z_n) = \sum_{j=1}^n z_j^k$, $z_j \ge 0$, and recall the Newton-Maclaurin inequality [8],

$$Z_1^2 \le n Z_2. \tag{6}$$

Applying (6) to the right-hand side of equality (5) and substituting into the resulting inequality an equality (4), we obtain

$$\left(\sum_{j=1}^{\beta_1} x_j(g-x_j)\right)^2 \le (\beta_1 - 1)\left[\sum_{j=1}^{\beta_1} x_j^2(g-x_j)^2 + 12\pi_5\right]. \tag{7}$$

On the other hand, according to (6), another inequality holds:

$$\left(\sum_{j=1}^{\beta_1} x_j(g-x_j)\right)^2 \le \beta_1 \sum_{j=1}^{\beta_1} x_j^2(g-x_j)^2. \tag{8}$$

Proposition 1. *Let a symmetric (not CI) semigroup S_5 be given with the Hilbert series $H(\mathsf{S}_5; z)$ according to (1). Then the following inequality holds:*

$$g \ge g_5, \qquad g_5 = \lambda(\beta_1)\sqrt[4]{\pi_5}, \qquad \lambda(\beta_1) = 4\sqrt[4]{\frac{3}{4}\frac{\beta_1 - 1}{\beta_1}}. \tag{9}$$

Proof. Inequality (8) holds always, while inequality (7) is not valid for every set $\{x_1, \ldots, x_{\beta_1}, g\}$. In order to make both inequalities consistent, we have to find a range for g where both inequalities (7) and (8) are satisfied. As we will see in (12), this requirement also constrains the admissible values of x_j. To provide both inequalities to be correct, it is enough to require

$$\frac{1}{\beta_1 - 1}\left(\sum_{j=1}^{\beta_1} x_j(g - x_j)\right)^2 - 12\pi_5 \geq \frac{1}{\beta_1}\left(\sum_{j=1}^{\beta_1} x_j(g - x_j)\right)^2. \qquad (10)$$

Making use of the notation in (6), rewrite inequality (10) for $X_1 = \sum_{j=1}^{\beta_1} x_j$ and $X_2 = \sum_{j=1}^{\beta_1} x_j^2$,

$$gX_1 - C \geq X_2, \qquad C = \sqrt{12\beta_1(\beta_1 - 1)\pi_5},$$

and combine it with inequality (6), namely, $X_2 \geq \beta_1^{-1} X_1^2$. Thus, we obtain

$$X_1^2 - \beta_1 g X_1 + \beta_1 C \leq 0. \qquad (11)$$

Represent the last inequality (11) in the following way,

$$\left(\frac{\beta_1 g}{2}\right)^2 \geq \beta_1 C + \left(X_1 - \frac{\beta_1 g}{2}\right)^2,$$

and obtain immediately the lower bound g_5 in accordance with (9). □

The lower bound for the Frobenius number is given by $F_5 = g_5 - \sigma_1$. Inequality (11) constrains the degrees x_j of the 1st syzygy for the symmetric (not CI) semigroup S_5,

$$\frac{\beta_1 g}{2}\left(1 - \sqrt{1 - \frac{g_5^2}{g^2}}\right) \leq X_1 \leq \frac{\beta_1 g}{2}\left(1 + \sqrt{1 - \frac{g_5^2}{g^2}}\right). \qquad (12)$$

Below we present twelve symmetric (not CI) semigroups S_5 with different Betti's numbers $\beta_1 = 7, 8, 9, 13$ and give a comparative Table 1 for their largest degree g of syzygies and its lower bound g_5:

$$\begin{array}{llll}
\beta_1 = 7, & A_1^7 = \langle 6, 10, 14, 15, 19\rangle, & A_2^7 = \langle 6, 10, 14, 17, 21\rangle, & A_3^7 = \langle 9, 10, 11, 13, 17\rangle, \\
\beta_1 = 8, & A_1^8 = \langle 6, 10, 14, 19, 23\rangle, & A_2^8 = \langle 8, 10, 13, 14, 19\rangle, & A_3^8 = \langle 8, 9, 12, 13, 19\rangle, \\
\beta_1 = 9, & A_1^9 = \langle 7, 12, 13, 18, 23\rangle, & A_2^9 = \langle 9, 12, 13, 14, 19\rangle, & A_3^9 = \langle 8, 11, 12, 15, 25\rangle, \\
\beta_1 = 13, & A_1^{13} = \langle 19, 23, 29, 31, 37\rangle, & A_2^{13} = \langle 19, 27, 28, 31, 32\rangle, & A_3^{13} = \langle 23, 28, 32, 45, 54\rangle.
\end{array}$$

