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ON THE NUMBER OF HYPER m-ARY PARTITIONS

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Abstract

For any $m > 2$, we provide a formula for the number of hyper m-ary partitions of a number n in terms of the number of hyperbinary paritions of integers obtained from the base-m representation of n. By applying a known formula for hyperbinary partitions, we are able to give a formula for the number of hyper m-ary partitions of n in terms of the the base-m representation of n. Finally, we show how to use our result to provide alternate proofs of known results about hyper m-ary partitions.

1. Introduction

Throughout this note, we let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ represent the set of natural numbers. For any $m \geq 2$, every natural number n has a unique base-m representation of the form $n = n_0 + n_1 m + \cdots + n_k m^k$ with $n_k \neq 0$. We express this more compactly as $n = (n_0, n_1, \dots, n_k)_m$ and use the convention that $n_i = 0$ if $i > k$.

For $m \geq 2$, we say a partition of $n \in \mathbb{N}$ is a hyper m-ary partition if the parts are all a power of m and each power of m appears at most m times. We let $h_m(n)$ represent the number of hyper m-ary partitions of n. For instance, the hyper 2-ary partitions (also called hyperbinary partitions) of 8 are

$$8, \quad 4+4, \quad 4+2+2, \quad 4+2+1+1$$

so that $h_2(8) = 4$.

Courtright and Sellers [4] describe some properties and congruences associated with hyper m-ary partitions following a result by Calkin and Wilf [3] demonstrating a connection between hyperbinary partitions and a tree that is used to enumerate the rationals. More recently, both Flowers and Lockard [7] and Eom, Jeong, and Song [5] have independently investigated trees associated with general hyper m-ary partitions. In [5], the authors also describe a variety of extensions of the work of Courtright and Sellers. Furthermore, in a separate paper, Flowers and Lockard [6] show that the sequence of hyper m_1-ary partitions is a subsequence of the sequence of hyper m_2-ary partitions whenever $m_1 \leq m_2$. In particular, they show $h_{m_1}(n) = h_{m_2}(\phi_{m_1,m_2}(n))$ where $\phi_{m_1,m_2}(n) = (n_0, n_1, \ldots, n_k)_{m_2}$ when $n = (n_0, n_1, \ldots, n_k)_{m_1}$. Consequently, it seems that the number of hyper m-ary partitions of n should somehow only depend on the 0 and 1 digits in the base-m representation of n. We extend the ideas of [5] to prove that this is in fact the case.

In Section 2, we introduce the relevant definitions and notation necessary to state our main result. In Section 3, we describe known facts about hyper m-ary partitions that allow us to prove our main result.

2. Preliminaries and Statement of the Main Result

Let $L = (L_0, L_1, \ldots, L_k)$ be a list of nonnegative integers. We say a substring of the form $(L_i, L_{i+1}, \ldots, L_j)$ is a *binary substring* if $L_\ell \in \{0, 1\}$ for all $i \leq \ell \leq j$. Now we say $(L_i, L_{i+1}, \ldots, L_j)$ is a *hyper-substring* if the following three conditions hold:

1. $(L_i, L_{i+1}, \ldots, L_j)$ is a binary substring;
2. $L_i = 0$; and
3. There is no binary substring of the form $(L_a, L_{a+1}, \ldots, L_i, L_{i+1}, \ldots, L_j)$ with $a < i$ and $L_a = 0$ or of the form $(L_i, L_{i+1}, \ldots, L_j, \ldots, L_{b-1}, L_b)$ with $b > j$.

Next, if $L = (L_0, L_1 \ldots, L_k)$ is a list and $(L_i, L_{i+1}, \ldots, L_j)$ is a hyper-substring of L, we say $(L_i, L_{i+1}, \ldots, L_j)$ is an *initial hyper-substring* if $i = 0$ and is a *terminal hyper-substring* if $j = k$. We note that if L is a binary list with $L_0 = 0$, then L is itself both an initial and terminal hyper-substring, and thus L itself is the only hyper-substring of L. We let $H(L)$ represent the multiset of all hyper-substrings of L. We note that $H(L)$ contains at most one terminal and at most one initial hyper-substring.

