JACOBI-TYPE CONTINUED FRACTIONS AND CONGRUENCES FOR BINOMIAL COEFFICIENTS

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Received: 2/11/17, Revised: 2/20/18, Accepted: 5/12/18, Published: 5/18/18

Abstract
We prove two new forms of Jacobi-type J-fraction expansions generating the binomial coefficients, $\binom{x+n}{n}$ and $\binom{x}{n}$, over all $n \geq 0$. Within the article we establish new forms of integer congruences for these binomial coefficient variations modulo any (prime or composite) $h \geq 2$ and compare our results with known identities for the binomial coefficients modulo primes p and prime powers p^k. We also prove new exact formulas for these binomial coefficient cases from the expansions of the h^{th} convergent functions to the infinite J-fraction series generating these coefficients for all n.

1. Introduction

1.1. Congruences for Binomial Coefficients Modulo Primes and Prime Powers

There are many well-known results providing congruences for the binomial coefficients modulo primes and prime powers. For example, we can state *Lucas's theorem* in the following form for p prime and $n, m \in \mathbb{N}$ where $n = n_0 + n_1 p + \cdots + n_d p^d$ and $m = m_0 + m_1 p + \cdots + m_d p^d$ for $0 \leq n_i, m_i < p$ [3]:

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_d}{m_d} \pmod{p}. \qquad \text{(Lucas' Theorem)}$$

We also have known results providing decompositions, or reductions of order, of the next binomial coefficients modulo the prime powers, p^k [5]:

$$\binom{np^k + n_0}{mp^k + m_0} \equiv \binom{np}{mp}\binom{n_0}{m_0} \pmod{p^k}.$$

Similarly, we can decompose linear and quadratic (and more general) polynomial inputs to the lower binomial coefficient indices as the congruence factors given by

[5, §3.2]:

$$\binom{np}{rp+s} \equiv (r+1)\binom{n}{r+1}\binom{p}{s} \pmod{p^2},$$
$$\binom{np}{mp^2+rp+s} \equiv (m+1)\binom{n}{m+1}\binom{p^2}{rp+s} \pmod{p^3}.$$

Within the context of this article, we are motivated to establish meaningful, non-trivial integer congruences for the two indeterminate binomial coefficient sequence variants, $\binom{x+n}{n}$ and $\binom{x}{n}$, modulo both prime and composite bases $h \geq 2$ by using new expansions of the Jacobi-type continued fractions generating these binomial coefficient variants that we prove in Section 2 and in Section 3. The next subsection establishes related known continued fractions and integer congruence properties for the general forms of Jacobi-type J-fractions of which our new results are special cases.

1.2. Jacobi-Type Continued Fraction Expansions

1.2.1. Continued Fractions Generating the Binomial Coefficients

In this article, we study new properties and congruence relations satisfied by the integer-order binomial coefficients through two specific, and apparently new, *Jacobi-type continued fraction* expansions of a formal power series in z. The expansions of these *J-fractions* are similar in form to a known *Stieltjes-type* continued fraction, or *S-fraction*, expansion given in Wall's book as [10, p. 343]

$$(1+z)^k = \cfrac{1}{1 - \cfrac{kz}{1 + \cfrac{\frac{1\cdot(1+k)}{1\cdot 2}z}{1 + \cfrac{\frac{1\cdot(1-k)}{2\cdot 3}z}{1 + \cfrac{\frac{2(2+k)}{3\cdot 4}z}{1+\cdots.}}}}} \qquad \text{(Binomial Series S-Fraction)}$$

1.2.2. Jacobi-Type J-Fractions for Generalized Factorial Functions

The loosely-termed "*non-exponential*" forms of a generalized class of binomial-coefficient-related *generalized factorial functions*, $p_n(\alpha, R)$, defined as,

$$p_n(\alpha, R) := R(R+\alpha)(R+2\alpha)\cdots(R+(n-1)\alpha),$$

where the special cases of $n! = p_n(-1, n)$, $\binom{x}{n} \equiv (-1)^n(-x)_n/n!$, and $p_n(\alpha, R)/n! \equiv (-\alpha)^n \cdot \binom{-R/\alpha}{n}$ when $(x)_n = x(x+1)\cdots(x+n-1)$ denotes the *Pochhammer symbol*, are studied in [9]. Exponential generating functions for the generalized symbolic product functions, $p_n(\alpha, R)$, with respect to n, are known and given in closed-form by

$$\sum_{n \geq 0} p_n(\alpha, R+1) \frac{z^n}{n!} = (1 - \alpha z)^{-(R+1)/\alpha},$$

where the corresponding generalized forms of the *Stirling numbers of the first kind*, or *α-factorial coefficients*, are defined by the generating functions [8, *cf.* §2-3]

$$\sum_{n \geq 0} [R^m] p_n(\alpha, R+1) \frac{z^n}{n!} = \frac{(-1)^m \alpha^{-m}}{m!} \times (1 - \alpha z)^{\frac{1}{\alpha}} \times \operatorname{Log}(1 - \alpha z)^m, \quad \text{for } m \in \mathbb{Z}^+.$$

The power series expansions of the ordinary generating functions for these factorial functions, which typically converge only for $z = 0$ when the parameter R is a function of n, are defined formally in [9] by infinite J-fractions of the form

$$\operatorname{Conv}_\infty(\alpha, R; z) = \cfrac{1}{1 - R \cdot z - \cfrac{\alpha R \cdot z^2}{1 - (R + 2\alpha) \cdot z - \cfrac{2\alpha(R+\alpha) \cdot z^2}{1 - (R + 4\alpha) \cdot z - \cfrac{3\alpha(R + 2\alpha) \cdot z^2}{\cdots}}}}.$$

These typically divergent ordinary generating functions are usually only expanded as power series in z by "*regularized*" *Borel* sums involving reciprocal powers of z (and αz) such as those given as examples in the introduction to [9]. This reference proves new identities for the generalized factorial functions $p_n(\alpha, R)$ by considering the coefficients of the rational h^{th} convergents to the divergent power series defined only formally by the previous equation which are $2h$–order accurate in approximating these generating functions. We employ a similar approach to enumerating new exact identities and congruence properties of the binomial coefficients in this article based on a formal treatment of the rational convergents to the generating functions defined by two special cases of the infinite Jacobi-type J-fractions defined in the next subsection.

1.2.3. Properties of the Expansions of General J-Fraction Series

More generally, Jacobi-type *J-fractions* correspond to power series defined by infinite continued fraction expansions of the form

$$J^{[\infty]}(z) = \cfrac{1}{1 - c_1 z - \cfrac{\mathrm{ab}_1 z^2}{1 - c_2 z - \cfrac{\mathrm{ab}_2 z^2}{\cdots}}}, \qquad \text{(J-Fraction Expansions)}$$

for some sequences $\{c_i\}_{i \geq 1}$ and $\{\mathrm{ab}_i\}_{i \geq 1}$, and some (typically formal) series variable $z \in \mathbb{C}$ [6, *cf.* §3.10] [10, 1, 2]. The formal series enumerated by special cases of the truncated and infinite continued fraction series of this form include *ordinary* (as opposed to typically closed-form *exponential*) *generating functions* for many one and two-index combinatorial sequences including the generalized factorial functions studied in [1, 2, 4, 9].

The h^{th} convergents, $\text{Conv}_h(z)$, to the J-fraction expansions of the form in the previous equation have several useful characteristic properties:

(A) The h^{th} convergent functions exactly enumerate the coefficients of the infinite continued fraction series as $[z^n]J^{[\infty]}(z) = [z^n]\,\text{Conv}_h(z)$ for all $0 \leq n < 2h$.

(B) The convergent function component numerator and denominator sequences are each easily defined recursively by the same second-order recurrence in h with differing initial conditions. The h^{th} convergent functions are always rational in z which implies an "*eventually periodic*" nature to their coefficients, as well as h-order finite difference equations satisfied by the coefficients of these truncated series approximations.