S_5	A_1^7	A_2^7	A_3^7	A_1^8	A_2^8	A_3^8	A_1^9	A_2^9	A_3^9	A_1^{13}	A_2^{13}	A_3^{13}
g	87	93	85	99	89	84	102	96	100	240	236	331
g_5	79.2	83.8	77.5	88.6	82.6	77.4	93.7	89.4	90.7	225.3	224.2	306.9

Table 1. The largest degree g of syzygies and its lower bound g_5 for symmetric (not CI) semigroups S_5 with different Betti's numbers $\beta_1 = 7, 8, 9, 13$.

The semigroups A_i^{13}, $i = 1, 2, 3$, were studied by H. Bresinsky [3]; the other semigroups were generated numerically with the help of the package "NumericalSgps" [4] by M. Delgado.

Comparing F_5 with the two known lower bounds of the Frobenius numbers F_{CI_5} and F_{NS_5} for the symmetric CI [7] and nonsymmetric [10] semigroups generated by five elements, respectively, we have

$$F_{CI_5} = 4\sqrt[4]{\pi_5} - \sigma_1, \qquad F_{NS_5} = \sqrt[4]{24\pi_5} - \sigma_1, \qquad F_{NS_5} < F_5 < F_{CI_5}.$$

2. Symmetric Numerical (Not CI) Semigroups S_5 With W Property

Watanabe [11] gave a construction of the numerical semigroup S_m generated by m elements starting with a semigroup S_{m-1} generated by $m-1$ elements and proved the following lemma.

Lemma 1. ([11]). *Let a numerical semigroup* $S_{m-1} = \langle \delta_1, \ldots, \delta_{m-1} \rangle$ *be given and* $a \in \mathbb{Z}_{>0}$, $a > 1$, $d_m > m$, *such that* $\gcd(a, d_m) = 1$, $d_m \in S_{m-1}$. *Consider a numerical semigroup* $S_m = \langle a\delta_1, \ldots, a\delta_{m-1}, d_m \rangle$ *and denote it by* $S_m = \langle aS_{m-1}, d_m \rangle$. *Then* S_m *is symmetric if and only if* S_{m-1} *is symmetric, and* S_m *is symmetric CI if and only if* S_{m-1} *is symmetric CI.*

For our purpose the following Corollary of Lemma 1 is important.

Corollary 1. *Let a numerical semigroup* $S_{m-1} = \langle \delta_1, \ldots, \delta_{m-1} \rangle$ *be given and* $a \in \mathbb{Z}_{>0}$, $a > 1$, $d_m > m$, *such that* $\gcd(a, d_m) = 1$, $d_m \in S_{m-1}$. *Consider a semigroup* $S_m = \langle aS_{m-1}, d_m \rangle$. *Then* S_m *is symmetric (not CI) if and only if* S_{m-1} *is symmetric (not CI).*

To utilize this construction we define the following property.

Definition 1. A symmetric (not CI) semigroup S_m has *the property* W if there exists another symmetric (not CI) semigroup S_{m-1} giving rise to S_m by the construction, described in Corollary 1.

Note that in Definition 1 the semigroup S_{m-1} does not necessarily possess the property W. The symbol W stands for Kei-ichi Watanabe. The next Proposition distinguishes the minimal *edim* of symmetric CI and symmetric (not CI) numerical semigroups with the property W.