Example 2.1. The hyper-substrings of the list

$$L = (\underbrace{0, 1}, 2, 8, 5, 4, 1, \underbrace{0, 1}, 3, 4, \underbrace{0, 1, 1}, 13, \underbrace{0, 1, 1})$$

are highlighted. Thus $H(L) = \{(0, 1), (0, 1), (0, 1, 1), (0, 1, 1)\}$. We allow both substrings $(0, 1)$ and $(0, 1, 1)$ to appear twice since $H(L)$ is a multiset. Moreover, note that the first $(0, 1)$ is an initial hyper-substring and the last $(0, 1, 1)$ is a terminal hyper-substring.

Next, let $L = (L_0, L_1, \ldots, L_k)$ be a fixed list. We define a map $e : H(L) \to \mathbb{N}$ as follows:

$$e(\alpha) = e((\alpha_0, \alpha_1, \ldots, \alpha_r)) = \begin{cases} (\alpha_0, \alpha_1, ..., \alpha_r, 1)_2 & \text{if } \alpha \text{ is not terminal;} \\ (\alpha_0, \alpha_1, ..., \alpha_r)_2 & \text{if } \alpha \text{ is terminal.} \end{cases}$$

In other words, we interpret a terminal hyper-substring as a base-2 representation of an integer and interpret every other hyper-substring as a base-2 representation of an integer after appending 1 to the substring. Then, we call the image of e, $B(L) := e(H(L))$, the multiset of *binary components* of L.

Example 2.2. We continue our example above with

$$L = (0, 1, 2, 8, 5, 4, 1, 0, 1, 3, 4, 0, 1, 1, 13, 0, 1, 1).$$

We apply e to $H(L)$ to get $e((0, 1)) = (0, 1, 1)_2 = 6$, $e((0, 1)) = (0, 1, 1)_2 = 6$, $e((0, 1, 1)) = (0, 1, 1, 1)_2 = 14$, and $e((0, 1, 1)) = (0, 1, 1)_2 = 6$ (since this last substring is terminal). Therefore, $B(L) = \{6, 6, 14, 6\}$, where we again allow 6 three times since $B(L)$ is a multiset.

Theorem 2.3. *Let $m \geq 2$ and let $n \in \mathbb{N}$ where $n = (n_0, n_1, \ldots, n_k)_m$. Then the number of hyper m-ary partitions of n, $h_m(n)$, is given by*

$$h_m(n) = \prod_{b \in B(N)} h_2(b),$$

where $N = (n_0, n_1, \ldots, n_k)$, and where we interpret the empty product as 1.

Example 2.4. Let $m = 16$ and $n = 314536187658680762896$. Then, the base 16 representation of n is $n = (0, 1, 2, 8, 5, 4, 1, 0, 1, 3, 4, 0, 1, 1, 13, 0, 1, 1)_{16}$. We note that the list $N = (0, 1, 2, 8, 5, 4, 1, 0, 1, 3, 4, 0, 1, 1, 13, 0, 1, 1)$ is the list L from the previous two examples. A computation using the recursion given in the next section shows $h_{16}(n) = 108$. We also can check that $\prod_{b \in B(N)} h_2(b) = h_2(6) \cdot h_2(6) \cdot h_2(14) \cdot h_2(6) = 3 \cdot 3 \cdot 4 \cdot 3 = 108$ (using data from sequence A002487 in the Online Encyclopedia of Integer Sequences [9] and the fact that $h_2(n) = A002487(n + 1)$).

For any integer, z, let $\overline{z} \in \{0, 1\}$ be the element congruent to z modulo 2. Northshield [8] shows that $h_2(b) = \sum_{k=0}^{b} \overline{\binom{k}{b-k}}$, where $\binom{n}{k}$ represents a binomial coefficient. Moreover, Ball et. al. [1] demonstrate that

$$\overline{\binom{k}{b-k}} = 1 \text{ if and only if } b - k \ll_2 k,$$

where $b - k \ll_2 k$ represents binary digital dominance: $x \ll_2 y$ if and only if $x_i \leq y_i$ for all i when $x = (x_0, x_1, \ldots, x_f)_2$ and $y = (y_0, y_1, \ldots, y_f)_2$ (consult [2] for more information about this partial order). Using these ideas, we get the following corollary to Theorem 2.3.