(C) If $M_h := \text{ab}_1 \cdots \text{ab}_h$ denotes the h^{th} *modulus* of the generalized J-fraction expansion defined above, the sequence of coefficients enumerated by the h^{th} convergent function is eventually periodic modulo M_h and satisfies a finite difference equation of order at most h [2, Thm. 1]. Moreover, if M_m is integer-valued and $h \geq 2$ divides M_m for some $m \geq h$, then $[z^n]J^{[\infty]} \equiv [z^n]\,\text{Conv}_m(z) \pmod{h}$ [4, §5.7].

Typically we are only interested in the congruences formed by the h^{th} convergent functions for the coefficients of z^n the infinite series, $J^{[\infty]}(z)$ in z for $0 \leq n \leq h$, which are in fact exactly enumerated by these convergent functions. Other properties of the convergent functions, such as finite difference equations satisfied by their coefficients, suggest other non-obvious and non-trivial properties of the resulting congruences guaranteed by the convergent functions.

1.3. Organization of the Article

We provide two new forms of infinite J-fractions generating the binomial coefficients, $\binom{x+n}{n}$ and $\binom{x}{n}$, for $n \geq 0$ and any non-zero indeterminate x in the next sections of the article. These new continued fraction results lead to new exact formulas and finite difference equations for these binomial coefficient variants, and new congruences for the binomial coefficients modulo *any* (prime or composite) integers $h \geq 2$ whenever

$$\frac{1}{2h}\binom{x+h-1}{h-1}\binom{x}{h-1}\Big/\binom{2h-3}{h-2}^2 \in \mathbb{Z}.$$

Addition formulas for these J-fractions imply new identities for reductions of order of the upper index, x, comparable to the statements of Lucas' theorem and the other results modulo primes p and prime powers p^k given in Section 1.1.

The proofs of these new continued fraction expansions mostly follow by inductive arguments applied to known recurrence relations for the h^{th} numerator and denominator convergent function sequences. The proofs that these infinite J-fraction expansions exactly enumerate our binomial coefficient variants of interest follow from the rationality of the h^{th} convergent functions in z for all $h \geq 1$. Consequences of the proofs of our main theorems include new exact formulas for the binomial coefficients, $\binom{x+n}{n}$ and $\binom{x}{n}$, and new congruence properties for these binomial coefficient variants. Since the proofs of the corresponding results in each case given in Section 2 and Section 3, respectively, are so similar, we give careful proofs of the results for the case of the J-fractions generating $\binom{x+n}{n}$ in the first section, and choose to only state the analogous results for the case of $\binom{x}{n}$ in the second section below for clarity of exposition.

h	$\mathrm{P}_{1,h}(x,z)$
0	0
1	1
2	$1+\frac{1-x}{3}z$
3	$1-\frac{2}{5}(x-2)z+\frac{1}{20}(x-2)(x-1)z^2$
4	$1-\frac{3}{7}(x-3)z+\frac{1}{14}(x-3)(x-2)z^2-\frac{1}{210}(x-3)(x-2)(x-1)z^3$
5	$1-\frac{4}{9}(x-4)z+\frac{1}{12}(x-4)(x-3)z^2-\frac{1}{126}(x-4)(x-3)(x-2)z^3+\frac{(x-4)(x-3)(x-2)(x-1)}{3024}z^4$
6	$1-\frac{5}{11}(x-5)z+\frac{1}{11}(x-5)(x-4)z^2-\frac{1}{99}(x-5)(x-4)(x-3)z^3+\frac{(x-5)(x-4)(x-3)(x-2)}{1584}z^4$ $-\frac{(x-5)(x-4)(x-3)(x-2)(x-1)}{55440}z^5$

h	$(2h-1)!\cdot\mathrm{P}_{1,h}(x,z)$
0	0
1	1
2	$6-2(x-1)z$
3	$120-48(x-2)z^2+6(x-2)(x-1)z^2$
4	$5040-2160(x-3)z+360(x-3)(x-2)z^2-24(x-3)(x-2)(x-1)z^3$
5	$362880-161280(x-4)z+30240(x-4)(x-3)z^2-2880(x-4)(x-3)(x-2)z^3$ $+120(x-4)(x-3)(x-2)(x-1)z^4$
6	$39916800-18144000(x-5)z+3628800(x-5)(x-4)z^2-403200(x-5)(x-4)(x-3)z^3$ $+25200(x-5)(x-4)(x-3)(x-2)z^4-720(x-5)(x-4)(x-3)(x-2)(x-1)z^5$

Table 1: The Numerator Convergent Functions, $\mathrm{P}_{1,h}(x,z)$, Generating the Binomial Coefficients, $\binom{x+n}{n}$

h	$\mathrm{Q}_{1,h}(x,z)$
1	$1+(x+1)z$
2	$1+\frac{2}{3}(x+2)z+\frac{1}{6}(x+1)(x+2)z^2$
3	$1+\frac{3}{5}(x+3)z+\frac{3}{20}(x+2)(x+3)z^2+\frac{1}{60}(x+1)(x+2)(x+3)z^3$
4	$1+\frac{4}{7}(x+4)z+\frac{1}{7}(x+3)(x+4)z^2+\frac{2}{105}(x+2)(x+3)(x+4)z^3$ $+\frac{1}{840}(x+1)(x+2)(x+3)(x+4)z^4$
5	$1+\frac{5}{9}(x+5)z+\frac{5}{36}(x+4)(x+5)z^2+\frac{5}{252}(x+3)(x+4)(x+5)z^3+\frac{5(x+2)(x+3)(x+4)(x+5)}{3024}z^4$ $+\frac{(x+1)(x+2)(x+3)(x+4)(x+5)}{15120}z^5$
6	$1+\frac{6}{11}(x+6)z+\frac{3}{22}(x+5)(x+6)z^2+\frac{2}{99}(x+4)(x+5)(x+6)z^3$ $+\frac{1}{528}(x+3)(x+4)(x+5)(x+6)z^4+\frac{(x+2)(x+3)(x+4)(x+5)(x+6)}{9240}z^5$ $+\frac{(x+1)(x+2)(x+3)(x+4)(x+5)(x+6)}{332640}z^6$

h	$(2h-1)!\cdot\mathrm{Q}_{1,h}(x,z)$
1	$1+(x+1)z$
2	$6+4(x+2)z+(x+1)(x+2)z^2$
3	$120+72(x+3)z+18(x+2)(x+3)z^2+2(x+1)(x+2)(x+3)z^3$
4	$5040+2880(x+4)z+720(x+3)(x+4)z^2+96(x+2)(x+3)(x+4)z^3$ $+6(x+1)(x+2)(x+3)(x+4)z^4$
5	$362880+201600(x+5)z+50400(x+4)(x+5)z^2+7200(x+3)(x+4)(x+5)z^3$ $+600(x+2)(x+3)(x+4)(x+5)z^4+24(x+1)(x+2)(x+3)(x+4)(x+5)z^5$
6	$39916800+21772800(x+6)z+5443200(x+5)(x+6)z^2+806400(x+4)(x+5)(x+6)z^3$ $+75600(x+3)(x+4)(x+5)(x+6)z^4+4320(x+2)(x+3)(x+4)(x+5)(x+6)z^5$ $+120(x+1)(x+2)(x+3)(x+4)(x+5)(x+6)z^6$

Table 2: The Denominator Convergent Functions, $\mathrm{Q}_{1,h}(x,z)$, Generating the Binomial Coefficients, $\binom{x+n}{n}$

2. J-Fraction Expansions for the Binomial Coefficients, $\binom{x+n}{n}$

Definition 1 (Component Sequences and Convergent Functions for $\binom{x+n}{n}$). For a non-zero indeterminate x, let the sequences, $c_{x,i}^{(1)}$ and $\mathrm{ab}_{x,i}^{(1)}$, be defined over $i \geq 1$ as follows:

$$c_{x,i}^{(1)} := -\frac{1}{(2i-1)(2i-3)}\left(1+2(i-2)i-x\right),$$

$$\mathrm{ab}_{x,i}^{(1)} := \begin{cases} -\frac{1}{4(2i-3)^2}(x-i+2)(x+i-1) & \text{if } i \geq 3 \\ -\frac{1}{2}x(x+1) & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

We then define the h^{th} convergent functions, $\mathrm{Conv}_{1,h}(x,z) := \mathrm{P}_{1,h}(x,z)/\,\mathrm{Q}_{1,h}(x,z)$, through the component numerator and denominator functions given recursively by

$$\mathrm{P}_{1,h}(x,z) = (1 - c_{x,h}^{(1)}z)\,\mathrm{P}_{1,h-1}(x,z) - \mathrm{ab}_{x,h}^{(1)}\,z^2\,\mathrm{P}_{1,h-1}(x,z) + [h=1]_{\delta} \tag{1}$$

$$\begin{aligned}\mathrm{Q}_{1,h}(x,z) = (1 - c_{x,h}^{(1)}z)\,\mathrm{Q}_{1,h-1}(x,z) - \mathrm{ab}_{x,h}^{(1)}\,z^2\,\mathrm{Q}_{1,h-1}(x,z)\\ + (1 - c_{x,1}^{(1)}z)\,[h=1]_{\delta} + [h=0]_{\delta}\,.\end{aligned}$$

Listings of the first several cases of the numerator and denominator convergent functions, $\mathrm{P}_{1,h}(x,z)$ and $\mathrm{Q}_{1,h}(x,z)$, are given in Table 1 and Table 2, respectively.