Proposition 2. *A minimal* edim *of symmetric (not CI) semigroup S_m with the property* W *is $m = 5$.*

Proof. All numerical semigroups of $edim = 2$ are symmetric CI, and all symmetric numerical semigroups of $edim = 3$ are CI [9]. Therefore, according to Definition 1, a minimal *edim* of symmetric CI semigroups with the property W is $edim = 3$. A minimal *edim* of numerical semigroups, which are symmetric (not CI), is $edim = 4$. Therefore, according to Definition 1 and Corollary 1, a minimal *edim* of symmetric (not CI) semigroups with the property W is $edim = 5$. □

According to Proposition 2, let us choose a symmetric (not CI) semigroup of $edim = 5$ with the property W and denote it by W_5 in order to distinguish it from other symmetric (not CI) semigroups S_5 (irrespective of the W property). Then the following containment holds: $\{\mathsf{W}_5\} \subset \{\mathsf{S}_5\}$. A minimal free resolution associated with semigroups W_5 was described recently ([1], section 4), where the degrees of all syzygies were also derived ([1], Corollary 12), e.g., its 1st Betti number is $\beta_1 = 6$.

Lemma 2. *Let two symmetric (not CI) semigroups $\mathsf{W}_5 = \langle a\mathsf{S}_4, d_5\rangle$ and $\mathsf{S}_4 = \langle \delta_1, \delta_2, \delta_3, \delta_4\rangle$ be given and $\gcd(a, d_5) = 1$, $d_5 \in \mathsf{S}_4$. Let the lower bound F_{5w} of the Frobenius number $F(\mathsf{W}_5)$ of the semigroup W_5 be represented as, $F_{5w} = g_{5w} - \left(a\sum_{j=1}^4 \delta_j + d_5\right)$. Then*

$$g_{5w} = a\left(\sqrt[3]{25\pi_4(\delta)} + d_5\right), \qquad \pi_4(\delta) = \prod_{j=1}^4 \delta_j. \tag{13}$$

Proof. Consider a symmetric (not CI) numerical semigroup S_4 generated by four integers (without the W property), and apply the recent result [6] on the lower bound F_4 of its Frobenius number, $F(\mathsf{S}_4)$, to obtain

$$F(\mathsf{S}_4) \geq F_4, \qquad F_4 = h_4 - \sum_{j=1}^4 \delta_j, \qquad h_4 = \sqrt[3]{25\pi_4(\delta)}. \tag{14}$$

The following relationship between the Frobenius numbers $F(\mathsf{W}_5)$ and $F(\mathsf{S}_4)$ was derived in [2]:

$$F(\mathsf{W}_5) = aF(\mathsf{S}_4) + (a-1)d_5. \tag{15}$$

Substituting two representations $F(\mathsf{S}_4) = h - \sum_{j=1}^4 \delta_j$ and $F(\mathsf{W}_5) = g - a\sum_{j=1}^4 \delta_j - d_5$ into equality (15), we obtain

$$g - a\sum_{j=1}^4 \delta_j - d_5 = ah - a\sum_{j=1}^4 \delta_j + (a-1)d_5 \qquad \to \qquad g = a(h + d_5). \tag{16}$$

Comparing the last equality in (16) with the lower bound of h in (14), we arrive at (13). □

Proposition 3. *Let two symmetric (not CI) semigroups* $\mathsf{W}_5 = \langle a\mathsf{S}_4, d_5 \rangle$ *and* $\mathsf{S}_4 = \langle \delta_1, \delta_2, \delta_3, \delta_4 \rangle$*, where* $\gcd(a, d_5) = 1$*,* $d_5 \in \mathsf{S}_4$*, be given with the Hilbert series* $H(\mathsf{W}_5; z)$ *according to (1). Then*

$$g_{5w} > g_5. \tag{17}$$

Proof. Keeping in mind that $\beta_1 = 6$ (see [1], Corollary 12), we determine g_5 for the semigroup W_5 according to (9),

$$g_5 = a\lambda(6)\sqrt[4]{\pi_4(\delta)d_5}, \qquad \lambda(6) = 4\sqrt[4]{5/8} \simeq 3.556,$$

and calculate the following ratio:

$$\rho = \frac{g_{5w}}{g_5} = \frac{1}{\lambda(6)} \frac{\sqrt[3]{25\pi_4(\delta)} + d_5}{\sqrt[4]{\pi_4(\delta)d_5}} = \frac{1}{\lambda(6)} \left(\sqrt[3]{25\eta} + \frac{1}{\eta} \right), \qquad \eta = \sqrt[4]{\frac{\pi_4(\delta)}{d_5^3}}.$$