Corollary 2.5. *Let $m \geq 2$ and let $n \in \mathbb{N}$ where $n = (n_0, n_1, \ldots, n_k)_m$. Then the* *number of hyper m-ary partitions of n is given by*

$$\begin{aligned} h_m(n) &= \prod_{b \in B(N)} \sum_{k=0}^{b} \overline{\binom{k}{b-k}} \\ &= \prod_{b \in B(N)} \sum_{k=0}^{b} [b-k \ll_b k] \\ &= \prod_{b \in B(N)} \sum_{\substack{0 \le k \le b \\ b-k \ll_2 k}} 1, \end{aligned}$$

where $N = (n_0, n_1, \ldots, n_k)$ *and* $[P]$ *represents the Iverson bracket (which outputs* 1 *if* P *is true and* 0 *if* P *is false).*

The elements $b \in B(N)$ have base-2 representations arising simply from the base-m representation of n, so the final part of this formula only requires the base-m representation of n.

The next corollary can be derived from the recurrence given below in Lemma 3.1, but now follows directly from Theorem 2.3.

Corollary 2.6. *Suppose that the base-m representation of n does not use the digit* 0*. Then* $h_m(n) = 1$.

Proof. By assumption $B(N) = \emptyset$ where N is the list of digits from the base-m representation of n. □

Theorem 2.3 also provides alternate proofs of the result given by Flowers and Lockard and one of the results given by Courtright and Sellers.

Corollary 2.7 (Flowers and Lockard). *Let* $N = (n_0, n_1, \ldots, n_k)$ *be a list of nonnegative integers and let* $m := \max\{n_i \mid 0 \le i \le k\}$*. Then, for any* $m_2 \ge m_1 > m$*, we have*

$$h_{m_1}(n) = h_{m_2}(n')$$

where $n = (n_0, n_1, \ldots, n_k)_{m_1}$ *and* $n' = (n_0, n_1, \ldots, n_k)_{m_2}$*.*

Corollary 2.8 (Courtright and Sellers). *Let* $m \ge 3$ *and* $j \ge 1$*. For all* $2 \le k \le m-1$ *and all* $n \ge 0$

$$h_m(km^{j-1} + nm^j) = jh_m(n).$$

Proof. Let $n = (n_0, n_1, \ldots, n_k)_m$, then $nm^j = (0, 0, \ldots, 0, n_0, n_1, \ldots, n_k)_m$, and so $x = km^{j-1} + nm^j = (0, 0, \ldots, 0, k, n_0, n_1, \ldots, n_k)_m$. We thus see that $B(X) = \{2^{j-1}\} \cup B(N)$ where $X = (0, 0, \ldots, 0, k, n_0, n_1, \ldots, n_k)$ and $N = (n_0, n_1, \ldots, n_k)$. Thus

$$h_m(x) = \prod_{b \in B(X)} h_2(b) = h_2(2^{j-1}) \cdot \prod_{b \in B(N)} h_2(b) = h_2(2^{j-1}) \cdot h_m(n),$$

and it is straightforward to check (see Lemma 3.1 below) that $h_2(2^{j-1}) = j$. □

Likewise, it is readily verified from Lemma 3.1 below that $h_2(2^j \cdot n) = jh_2(n - 1) + h_2(n)$. With this fact, we have the following corollary in general.

Corollary 2.9 (Eom, Jeong, Sohn). *Let n and j be positive integers. Then*

$$h_m(nm^j - 1) = h_m(n - 1).$$

Proof. As in the previous proof, we note that $nm^j = (0, 0, \dots, 0, n_0, n_1, \dots, n_k)_m$ when $n = (n_0, n_1, \dots, n_k)_m$. Now, if i is the least index with $n_i \neq 0$, then, $nm^j - 1 = (m - 1, m - 1, \dots, m - 1, m - 1, \dots, m - 1, n_i - 1, n_{i+1}, \dots, n_k)_m$ and $n - 1 = (m-1, \dots, m-1, n_i-1, n_{i+1}, \dots, n_k)_m$. From this, we can verify that $B(nm^j - 1) = B(n - 1)$, and then the result follows from Theorem 2.3. □

Corollary 2.9, coupled with Lemma 3.1 from the next section, allow Eom, Jeong, and Sohn [5] to prove, by induction, a generalization of Corollary 2.8 which states that $h_m(nm^j) = jh_m(n - 1) + h_m(n)$ for all positive integers n and j.