Proposition 1 (Formulas for the Numerator Convergent Functions). *For all $h \geq 1$ and indeterminate x, the numerator convergent functions,* $\mathrm{P}_{1,h}(x,z)$*, have the following formulas:*

$$\begin{aligned}\mathrm{P}_h(x,z) &= \sum_{n=0}^{h-1} \binom{x+n-h}{n} \frac{(h-1)!}{(h-1-n)!} \cdot \frac{(2h-1-n)!}{(2h-1)!} \cdot (-z)^n \\ &= \sum_{n=0}^{h-1} \binom{x+n-h}{n} \binom{h-1}{n} \binom{2h-1}{n}^{-1} \cdot (-z)^n.\end{aligned}$$

Proof. The proof follows from (1) of Definition 1 by induction on h. The claimed formula holds for $h=0,1,2$ by the computations given in Table 1. Suppose that $h \geq 3$ and that the two equivalent formulas stated in the proposition for $\mathrm{P}_{1,k}(x,z)$ are correct for all $k < h$. In particular, the stated formula holds when $k = h-1, h-2$. Then by expanding (1) using our hypothesis, we have that

$$\begin{aligned}\mathrm{P}_{1,k}(x,z) = &\left(1 + \frac{(1+2h(h-2)-x)z}{(2h-1)(2h-3)}\right) \times \sum_{n=0}^{h-2} \binom{x+n+1-h}{n} \binom{h-2}{n} \binom{2h-3}{n}^{-1} (-z)^n \\ &+ \frac{(x+2-h)(x+h-1)z^2}{4(2h-3)^2} \times \sum_{n=0}^{h-3} \binom{x+n+2-h}{n} \binom{h-3}{n} \binom{2h-5}{n}^{-1} (-z)^n,\end{aligned}$$

which then implies the following expansions:

$$\begin{aligned}
\mathrm{P}_{1,k}(x,z) &= \sum_{n=0}^{h-1} \binom{x+n+1}{n}\binom{h-2}{n}\binom{2h-3}{n}^{-1}(-z)^n \\
&\quad - \frac{(1+2h(h-2)-x)}{(2h-1)(2h-3)} \sum_{n=1}^{h-1} \binom{x+n-h}{n-1}\binom{h-2}{n-1}\binom{2h-3}{n-1}^{-1}(-z)^n \\
&\quad + \frac{(x+2-h)(x+h-1)}{4(2h-3)^2} \sum_{n=2}^{h-1} \binom{x+n-h}{n-2}\binom{h-3}{n-2}\binom{2h-5}{n-2}^{-1}(-z)^n \\
&= 1 + \left(\binom{x+2-h}{1}\binom{h-2}{1}\binom{2h-3}{1}^{-1} - \frac{(1+2h(h-2)-x)}{(2h-1)(2h-3)} \right)(-z) \\
&\quad + \left(\frac{(x+2-h)(x+h-1)}{4(2h-3)^2} \binom{x-1}{h-3}\binom{h-3}{h-3}\binom{2h-5}{h-3}^{-1} \right. \\
&\qquad \left. - \binom{x-1}{h-2}\binom{h-2}{h-2}\binom{2h-3}{h-2}^{-1} \right)(-z)^{h-1} \\
&\quad + \sum_{i=2}^{h-2} \binom{x+n-h}{n}\binom{h-1}{n}\binom{2h-1}{n}^{-1} \left(\frac{2(2h-1)(h-n-1)(x+n+1-h)}{(2h-2-n)(2h-1-n)(x+1-h)} \right. \\
&\qquad - \frac{2n(1+2h(h-2)-x)}{(2h-3)(2h-1-n)(x+1-h)} \\
&\qquad \left. + \frac{n(n-1)(2h-1)(x+2-h)(x-1+h)}{(2h-3)(2h-2-n)(2h-1-n)(x+2-h)(x+1-h)} \right)(-z)^n.
\end{aligned}$$

Finally, we can simplify the last equation into the form of

$$\begin{aligned}
\mathrm{P}_{1,k}(x,z) &= 1 + \binom{x+1-h}{1}\binom{h-1}{1}\binom{2h-1}{1}^{-1}(-z) + \binom{x-1}{h-1}\binom{h-1}{h-1}\binom{2h-1}{h-1}^{-1}(-z)^{h-1} \\
&\quad + \sum_{n=2}^{h-2} \binom{x+n-h}{n}\binom{h-1}{n}\binom{2h-1}{n}^{-1}(-z)^n.
\end{aligned}$$

The next identity is used to prove Theorem 2 below. In particular, the coefficients of the powers of z^n on the right-hand-sides of the stated formulas satisfy

$$\begin{aligned}
&(-1)^n \binom{x+n-h}{n}\binom{h-1}{n}\binom{2h-1}{n}^{-1} \\
&\qquad = \sum_{i=0}^{n} \binom{x+n-i}{n-i}\binom{x+h}{i} \frac{h!}{(h-i)!}\frac{(2h-1-i)!}{(2h-1)!}(-1)^{n-i},
\end{aligned} \tag{2}$$

which is proved directly by summing the corresponding hypergeometric terms exactly with *Mathematica*. □

Proposition 2 (Formulas for the Denominator Convergent Functions). *For all $h \geq 0$ and fixed indeterminates x, we have that the denominator convergent* *functions,* $\mathrm{Q}_{1,h}(x,z)$*, satisfy the following formula:*

$$\mathrm{Q}_h(x,z) = \sum_{i=0}^{h} \binom{x+h}{i} \cdot \frac{h!}{(h-i)!} \cdot \frac{(2h-1-i)!}{(2h-1)!} \cdot z^i.$$

Proof. The proof again follows from (1) of Definition 1 by induction on h. The claimed formula holds when $h = 0, 1, 2$ by the computations given in Table 2. Next, we suppose that $h \geq 3$ and that the formula stated in the proposition for $\mathrm{Q}_{1,k}(x,z)$ is correct for all $k < h$. In particular, the stated formulas hold when $k = h-1, h-2$. Then by expanding (1) using our inductive hypothesis, we have that

$$\begin{aligned}
\mathrm{Q}_{1,h}(x,z) &= \sum_{i=0}^{h-1} \binom{x+h-1}{i} \frac{(h-1)!}{(h-1-i)!} \frac{(2h-3-i)!}{(2h-3)!} z^i \\
&\quad + \frac{(1+2h(h-2)-x)}{(2h-1)(2h-3)} \sum_{i=1}^{h} \binom{x+h-1}{i-1} \frac{(h-1)!}{(h-i)!} \frac{(2h-2-i)!}{(2h-3)!} z^i \\
&\quad + \frac{(x+1-h)(x+h-1)}{4(2h-3)^2} \sum_{i=2}^{h} \binom{x+h-2}{i-2} \frac{(h-2)!}{(h-i)!} \frac{(2h-3-i)!}{(2h-5)!} z^i \\
&= 1 + \binom{x+h}{1} \frac{hz}{(2h-1)} + \binom{x+h}{h} \frac{h!(h-1)!z^h}{(2h-1)!} \\
&\quad + \sum_{i=2}^{h-1} \binom{x+h}{i} \frac{h!}{(h-i)!} \frac{(2h-1-i)!}{(2h-1)!} z^i \Bigg(\frac{2(h-1)(2h-1)(h-i)(x+h-i)}{h(2h-2-i)(2h-1-i)(h+x)} \\
&\qquad + \frac{2(h-1)i(1+2h(h-2)-x)}{h(2h-3)(2h-1-i)(h+x)} \\
&\qquad + \frac{(h-2)(2h-1)(i(i-1)(x+2-h)}{h(2h-3)(2h-2-i)(2h-1-i)(h+x)} \Bigg) \\
&= \sum_{i=0}^{h} \binom{x+h}{i} \frac{h!}{(h-i)!} \frac{(2h-1-i)!}{(2h-1)!} z^i.
\end{aligned}$$