The function $\rho(\eta)$ is positive for $\eta > 0$ and has an absolute minimum

$$\rho(\eta_m) \simeq 1.1033, \qquad \eta_m \simeq 1.01943, \qquad \pi_4(\delta) \simeq 1.08 d_5^3.$$

In other words, we arrive at $\rho(\eta_m) > 1$, which proves the Proposition. □

We present ten symmetric (not CI) semigroups A_j^6 with the W property and the Betti number $\beta_1 = 6$. All semigroups A_j^6 are built according to Lemma 1 and based on symmetric (not CI) semigroups S_4, generated by four integers [7]. We give a comparative Table 2 for their g, lower bounds g_{5w} and g_5, and the parameter η.

$$\begin{aligned}
&A_1^6 = \langle 14, 15, 16, 18, 26 \rangle, \quad A_2^6 = \langle 20, 21, 24, 27, 39 \rangle, \quad A_3^6 = \langle 10, 11, 12, 14, 16 \rangle,\\
&A_4^6 = \langle 14, 16, 17, 18, 26 \rangle, \quad A_5^6 = \langle 16, 21, 26, 30, 34 \rangle, \quad A_6^6 = \langle 23, 24, 39, 45, 51 \rangle,\\
&A_7^6 = \langle 35, 40, 41, 45, 65 \rangle, \quad A_8^6 = \langle 302, 305, 308, 314, 316 \rangle,\\
&A_9^6 = \langle 302, 308, 314, 315, 316 \rangle, \quad A_{10}^6 = \langle 453, 462, 469, 471, 474 \rangle,
\end{aligned}$$

W_5	A_1^6	A_2^6	A_3^6	A_4^6	A_5^6	A_6^6	A_7^6	A_8^6	A_9^6	A_{10}^6
g	142	228	92	146	218	333	485	9120	9140	14172
g_{5w}	139.4	224.1	91.5	143.4	216.4	330.6	478.6	5478.1	5498.1	8709.2
g_5	125.9	203.0	82.9	129.9	194.3	298.2	404.8	4606.8	4644.1	7695.0
η	1.180	0.951	1.060	1.075	1.301	1.215	0.555	2.123	2.073	1.538

Table 2. The largest degree g of syzygies, its lower bounds g_5 and g_{5w} and the parameter η for symmetric (not CI) semigroups W_5 with the Betti number $\beta_1 = 6$.

Acknowledgement. The author is thankful to M. Delgado for providing him with various examples of symmetric semigroups generated by five elements which were found numerically with the help of the package "NumericalSgps" [4]. The author is grateful to the managing editor for valuable suggestions to improve the manuscript.

References

[1] V. Barucci and R. Fröberg, The minimal graded resolution of some Gorenstein rings, http://lanl.arxiv.org/pdf/1211.4747v1

[2] A. Brauer, J.E. Shockley, On a problem of Frobenius, *J. Reine Angew. Math.*, **211** (1962), 215-220.

[3] H. Bresinsky, Monomial Gorenstein ideals, *Manuscripta Math.*, **29** (1979), 159-181.

[4] M. Delgado, P.A. García-Sánchez and J. Morais, GAP Package NumericalSgps, Version 1.1.5, https://gap-packages.github.io/numericalsgps/

[5] L.G. Fel, Duality relation for the Hilbert series of almost symmetric numerical semigroups, *Israel J. Math.*, **185** (2011), 413-444.

[6] L.G. Fel, On Frobenius numbers for symmetric (not complete intersection) semigroups generated by four elements, *Semigroup Forum*, **93** (2016), 423-426.

[7] L.G. Fel, Restricted partition functions and identities for degrees of syzygies in numerical semigroups, *Ramanujan J.*, **43** (2017), 465-491.

[8] G.H. Hardy, J.E. Littlewood and G. Polya, *Inequalities*, Univ. Press, Cambridge, 1959.

[9] J. Herzog, Generators and relations of Abelian semigroups and semigroup rings, *Manuscripta Math.*, **3** (1970), 175-193.

[10] H. G. KillingbergtrØ, Betjening av figur i Frobenius' problem, *Normat (Nordisk Matematisk Tidskrift)*, **2** (2000), 75-82, (In Norwegian)

[11] K. Watanabe, Some examples of 1-dim Gorenstein domains, *Nagoya Math. J.*, **49** (1973), 101-109.