3. Proof of Theorem 2.3

We let $m \geq 2$ and $n \in \mathbb{N}$ with $n = (n_0, n_1, n_2, \dots, n_k)_m$. If $n = qm + r$ where $0 \leq r < m$, then $n_0 = r$ and $q = (n_1, n_2, \dots, n_k)_m$. Equipped with this knowledge, we have the following lemmas, which allow us to prove our result.

Lemma 3.1 (Courtright and Sellers). *Let $m \geq 2$. Then $h_m(0) = 1$, and for $n \geq 1$, if $n = mq + r$ with $0 \leq r < m$, then*

$$h_m(n) = \begin{cases} h_m(q) & \text{if } 0 < r < m, \\ h_m(q) + h_m(q - 1) & \text{if } r = 0. \end{cases}$$

Lemma 3.2. *Let $L = (L_0, L_1, \dots, L_k)$ and $L' = (X_0, L_1, \dots, L_k)$ where $X_0 \neq 0$ and $L_0 \neq 0$, then $B(L) = B(L')$.*

Proof. This is clear by definition since neither L nor L' have an initial hyper-substring, which means $B(L) = B((L_1, \dots, L_k)) = B(L')$. □

Lemma 3.3. *If $L = (0, L_1, L_2, \dots, L_k)$ is a binary list ($L_i \in \{0, 1\}$ for all i), then $L' = (L_1, L_2, \dots, L_k)$ is either the empty list or the list of all 1's (and thus has no hyper-substring), or L' has a unique hyper-substring.*

Proof. If $L' = (L_1, L_2, \dots, L_k)$ is not empty and is not the list of all 1's, then let j be the minimal index such that $L_j = 0$. Then $(L_j, \dots, L_k)$ is the only hyper-substring of L'. □

According to the previous lemma, if $L = (0, L_1, L_2, \ldots, L_k)$ is a binary string, then we define $\tilde{L}$ be the unique hyper-substring of $L' = (L_1, L_2, \ldots, L_k)$ (where $\tilde{L}$ is interpreted as the empty list if L' is empty or the list of all 1's). According the proof, either $\tilde{L} := ()$ or $\tilde{L} := (L_j, L_{j+1}, \ldots, L_k)$ where j is the least index $1 \leq j \leq k$ such that $L_j = 0$.

Lemma 3.4. *Let $N = (n_0, n_1, n_2, \ldots, n_k)$ be a list, and suppose that N has an initial hyper-substring $\alpha = (n_0, n_1, ..., n_i)$ where $i < k$. Then $h_2(e(\alpha)) = h_2(e(\tilde{\alpha})) + h_2(e(\tilde{\alpha}) - 1)$ where $\tilde{\alpha}$ is the hyper-substring of α afforded by Lemma 3.3.*

Proof. This is a restatement of Lemma 3.1 when $m = 2$ in terms of the initial hyper-substring of N since $e(\alpha) = 2e(\alpha')$ where $\alpha' = (n_1, ..., n_i)$ along with the fact that $e(\alpha') = e(\tilde{\alpha})$, which follows either by Lemma 3.2 or the fact that $\alpha' = \tilde{\alpha}$ if $n_1 = 0$. □

Lemma 3.5. *Let $N = (n_0, n_1, n_2, \ldots, n_k)$ be a list. If $Q = (n_1, n_2, \ldots, n_k)$, then exactly one of the following occurs.*

- *N does not have an initial hyper-substring, and so $B(N) = B(Q)$;*
- *N has an initial hyper-substring, $\alpha = (n_0, n_1, ..., n_i)$, and*

$$B(N) \setminus \{e(\alpha)\} = B(Q) \setminus \{e(\tilde{\alpha})\}$$

where $\tilde{\alpha}$ is the hyper-substring afforded by Lemma 3.3 and is empty if no such substring exists.