□

The expansions of the J-fractions and, more generally, for any continued fraction whose convergents are defined by the ratio of terms defined recursively as in (1), provide additional recurrence relations and exact finite sums for the h^{th} convergent functions, $\mathrm{Conv}_{1,h}(x,z)$, given by [6, §1.12]:

$$\begin{aligned}
\mathrm{Conv}_{1,h}(x,z) &= \mathrm{Conv}_{1,h-1}(x,z) + \frac{(-1)^{h-1}\,\mathrm{ab}_{x,2}\,\mathrm{ab}_{x,3}\cdots\mathrm{ab}_{x,h}\,z^{2h-2}}{\mathrm{Q}_{1,h}(x,z)\,\mathrm{Q}_{1,h-1}(x,z)} \\
&= \frac{1}{1+(x+1)z} + \sum_{i=2}^{h} \frac{(-1)^{i-1}\binom{x+i-1}{i-1}\binom{x}{i-1}\binom{2i-3}{i-2}^{-2} z^{2i-2}}{2\cdot \mathrm{Q}_{1,i}(x,z)\,\mathrm{Q}_{1,i-1}(x,z)}.
\end{aligned}$$

Theorem 2 (Main Theorem I). *For integers $h \geq 2$, we have that the h^{th} convergent functions,* $\mathrm{Conv}_{1,h}(x,z)$*, exactly generate the binomial coefficients* $\binom{x+n}{n}$ *for all* $0 \leq n \leq h$ *as*

$$[z^n]\,\mathrm{Conv}_{1,h}(x,-z) = \binom{x+n}{n}.$$

Proof. Since $\mathrm{Conv}_{1,h}(x,z)$ is rational in z for all $h \geq 2$, Proposition 1 and Proposition 2 imply the following finite difference equations for the coefficients of the convergent functions, $\mathrm{Conv}_{1,h}(x,z)$, when $n \geq 0$ [4, §2.3]:

$$[z^n]\,\mathrm{Conv}_{1,h}(x,-z) = -\sum_{i=1}^{\min(n,h)} \binom{x+h}{i} [z^{n-i}]\,\mathrm{Conv}_{1,h}(x,z) \times \frac{h!}{(h-i)!}\frac{(2h-1-i)!}{(2h-1)!}(-1)^i + [z^n]\,\mathrm{P}_{1,h}(x,-z)\,[0 \leq n < h]_\delta\,. \tag{3}$$

We must show that $[z^n]\,\mathrm{Conv}_{1,h}(x,-z) = \binom{x+n}{n}$ in the separate cases where $0 \leq n < h$ and when $n = h$. For the first case, we use induction on n to show our result. Since $[z^0]F(z)/G(z) = F(0)/G(0)$ for any functions, $F(z)$ and $G(z)$, with a power series expansion in z about zero, we have by the two propositions above that $[z^0]\,\mathrm{Conv}_{1,h}(x,z) = \binom{x+0}{0} \equiv 1$ for all $h \geq 2$.

Next, we suppose that for $h > n \geq 1$ and all $k < n$, $[z^k]\,\mathrm{Conv}_{1,h}(x,-z) = \binom{x+k}{k}$. Since $0 \leq n-i < n$ for all i in the right-hand-side sum of (3), we can apply the inductive hypothesis, combined with the observation in (2) from the proof of Proposition 1, to the recurrence relation in the previous equation to obtain that when $n < h$ we have

$$\begin{aligned}
[z^n]\,\mathrm{Conv}_{1,h}(x,-z) &= -\sum_{i=1}^{n} \binom{x+h}{i}\binom{x+n-i}{n-i} \frac{h!}{(h-i)!}\frac{(2h-1-i)!}{(2h-1)!}(-1)^i \\
&\quad + \sum_{i=0}^{n} \binom{x+h}{i}\binom{x+n-i}{n-i} \frac{h!}{(h-i)!}\frac{(2h-1-i)!}{(2h-1)!}(-1)^i \\
&= \binom{x+n}{n}.
\end{aligned}$$

To prove the claim in the first special case of (3) where $[z^n]\,\mathrm{P}_{1,h}(x,-z) \equiv 0$, *i.e.,* precisely when $n = h$, we use an alternate approach to evaluating these sums by exactly summing with *Mathematica* as

$$\begin{aligned}
[z^n]\,\mathrm{Conv}_{1,h}(x,-z) &= -\sum_{i=1}^{n} \binom{x+h}{i}\binom{x+n-i}{n-i} \frac{h!}{(h-i)!}\frac{(2h-1-i)!}{(2h-1)!}(-1)^i \\
&= \binom{x+n}{n}\left(1 - {}_3F_2(-h,-n,-(x+h);1-2h,-(x+n);1),\right)
\end{aligned}$$

where ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ denotes the *generalized hypergeometric function* whose coefficients when expanded in powers of z are given by the next *Pochhammer* *symbol* products [6, §16]:

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k \geq 0} \frac{(a_1)_k \cdots (a_p)_k}{k! \cdot (b_1)_k \cdots (b_q)_k} z^k. $$

When $h = n$, the terms contributed by the generalized hypergeometric function in the previous equation are zero-valued[1]. Therefore when $n = h$, we have that $[z^n] \operatorname{Conv}_{1,h}(x, z) = \binom{x+n}{n}$. □

We can actually prove a stronger statement of Theorem 2 which provides that $[z^n] \operatorname{Conv}_{1,h}(x, z) = \binom{x+n}{n}$ for all $0 \leq n < 2h$, though for the purposes of showing that our J-fraction expansions defined by Definition 1 are correct, the proof of the theorem given above suffices to show the result. One consequence of the theorem is that we have the following finite sums exactly generating, $\binom{x+n}{n}$, for any x and all $n \geq 0$, *i.e.,* in the case of (3) where $h = n$:

$$ \binom{x+n}{n} = \sum_{i=1}^{n} \binom{x+n}{i} \binom{x+n-i}{n-i} \binom{n}{i} \binom{2n-1}{i}^{-1} (-1)^{i+1} + [n=0]_\delta . $$

Corollary 1 (Congruences for $\binom{x+n}{n}$ Modulo Integers $h \geq 2$). *For $h \geq 2$, let $\lambda_h(x) := \prod_{i=2}^{h} \mathrm{ab}_{x,i}$, or equivalently, let $\lambda_h(x) \equiv \frac{(-1)^{h-1}}{2} \binom{x+h-1}{h-1} \binom{x}{h-1} / \binom{2h-3}{h-2}^2$. For all $h \geq 2$, $m \geq h$, $0 \leq x \leq h$, and $n \geq 0$, whenever $\lambda_m(x) \in \mathbb{Z}$ and $h \mid \lambda_m(x)$, we have the following congruences congruences for the binomial coefficients modulo h:*

$$ \begin{aligned} \binom{x+n}{n} \equiv \sum_{i=1}^{n} \binom{x+m}{i} \binom{x+n-i}{n-i} \cdot \frac{m!}{(m-i)!} \cdot \frac{(2m-1-i)!}{(2m-1)!} (-1)^{i+1} & \\ + \binom{x+n-m}{n} \binom{m-1}{n} \binom{2m-1}{n}^{-1} [0 \leq n < m]_\delta & \pmod{h}. \end{aligned} $$

Proof. The result is an immediate corollary of (3) from the proof of Theorem 2 and the J-fraction coefficient congruence properties cited in property (C) of Section 1.2 in the introduction [2] [4, *cf.* §5.7]. □

Remark 3 (Exact Congruences for the Binomial Coefficients). We conjecture that in fact for all $h \geq 2$, $h > n \geq 0$, and $x < h$, which typically corresponds to the cases of the binomial coefficients that we actually wish to evaluate modulo

[1] *i.e.,* since $(-n)_{n+1+k} = 0$ for all $k \geq 0$ and where for all integers $n \geq 1$ the next hypergeometric sum is evaluated exactly with *Mathematica* as

$$ \sum_{k=0}^{n} \binom{n}{k}^2 \times \frac{(-1)^k k! \cdot (2n-1-k)!}{(2n-1)!} = 0. $$

h, that

$$\binom{x+n}{n} \equiv \sum_{i=1}^{h} \binom{x+h}{i}\binom{x+n-i}{n-i} \cdot \frac{h!}{(h-i)!} \cdot \frac{(2h-1-i)!}{(2h-1)!}(-1)^{i+1} \pmod{h}, \tag{4}$$

at a minimum for any x such that h evenly divides $\lambda_h(x)$ as an integer factor, though other choices of the indeterminate x may correctly yield congruence modulo h. Given this restriction on the x modulo each h, we then define the corresponding index sets

$$\mathcal{M}_h := \left\{ x \in \mathbb{Z} : \frac{1}{2h}\binom{x+h-1}{h-1}\binom{x}{h-1}\binom{2h-3}{h-2}^{-2} \in \mathbb{Z} \right\}.$$

The first few particular special cases of these restricted index sets include

$$\begin{aligned}
\mathcal{M}_2 &= \left\{ x : \frac{x(x+1)}{4} \in \mathbb{Z} \right\} = \{x : x \equiv 0, 3 \bmod 4\} \\
\mathcal{M}_3 &= \left\{ x : \frac{(x-1)x(x+1)(x+2)}{216} \in \mathbb{Z} \right\} \\
&= \{x : x \equiv 0, 1, 7, 10, 16, 19, 25, 26 \bmod 27\} \\
\mathcal{M}_4 &= \left\{ x : \frac{(x-1)(x-1)x(x+1)(x+2)(x+3)}{28800} \in \mathbb{Z} \right\} \\
&= \{x : x \equiv 0, 1, 2, 7, 12, 17, 22, 23, 24 \bmod 25\} \\
&\qquad \cap \{x : x \equiv 0, 1, 2, 13, 14, 17, 18, 29, 30, 31 \bmod 32\} \\
\mathcal{M}_5 &= \left\{ x : \frac{(x-2)(x-1)(x-1)x(x+1)(x+2)(x+3)(x+4)}{7056000} \in \mathbb{Z} \right\} \\
&= \{x : x \equiv 0, 1, 2, 3, 21, 22, 23, 24 \bmod 25\} \\
&\qquad \cap \{x : x \equiv 0, 1, 2, 3, 10, 17, 24, 31, 38, 45, 46, 47, 48 \bmod 49\}.
\end{aligned}$$

Now using these results, we can expand these special case congruences for $\binom{x+n}{n}$ modulo 2, 3, 4, and 5 as follows:

$$\binom{x+n}{n} \equiv \frac{2(x+2)}{3}\binom{x+n-1}{n-1} - \frac{(x+1)(x+2)}{6}\binom{x+n-2}{n-2} \pmod{2}, \quad \text{for all } x \in \mathcal{M}_2$$

$$\begin{aligned}
\binom{x+n}{n} &\equiv \frac{3(x+3)}{5}\binom{x+n-1}{n-1} - \frac{3(x+2)(x+3)}{20}\binom{x+n-2}{n-2} \\
&\quad + \frac{(x+1)(x+2)(x+3)}{60}\binom{x+n-3}{n-3} \pmod{3}, \quad \text{for all } x \in \mathcal{M}_3
\end{aligned}$$

$$\begin{aligned}\binom{x+n}{n} &\equiv \frac{4(x+4)}{7}\binom{x+n-1}{n-1} - \frac{(x+3)(x+4)}{7}\binom{x+n-2}{n-2} \\ &\quad + \frac{2(x+2)(x+3)(x+4)}{105}\binom{x+n-3}{n-3} \\ &\quad - \frac{(x+1)(x+2)(x+3)(x+4)}{840}\binom{x+n-4}{n-4} \qquad \pmod{4}, \\ &\qquad\qquad \text{for all } x \in \mathcal{M}_4 \\ \binom{x+n}{n} &\equiv \frac{5(x+5)}{9}\binom{x+n-1}{n-1} - \frac{5(x+4)(x+5)}{56}\binom{x+n-2}{n-2} \\ &\quad + \frac{5(x+3)(x+4)(x+5)}{252}\binom{x+n-3}{n-3} \\ &\quad - \frac{5(x+2)(x+3)(x+4)(x+5)}{3024}\binom{x+n-4}{n-4} \\ &\quad + \frac{(x+1)(x+2)(x+3)(x+4)(x+5)}{15120}\binom{x+n-5}{n-5} \qquad \pmod{5}, \\ &\qquad\qquad \text{for all } x \in \mathcal{M}_5.\end{aligned}$$

We also notice that once we know the restricted sets $\mathcal{M}_h$ for any $h \geq 2$, the congruences expanded above provide us with new information about the *central binomial coefficients*, $\binom{2n}{n}$ for $n \in \mathcal{M}_h \cap \mathbb{N}$, modulo prime and composite choices of h.

Example 4 (Divisors of the Central Binomial Coefficients). One famous example of a problem we can approach using the new machinery proved in this article is stated in [7, §4]. The problem (and its prize) due to Ron Graham is to determine whether there are infinitely many n such that $\binom{2n}{n}$ is relatively prime to 105, or equivalently, to find the sequence of n such that none of the individual prime factors divide the n^{th} central binomial coefficient: $3, 5, 7 \nmid \binom{2n}{n}$. Two of the congruences explicitly expanded in the previous equations imply new generating functions for two of the three cases we need to consider in Graham's problem. For any integer modulus $h \geq 2$, we set

$$\mathcal{B}_{\text{cent},h}(z) := \sum_{n \in \mathcal{M}_p \cap \mathbb{N}} \left\{ \binom{2n}{n} \pmod{p} \right\} z^n.$$

We can easily compute (i.e., using *Mathematica*) that the non-identically-zero coefficients of the next generating functions, denoted by $\widetilde{\mathcal{B}}_{\text{cent},h}(z)$, are congruent to the respective coefficients of $\mathcal{B}_{\text{cent},h}(z)$ defined in the last equation. For $h \in \{2, 3\}$, particular concrete examples of these generating functions are expanded as

$$\begin{aligned}\widetilde{\mathcal{B}}_{\text{cent},2}(z) &= \frac{1}{2} - \frac{5}{32\hat{\mathcal{B}}(z)} + \frac{17}{32}\hat{\mathcal{B}}(z) + \frac{5}{32}\hat{\mathcal{B}}(z)^3 - \frac{1}{32}\hat{\mathcal{B}}(z)^5 \\ \widetilde{\mathcal{B}}_{\text{cent},3}(z) &= \frac{5-z}{10} - \frac{63}{256\hat{\mathcal{B}}(z)} + \frac{43}{64}\hat{\mathcal{B}}(z) + \frac{63}{640}\hat{\mathcal{B}}(z)^3 - \frac{9}{320}\hat{\mathcal{B}}(z)^5 + \frac{1}{256}\hat{\mathcal{B}}(z)^7,\end{aligned}$$

in powers of the ordinary generating function for the central binomial coefficients defined by

$$\hat{\mathcal{B}}(z) := \sum_{n\geq 0}\binom{2n}{n}z^n = \frac{1}{\sqrt{1-4z}}.$$

It is easy to see by induction from (4) that for higher-order $h > 5$, a general formula for the expansions of the generating functions $\widetilde{\mathcal{B}}_{\mathrm{cent},h}(z)$ can be expressed as a sum of a rational function of z of small degree less than h plus a finite sum of odd powers of $\hat{\mathcal{B}}(z)^m$ for $m \in [-1, 2h+1] \cap \mathbb{Z} \setminus 2\mathbb{Z}$. We can then obtain formulas for "most" of the $\binom{2n}{n}$ mod h when $n \in \mathcal{M}_h$ by *Faà di Bruno's formula* or directly by taking the Cauchy products corresponding to multiple convolutions of the central binomial generating function defined above.