Proof. This is clear by construction. □

Lemma 3.6. *Let $m \geq 2$ be fixed. Let $Z = (z_1, \ldots, z_k)$ be a list with $z_a \geq 2$ for some a. We also assume Z has an initial hyper-substring $\beta = (z_1, \ldots, z_i)$ with $i \geq 1$. Let j be the minimal index with $z_j \geq 1$ and let*

$$Y = (\underbrace{m-1, m-1, \ldots, m-1}_{j-1}, z_j - 1, z_{j+1}, \ldots, z_k).$$

1. *If $0 \leq j \leq i$, then $z_j = 1$ and $B(Z) \setminus \{e(\beta)\} = B(Y) \setminus \{e(\beta')\}$ where $\beta' = (z_j - 1, z_{j+1}, \ldots, z_i) = (0, z_{j+1}, \ldots, z_i)$ is the first hyper-substring of Y. Moreover, in this case $h_2(e(\beta) - 1) = h_2(e(\beta'))$.*

2. *If $j > i$, then $j = i + 1$, and $z_j = z_{i+1} \geq 2$, so that $z_j - 1 \geq 1$. Then*

$$B(Z) \setminus \{e(\beta)\} = B(Y)$$

and $h_2(e(\beta) - 1) = h_2(2^{j-1} - 1) = 1$.

Proof. For part (1), we clearly have $B(Z) \setminus \{e(\beta)\} = B(Y) \setminus \{e(\beta')\}$ since β is the initial hyper-substring of Z and β' is the first hyper-substring of Y. We also know that $\beta = (0, 0, \ldots, 0, 1, z_{j+1} ..., z_i)$ and $\beta' = (0, z_{j+1}, ..., z_i)$. Then $e(\beta) - 1 = (1, 1, \ldots, 1, 0, z_{j+1}, \ldots, z_i, 1)_2$, and $e(\beta') = (0, z_{j+1}, \ldots, z_i, 1)_2$. Apply Lemma 3.1 repeatedly to conclude that $h_2(e(\beta) - 1) = h_2(e(\beta'))$.

For part (2), we again clearly have $B(Z) \setminus \{e(\beta)\} = B(Y)$ since β is the initial hyper-substring of Z and Y has no hyper-substrings occurring before index i. In this case we have $\beta = (0, 0, \ldots, 0)$ so that $e(\beta) = (0, 0, \ldots, 0, 1)_2 = 2^{j-1}$ and $e(\beta) - 1 = 2^{j-1} - 1$. Finally, the recursion from Lemma 3.1 tells us that $h_2(2^{j-1} - 1) = 1$. □

We are now ready to prove our main theorem. For $m \geq 2$, we let $P_m(n)$ be the function defined by

$$P_m(n) = \prod_{b \in B(N)} h_2(b),$$

where $N = (n_0, n_1, \ldots, n_k)$ when $n = (n_0, n_1, \ldots, n_k)_m$. Using this terminology, to prove Theorem 2.3, we must show that $P_m(n) = h_m(n)$.

Proof of Theorem 2.3. We will show that P_m satisfies the same recursion as h_m. Let $n = (n_0, n_1, \ldots, n_k)_m$ and assume that $n = mq + r$ with $0 \leq r < m$. Then, $n_0 = r$ and $q = (n_1, n_2 \ldots, n_k)_m$. Furthermore, let $N = (n_0, n_1, n_2, \ldots, n_k)$ and $Q = (n_1, n_2, \ldots, n_k)$. We consider the following two cases.

`Case 1.` Assume $r = n_0 \neq 0$ so that Lemma 3.5 implies that $B(N) = B(Q)$. Thus

$$P_m(n) = \prod_{b \in B(N)} h_2(b) = \prod_{b \in B(Q)} h_2(b) = P_m(q).$$

`Case 2.` Assume $r = n_0 = 0$. Then let $\alpha = (n_0, n_1, \ldots, n_i)$ be the initial hyper-substring of N. Then we have two subcases.