Corollary 2 (Reduction of Order of the Binomial Coefficients). *For integers r, p such that $p \geq r \geq 0$, let the sequences, $k_{r,p}(x)$, be defined as*

$$k_{r,p}(x) = (-1)^{p-r}\binom{x+p}{p-r}\binom{2r}{r}\binom{p+r}{r}^{-1}.$$

For all integers $p, q \geq 0$, we have an addition theorem for the binomial coefficients given by

$$(-1)^{p+q}\binom{x+p+q}{p+q} = k_{0,p}(x)k_{0,q}(x) + \sum_{i=1}^{\min(p,q)} \lambda_{i+1}(x)k_{i,p}(x)k_{i,q}(x),$$

where $\lambda_h(x)$ is defined as in Corollary 1.

Proof. We can prove the result using the matrix method from [1, §1, p. 133] and [10, §11]. In particular, for integers $p \geq r \geq 0$, let the functions, $\widetilde{k}_{r,p}(x)$, denote the solutions to the matrix equation

$$\begin{aligned}&\begin{bmatrix} \widetilde{k}_{0,1}(x) & \widetilde{k}_{1,1}(x) & 0 & 0 & \cdots \\ \widetilde{k}_{0,2}(x) & \widetilde{k}_{1,2}(x) & \widetilde{k}_{2,2}(x) & 0 & \cdots \\ \widetilde{k}_{0,3}(x) & \widetilde{k}_{1,3}(x) & \widetilde{k}_{2,3}(x) & \widetilde{k}_{3,3}(x) & \cdots \\ & \cdots & & & \end{bmatrix} \\ &\qquad = \begin{bmatrix} \widetilde{k}_{0,0}(x) & 0 & 0 & 0 & \cdots \\ \widetilde{k}_{0,1}(x) & \widetilde{k}_{1,1}(x) & 0 & 0 & \cdots \\ \widetilde{k}_{0,1}(x) & \widetilde{k}_{1,1}(x) & \widetilde{k}_{2,2}(x) & 0 & \cdots \\ & & \cdots & & \end{bmatrix}\begin{bmatrix} c_{x,1} & 1 & 0 & 0 & \cdots \\ \mathrm{ab}_{x,2} & c_{x,2} & 1 & 0 & \cdots \\ 0 & \mathrm{ab}_{x,3} & c_{x,3} & 1 & \cdots \\ & & \cdots & & \end{bmatrix},\end{aligned} \tag{5}$$

where here we define $\widetilde{k}_{0,p}(x) := \binom{x+p}{p}$ for all $p \geq 0$. We claim that for all integers $p, r \geq 0$ with $p \geq r \geq 0$, the two functions, $k_{r,p}(x)$ and $\widetilde{k}_{r,p}(x)$, are equal. To prove the claim, we notice that for fixed p, equation (5) implies recurrence relations over r given by

$$\begin{aligned}
\widetilde{k}_{1,p}(x) &= \frac{1}{\mathrm{ab}_{x,2}}\left(\widetilde{k}_{0,p+1}(x) - c_{x,1}\cdot\widetilde{k}_{0,p}(x)\right),\ p \geq 1 \\
\widetilde{k}_{r+1,p}(x) &= \frac{1}{\mathrm{ab}_{x,r+2}}\left(\widetilde{k}_{r,p+1}(x) - \widetilde{k}_{r-1,p}(x) - c_{x,r+1}\cdot\widetilde{k}_{r,p}(x)\right),\ p > r \geq 1.
\end{aligned}$$

The first recurrence relation provides that for $p \geq 1$

$$\begin{aligned}
\widetilde{k}_{1,p} &= -\frac{2}{x(x+1)}\left((-1)^{p+1}\binom{x+p+1}{p+1} + (-1)^p(x+1)\binom{x+p}{p}\right) \\
&= \frac{2(-1)^{p-1}(p+x)!}{(p+1)(x+1)!(p-1)!}\left(\frac{(x+1)(p+1)}{px} - \frac{(x+p+1)}{px}\right) \\
&= \frac{2(-1)^{p-1}}{(p+1)}\binom{x+p}{p-1},
\end{aligned}$$

which is the same as the formula for $k_{1,p}(x)$ stated above. We can then use the second recurrence relation to complete the proof of the claim by induction on r. When $r = 0, 1$, we have that the two formulas for $k_{r,p}(x)$ and $\widetilde{k}_{r,p}(x)$ coincide. We suppose that the claim is true for some $r \geq 1$, which by the previous recurrence relation in turn implies that

$$\begin{aligned}
&\widetilde{k}_{r+1,p}(x) \\
&= \frac{-4\cdot(2r+1)^2}{(x-r)(x+r+1)}\Bigg((-1)^{p-1-r}\binom{x+p+1}{p+1-r}\binom{2p+2}{p+1}\binom{r+p+1}{p+1}^{-1} \\
&\qquad - (-1)^{r-1-p}\binom{x+p}{p+1-r}\binom{2r-2}{r-1}\binom{p+r-1}{r-1}^{-1} \\
&\qquad + \frac{(-1)^{p-1-r}(1+2(r-1)(r+1)-x)}{(2r+1)(2r-1)}\binom{x+p}{p-r}\binom{2p}{p}\binom{r+p}{p}^{-1}\Bigg) \\
&= (-1)^{p-r-1}\binom{x+p}{p-1-r}\binom{2r+2}{r+1}\binom{p+r+1}{r+1}^{-1} \times \frac{4(2r+1)^2}{(r-x)(x+r+1)} \times \Bigg(\\
&\qquad \frac{(p+1)(p+1+x)(r+1+x)}{2(p-r)(p+1-r)(2r-1)(2r+1)} - \frac{(p+r)(p+1+r)(r+x)(r+1+x)}{4(p-r)(p+1-r)(2r-1)(2r+1)} \\
&\qquad - \frac{(1+2(r-1)(r+1)-x)(p+1+r)(r+1+x)}{2(p-r)(2r-1)(2r+1)^2}\Bigg) \\
&= (-1)^{p-r-1}\binom{x+p}{p-1-r}\binom{2r+2}{r+1}\binom{p+r+1}{r+1}^{-1}.
\end{aligned}$$

Since the two formulas are equivalent, we obtain the next form of the *addition formula*, or alternately, a formula providing a "*reduction*" of the order of the upper and lower indices to the binomial coefficients, $\binom{x+n}{n}$, given by the expansion in the references in the following forms for all integers $p, q \geq 0$:

$$\begin{aligned}(-1)^{p+q}\binom{x+p+q}{p+q} &= k_{0,p}k_{0,q} + \mathrm{ab}_{x,2}\cdot k_{1,p}k_{1,q} + \mathrm{ab}_{x,2}\,\mathrm{ab}_{x,3}\cdot k_{2,p}k_{2,q} + \cdots \\ &= (-1)^{p+q}\binom{x+p}{p}\binom{x+q}{q} \\ &\quad + \sum_{i=1}^{\min(p,q)}(-1)^{p+q}\lambda_{i+1}(x)\binom{x+p}{p-i}\binom{x+q}{q-i}\binom{2i}{i}^2\binom{p+i}{i}^{-1}\binom{q+i}{i}^{-1}.\end{aligned}$$

□

Example 5. We notice that this result provides *exact* finite sum expansions of the binomial coefficients, $\binom{x+n}{n}$, for any x and $n \geq 0$ which hold modulo any integers $h \geq 2$ (prime or composite), and compare the reduction in the upper and lower coefficient indices to the congruence result provided by Lucas' theorem and its related variants stated in the introduction. For a concrete example, let $p = q = 2$ in Corollary 2. Then we obtain that

$$\begin{aligned}\binom{x+4}{4} &= \frac{(x+1)^2(x+2)^2}{4} - \frac{2}{9}(x+2)^2(x+1)x + \frac{(x+2)(x+1)x(x-1)}{72} \\ &= \frac{(x+4)(x+3)(x+2)(x+1)}{24}.\end{aligned}$$

Furthermore, if we let $n \in \mathbb{Z}^+$ and specialize $x = p = q := n$ in the corollary above, we are able to obtain the new sum

$$\begin{aligned}\binom{3n}{2n} &= \binom{2n}{n}^2 + 2 \times \sum_{i=1}^{n}(-1)^i\binom{n}{i}\binom{n+i}{i}^{-1}\binom{2n}{n-i}^2 \\ &= \binom{2n}{n}^2 - 2n(n+1)\cdot\binom{2n}{n-1}^2 - 2\left(\frac{6}{n+2} - \frac{2}{n+1} - 1\right)\binom{2n}{n-2}^2 - \cdots.\end{aligned}$$

Related identities and congruences are formulated similarly by taking each of x, p, q to be integer multiples of the positive integer $n \geq 1$ specified in the previous example.

3. J-Fraction Expansions for the Binomial Coefficients, $\binom{x}{n}$

We can construct (and formally prove) similar convergent-based J-fraction constructions enumerating the binomial coefficients, $\binom{x}{n}$, for an indeterminate x and $n \geq 0$. Since the proofs for the propositions, theorem, and corollaries in this section are almost identical to those given in the case of the binomial coefficient variants, $\binom{x+n}{n}$, in Section 2, we omit the proofs of the next results stated below. We begin with the definition of the component sequences and convergent functions corresponding to the J-fractions generating the binomial coefficients, $\binom{x}{n}$, in this case.

Definition 6 (Component Sequences and Convergent Functions for $\binom{x}{n}$). For a fixed non-zero indeterminate x and $i \geq 1$, we define the component sequences, $c_{x,i}^{(2)}$ and $\mathrm{ab}_{x,i}^{(2)}$, as follows:

$$c_{x,i}^{(2)} := -\frac{1}{(2i-1)(2i-3)}\left(x + 2(i-1)^2\right)$$

$$\mathrm{ab}_{x,i}^{(2)} := \begin{cases} -\frac{1}{4(2i-3)^2}(x-i+2)(x+i-1) & \text{if } i \geq 3 \\ -\frac{1}{2}x(x+1) & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

For $h \geq 0$, we define the h^{th} convergent functions, $\mathrm{Conv}_{2,h}(x,z) := \mathrm{P}_{2,h}(x,z)/\,\mathrm{Q}_{2,h}(x,z)$, recursively through the component functions in the forms of

$$\begin{aligned} \mathrm{P}_{2,h}(x,z) &= (1 - c_{x,h}^{(2)}z)\,\mathrm{P}_{2,h-1}(x,z) - \mathrm{ab}_{x,h}^{(2)}\,z^2\,\mathrm{P}_{2,h-1}(x,z) + [h=1]_\delta \qquad (6) \\ \mathrm{Q}_{2,h}(x,z) &= (1 - c_{x,h}^{(2)}z)\,\mathrm{Q}_{2,h-1}(x,z) - \mathrm{ab}_{x,h}^{(2)}\,z^2\,\mathrm{Q}_{2,h-1}(x,z) \\ &\quad + (1 - c_{x,1}^{(2)}z)\,[h=1]_\delta + [h=0]_\delta\,. \end{aligned}$$

Listings of the first several cases of the numerator and denominator convergent functions, $\mathrm{P}_{2,h}(x,z)$ and $\mathrm{Q}_{2,h}(x,z)$, are given in Table 3 and Table 4, respectively.

h	$\mathrm{P}_{2,h}(x,z)$
0	0
1	1
2	$1 + \frac{x+2}{3}z$
3	$1 + \frac{2}{5}(x+3)z + \frac{1}{20}(x+2)(x+3)z^2$
4	$1 + \frac{3}{7}(x+4)z + \frac{1}{14}(x+3)(x+4)z^2 + \frac{1}{210}(x+2)(x+3)(x+4)z^3$
5	$1 + \frac{4}{9}(x+5)z + \frac{1}{12}(x+4)(x+5)z^2 + \frac{1}{126}(x+3)(x+4)(x+5)z^3$ $+ \frac{(x+2)(x+3)(x+4)(x+5)}{3024}z^4$
6	$1 + \frac{5}{11}(x+6)z + \frac{1}{11}(x+5)(x+6)z^2 + \frac{1}{99}(x+4)(x+5)(x+6)z^3$ $+ \frac{(x+3)(x+4)(x+5)(x+6)}{1584}z^4 + \frac{(x+2)(x+3)(x+4)(x+5)(x+6)}{55440}z^5$

Table 3: The Numerator Convergent Functions, $\mathrm{P}_{2,h}(x,z)$, Generating the Binomial Coefficients, $\binom{x}{n}$

h	$\mathrm{Q}_{2,h}(x,z)$
1	$1-xz$
2	$1-\frac{2}{3}(x-1)z+\frac{1}{6}(x-1)xz^2$
3	$1-\frac{3}{5}(x-2)z+\frac{3}{20}(x-2)(x-1)z^2-\frac{1}{60}(x-2)(x-1)xz^3$
4	$1-\frac{4}{7}(x-3)z+\frac{1}{7}(x-3)(x-2)z^2-\frac{2}{105}(x-3)(x-2)(x-1)z^3$ $+\frac{1}{840}(x-3)(x-2)(x-1)xz^4$
5	$1-\frac{5}{9}(x-4)z+\frac{5}{36}(x-4)(x-3)z^2-\frac{5}{252}(x-4)(x-3)(x-2)z^3$ $+\frac{5(x-4)(x-3)(x-2)(x-1)}{3024}z^4-\frac{(x-4)(x-3)(x-2)(x-1)x}{15120}z^5$
6	$1-\frac{6}{11}(x-5)z+\frac{3}{22}(x-5)(x-4)z^2-\frac{2}{99}(x-5)(x-4)(x-3)z^3$ $+\frac{1}{528}(x-5)(x-4)(x-3)(x-2)z^4-\frac{(x-5)(x-4)(x-3)(x-2)(x-1)}{9240}z^5$ $+\frac{(x-5)(x-4)(x-3)(x-2)(x-1)x}{332640}z^6$

Table 4: The Denominator Convergent Functions, $\mathrm{Q}_{2,h}(x,z)$, Generating the Binomial Coefficients, $\binom{x}{n}$

Proposition 3 (Convergent Function Formulas). *For all $h \geq 2$ and a fixed indeterminate x, the numerator and denominator convergent functions, $\mathrm{P}_{2,h}(x,z)$ and $\mathrm{Q}_{2,h}(x,z)$, each satisfy the following respective formulas:*

$$\begin{aligned}\mathrm{P}_{2,h}(x,z) &= \sum_{n=0}^{h-1}\binom{x+h}{n}\frac{(h-1)!}{(h-1-n)!}\frac{(2h-1-n)!}{(2h-1)!}z^n \\ &= \sum_{n=0}^{h-1}\binom{x+h}{n}\binom{h-1}{n}\binom{2h-1}{n}^{-1}z^n \\ \mathrm{Q}_{2,h}(x,z) &= \sum_{i=0}^{h}\binom{x-h+i}{i}\frac{h!}{(h-i)!}\frac{(2h-1-i)!}{(2h-1)!}(-z)^i.\end{aligned}$$

Theorem 7 (Main Theorem II). *For all integers $h \geq 2$ and $0 \leq n \leq h$, we have that the h^{th} convergent functions, $\mathrm{Conv}_{2,h}(x,z)$, exactly generate the binomial coefficients $\binom{x}{n}$ as $[z^n]\,\mathrm{Conv}_{2,h}(x,z) = \binom{x}{n}$.*