`Subcase a.` Assume that $i < k$ and then let j be the minimal index such that $n_j \geq 1$; we note that $q - 1 = (m-1, m-1, \ldots, m-1, n_j - 1, n_{j+1}, \ldots, n_k)_m$. Let Q' be the corresponding list $Q' = (m-1, m-1 \ldots, m-1, n_j - 1, n_{j+1}, \ldots, n_k)$. By Lemmas 3.5 and 3.4, we have

$$\begin{aligned} P_m(n) = \prod_{b \in B(N)} h_2(b) &= h_2(e(\alpha)) \cdot \prod_{b \in B(Q) \setminus \{e(\tilde{\alpha})\}} h_2(b) \\ &= (h_2(e(\tilde{\alpha})) + h_2(e(\tilde{\alpha}) - 1)) \cdot \prod_{b \in B(Q) \setminus \{e(\tilde{\alpha})\}} h_2(b) \\ &= \prod_{b \in B(Q)} h_2(b) + (h_2(e(\tilde{\alpha}) - 1)) \cdot \prod_{b \in B(Q) \setminus \{e(\tilde{\alpha})\}} h_2(b) \\ &= \prod_{b \in B(Q)} h_2(b) + \prod_{b \in B(Q')} h_2(b) \\ &= P_m(q) + P_m(q-1). \end{aligned}$$

where the second-to-last equality holds by Lemma 3.6 (the two different cases may occur but result in the same outcome), where Q assumes the role of Z and Q' assumes the role of Y.

Subcase b. Assume that $i = k$ so that N is a binary list. Let

$$n' = (n_0, n_1, n_2, \ldots, n_k)_2 \text{ and } q' = (n_1, n_2, \ldots, n_k)_2.$$

Next, let j be the minimal index with $n_j = 1$ (so that $j \geq 1$). Then $n = (0, 0, \ldots, 0, 1, n_{j+1}, \ldots, n_k)_m$ and $n' = (0, 0, \ldots, 0, 1, n_{j+1}, \ldots, n_k)_2$ so that $q = (0, \ldots, 0, 1, n_{j+1}, \ldots, n_k)_m$ and $q' = (0, \ldots, 0, 1, n_{j+1}, \ldots, n_k)_2$. Thus, we have

$$\begin{aligned} q - 1 &= (m-1, \ldots, m-1, 0, n_{j+1}, \ldots, n_k)_m, \text{ and} \\ q' - 1 &= (1, \ldots, 1, 0, n_{j+1}, \ldots, n_k)_2. \end{aligned}$$

Let L_1 and L_2 be the corresponding lists; that is,

$$\begin{aligned} L_1 &= (m-1 \ldots, m-1, 0, n_{j+1}, \ldots, n_k), \text{ and} \\ L_2 &= (1, \ldots, 1, 0, n_{j+1}, \ldots, n_k). \end{aligned}$$

We note that if $\overline{L}$ is defined by $\overline{L} = (0, n_{j+1}, \ldots, n_k)$, then note that $B(L_1) = B(\overline{L}) = B(L_2)$, which follows by definition or by Lemma 3.2. We also know that $B(\overline{L}) = \{e(\overline{L})\} = \{\ell\}$, where $\ell = (0, n_{j+1}, \ldots, n_k)_2$ since $\overline{L}$ is both an initial and terminal hyper-substring of $\overline{L}$.

Now, since N is a binary list and is the initial and terminal hyper-substring of itself, we have $P_m(n) = h_2(e(N)) = h_2(n') = h_2(q') + h_2(q'-1)$ by Lemma 3.1. Furthermore, the same recursion implies that $h_2(q'-1) = h_2(\ell)$ and that $h_2(q') = \prod_{b \in B(Q)} h_2(b)$ (since there is only one binary component of this list for q'). Hence, we see that

$$\begin{aligned} P_m(q) + P_m(q-1) &= \prod_{b \in B(Q)} h_2(b) + \prod_{b \in B(L_1)} h_2(b) \\ &= h_2(q') + h_2(\ell) \\ &= h_2(q') + h_2(q'-1) \\ &= h_2(n') \\ &= h_2(e(N)) = P_m(n). \end{aligned}$$

Thus, we have seen that $P_m(mq) = P_m(q) + P_m(q-1)$ and $P_m(mq+r) = P_m(q)$ when $1 \leq r \leq m-1$. Finally, we check that $P_m(0) = 1$, which is the same initial condition for h_m. Since h_m and P_m satisfy the same recursion with the same initial conditions, we must have $h_m(n) = P_m(n)$ for all n, which proves Theorem 2.3. □

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