As in Section 2, we remark that one consequence of the second main theorem resulting from the proposition above is that we have a new proof of the next exact formula generating the binomial coefficients, $\binom{x}{n}$, for any x and all $n \geq 0$:

$$\binom{x}{n} = \sum_{i=1}^{n}\binom{x-n+i}{i}\binom{x}{n-i}\binom{n}{i}\binom{2n-1}{i}^{-1}(-1)^{i+1} + [n=0]_{\delta}\,.$$

Corollary 3 (Congruences for $\binom{x}{n}$ Modulo Integers $h \geq 2$). *Let the h^{th} modulus, $\lambda_h(x)$, of the J-fraction defined by Definition 6 correspond to the definitions* *given in Corollary 1. For all $h \geq 2$, $m \geq h$, $0 \leq x \leq h$, and $n \geq 0$, whenever $\lambda_m(x) \in \mathbb{Z}$ and $h \mid \lambda_m(x)$, we have the following congruences congruences for the binomial coefficients, $\binom{x}{n}$, modulo h:*

$$\begin{aligned}\binom{x}{n} &\equiv \sum_{i=1}^{n} \binom{x-m+i}{i}\binom{x}{n-i} \cdot \frac{m!}{(m-i)!} \cdot \frac{(2m-1-i)!}{(2m-1)!}(-1)^{i+1} \\ &\quad + \binom{x+m}{n}\binom{m-1}{n}\binom{2m-1}{n}^{-1} [0 \leq n < m]_\delta \qquad \pmod{h}.\end{aligned}$$

Remark 8 (Exact Congruences for Special Cases). We again conjecture that for all $h \geq 2$ and all $x, n \geq 0$, we have congruences for the binomial coefficients, $\binom{x}{n}$, of the form

$$\begin{aligned}\binom{x}{n} &\equiv \sum_{i=1}^{h} \binom{x-h+i}{i}\binom{x}{n-i}\binom{h}{i}\binom{2h-1}{i}^{-1}(-1)^{i+1} \\ &\quad + \binom{x+h}{n}\binom{h-1}{n}\binom{2h-1}{n}^{-1} \qquad \pmod{h}, \\ &\qquad \text{for all } x \in \mathcal{M}_h,\end{aligned}$$

where the sets $\mathcal{M}_h$ are defined as in Remark 3 of the previous section. We can then expand the first several special cases of these congruences using this result as follows:

$$\binom{x}{n} \equiv \frac{2(x-1)}{3}\binom{x}{n-1} - \frac{x(x-1)}{6}\binom{x}{n-2} + \frac{(n-2)(n-3)}{6}\binom{x+2}{n}[n \leq 1]_\delta \qquad \pmod{2}, \quad \text{for all } x \in \mathcal{M}_2$$

$$\begin{aligned}\binom{x}{n} &\equiv \frac{3(x-2)}{5}\binom{x}{n-1} - \frac{3(x-1)(x-2)}{20}\binom{x}{n-2} + \frac{x(x-1)(x-2)}{60}\binom{x}{n-3} \\ &\quad - \frac{(n-3)(n-4)(n-5)}{60}\binom{x+3}{n}[n \leq 2]_\delta \qquad \pmod{3}, \quad \text{for all } x \in \mathcal{M}_3\end{aligned}$$

$$\begin{aligned}\binom{x}{n} &\equiv \frac{4(x-3)}{7}\binom{x}{n-1} - \frac{(x-2)(x-3)}{7}\binom{x}{n-2} + \frac{2(x-1)(x-2)(x-3)}{105}\binom{x}{n-3} \\ &\quad - \frac{x(x-1)(x-2)(x-3)}{840}\binom{x}{n-4} \\ &\quad + \frac{(n-4)(n-5)(n-6)(n-7)}{840}\binom{x+4}{n}[n \leq 3]_\delta \qquad \pmod{4}, \quad \text{for all } x \in \mathcal{M}_4\end{aligned}$$

$$\begin{aligned}\binom{x}{n} &\equiv \frac{5(x-4)}{9}\binom{x}{n-1} - \frac{5(x-3)(x-4)}{36}\binom{x}{n-2} + \frac{5(x-2)(x-3)(x-4)}{252}\binom{x}{n-3} \\ &\quad - \frac{5(x-1)(x-2)(x-3)(x-4)}{3024}\binom{x}{n-4} \\ &\quad - \frac{x(x-1)(x-2)(x-3)(x-4)}{15120}\binom{x}{n-5} \\ &\quad - \frac{(n-5)(n-6)(n-7)(n-8)(n-9)}{15120}\binom{x+5}{n}[n \leq 4]_\delta \qquad \pmod{5}, \quad \text{for all } x \in \mathcal{M}_5.\end{aligned}$$

Corollary 4 (Reduction Formulas for $\binom{x}{n}$). *Let the functions, $k_{r,s}(x)$ be defined for all $s \geq r \geq 0$ as*

$$k_{r,s}(x) = \binom{x-r}{s-r}\binom{2r}{r}\binom{r+s}{r}^{-1}.$$

We have a corresponding addition, or lower index reduction, formula for the binomial coefficients, $\binom{x}{p+q}$, given by the following equations for integers $p, q \geq 0$:

$$\begin{aligned}\binom{x}{p+q} &= k_{0,p}(x)k_{0,q}(x) + \sum_{i=1}^{\min(p,q)} \lambda_{i+1}(x)k_{i,p}(x)k_{i,q}(x) \\ &= \binom{x}{p}\binom{x}{q} + \sum_{i=1}^{\min(p,q)} \frac{(-1)^i \binom{x-i}{p-i}\binom{x-i}{q-i}\binom{x+i}{i}\binom{x}{i}\binom{2i}{i}^2}{2 \cdot \binom{p+i}{i}\binom{q+i}{i}\binom{2i-1}{i-1}^2}.\end{aligned}$$

We can again compare the statements of the two summation formulas in the corollary to the binomial coefficient expansions in the statement of Lucas' theorem from the introduction modulo any prime p.

4. Conclusions

We have established the forms of two new Jacobi-type J-fraction expansions providing ordinary generating functions for the binomial coefficients, $\binom{x+n}{n}$ and $\binom{x}{n}$, for any x and all $n \geq 0$. The key ingredients to the proofs of these series expansions are the rationality of the h^{th} convergents, $\mathrm{Conv}_{i,h}(x,z)$, for all $h \geq 1$ and closed-form formulas for the corresponding numerator and denominator function sequences, which are finite-degree polynomials in z with $\deg_z\{\mathrm{P}_{i,h}(x,z)\} = h-1$ and $\deg_z\{\mathrm{Q}_{i,h}(x,z)\} = h$ for all $h \geq 1$ when $i = 1, 2$. The finite difference equations implied by the rationality of the h^{th} convergent functions at each h lead to new exact formulas for, and new congruences satisfied by, the two binomial coefficient variants studied within the article. We note that unlike the J-fraction expansions enumerating the generalized factorial functions studied in [9], the h^{th} modulus, $\lambda_h(x)$, of these J-fractions defined in Corollary 1 is not strictly integer-valued for all x and h, which complicates the formulations of the new congruence properties for the binomial coefficient variants considered in the article.

We also compare the two forms of the addition, or *reduction* of index, formulas for these coefficients stated in Corollary 2 and in Corollary 4 to the forms of Lucas's theorem and to the prime power congruences cited in the introduction, which similarly reduce the upper and lower indices to these sequences with respect to powers of prime moduli. New research directions based on the results in this article include further study of the binomial coefficient congruences given in Section 2 and Section 3 modulo composite integers $h \geq 4$. In particular, one direction for future research based on these topics is to find exact formulas for doubly-indexed sequences of multipliers, $\widetilde{m}_{i,h,n}$, such that $\binom{x+n}{n} \equiv \widetilde{m}_{1,h,n}[z^n]\,\mathrm{Conv}_{1,h}(x,z) \pmod{h}$ and $\binom{x}{n} \equiv \widetilde{m}_{2,h,n}[z^n]\,\mathrm{Conv}_{2,h}(x,z) \pmod{h}$ for all integers $x, n \geq 0$. This application is surely a non-trivial task worthy of further study and consideration based on the results we prove here.

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