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AN IMPROVEMENT OF AN INEQUALITY OF OCHEM AND RAO CONCERNING ODD PERFECT NUMBERS

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Abstract

Let $\Omega(n)$ denote the total number of prime divisors of n (counting multiplicity) and let $\omega(n)$ denote the number of distinct prime divisors of n. Various inequalities have been proved relating $\omega(N)$ and $\Omega(N)$ when N is an odd perfect number. We improve on these inequalities. In particular, we show that if $(3, N) = 1$, then $\Omega(N) \geq \frac{8}{3}\omega(N) - \frac{7}{3}$, and if $3 \mid N$ then $\Omega(N) \geq \frac{21}{8}\omega(N) - \frac{39}{8}$.

1. Introduction

Let $\Omega(n)$ denote the total number of prime divisors of n (counting multiplicity) and let $\omega(n)$ denote the number of distinct prime divisors of n.

Let N be an odd perfect number. Ochem and Rao [2] have proved that N must satisfy

$$\Omega(N) \geq \frac{18\omega(N) - 31}{7} \tag{1}$$

and

$$\Omega(N) \geq 2\omega(n) + 51. \tag{2}$$

Note that Ochem and Rao's second inequality is stronger than the first as long as $\omega(N) \leq 96$. Nielsen [1] has shown that $\omega(n) \geq 10$.

In this note, we improve Ochem and Rao's first inequality. In particular we have:

Theorem 1. *If N is an odd perfect number with $3 \nmid N$ then,*

$$\Omega(N) \geq \frac{8}{3}\omega(N) - \frac{7}{3}. \tag{3}$$

If N is an odd perfect number with $3 \mid N$ then,

$$\Omega(N) \geq \frac{21}{8}\omega(N) - \frac{39}{8}. \tag{4}$$

Inequality 3 is always better than inequality 1, while inequality 4 becomes stronger than inequality 1 when $\omega(N) \geq 9$ and thus for all odd perfect numbers by Nielsen's result.

Note that, if one only uses Ochem and Rao's original system of inequalities but assumes that $3 \nmid N$, then one can improve the constant term of inequality 1 but one still has a linear coefficient of 18/7. Thus, both cases here do represent non-trivial improvement.

2. Proof of the Main Results

Our method of proof is very similar to that of Ochem and Rao. They created a series of linear inequalities involving the number of different types of prime factors (both total and distinct) of N, and showed that the linear system in question forced a certain lower bound. We will use a similar method, but with additional inequalities.

Euler proved that N must have the form $N = q^e m^2$ where q is a prime such that $q \equiv e \equiv 1 \pmod 4$, $(q, m) = 1$. Traditionally q is called the special prime. Note that from Euler's result one immediately has $\Omega(N) \geq 2\omega(N) - 1$. For the remainder of this paper, we will assume that N is an odd perfect number with q, e and m given as above. We we will write $\Omega(N)$ as just Ω and $\omega(N)$ as ω.

The following Lemma is the primary insight that allows us to have a system that is tighter than that of Ochem and Rao.

Lemma 1. *Let a and b be distinct odd primes and p a prime such that $p \mid (a^2+a+1)$ and $p \mid (b^2+b+1)$. If $a \equiv b \equiv 2$ (mod 3), then $p \leq \frac{a+b+1}{5}$. If $a \equiv b \equiv 1$ (mod 3) then $p \leq \frac{a+b+1}{3}$.*

Proof. We will prove this when $a \equiv b \equiv 2 \pmod 3$ (the 1 mod 3 proof is nearly identical). Without loss of generality, assume that $a > b$. Note that one must have $p \equiv 1 \pmod 3$. We have

$$p \mid (a^2 + a + 1) - (b^2 + b + 1) = (a - b)(a + b + 1).$$

So either $p \mid (a - b)$ or $p \mid (a + b + 1)$. In the first case, we note that $6 \mid (a - b)$, so $6p \mid (a - b)$ and thus

$$p \leq \frac{a-b}{6} \leq \frac{a+b+1}{5}.$$

In the second case, we have that $p \mid (a + b + 1)$ gives us $pk = a + b + 1$ for some k with $k \equiv 5 \pmod 6$ and so $k \geq 5$. Thus,

$$p \leq \frac{a+b+1}{5}.$$

Thus, in both cases we have the desired inequality. $\square$

Note that we do not have a version of Lemma 1 when $a \equiv 1 \pmod 3$ and $b \equiv 2 \pmod 3$. In that case, one cannot do better $p \leq a+b+1$. The specific example of case of $a = 7$, $b = 11$ and $p = 19$ shows that one can in fact have $p = a+b+1$. In order to use our lemma we need an inequality strong enough that we can conclude that $p < \max(a,b)$, and so $p \leq a+b+1$. will be insufficient.

We will also need the following result from [2].

Lemma 2 (Ochem and Rao). *Let p, q and r be positive integers. If $p^2+p+1 = r$ and $q^2+q+1 = 3r$, then p is not an odd prime.*

Now, for the proof of the main result we will write

$$S = \prod_{p||m, p \neq 3} p$$

and

$$T = \prod_{p^2|m, p \neq 3} p.$$

We will set $S = S_1 S_2 S_3$, where a prime p appears in S_i for $1 \leq i \leq 2$ if $\sigma(p^2)$ is a product of i primes. S_3 will contain all the primes of S where $\sigma(p^2)$ has at least 3 prime factors. We will write $s = \omega(S)$ and write $t = \omega(T)$. We define s_1, s_2 and s_3 similarly.

We will write $S_{i,j}$ to be the set of primes in S_i which are $j \pmod 3$. We will as in the previous paragraph use lower case letters to denote the number of primes. That is, we set $s_{i,j} = \omega(S_{i,j})$ and will note that $s_{1,1} = 0$. Thus, we do not need to concern ourselves with this split for S_1; since all primes in S_1 are $2 \pmod 3$ there is no need to split S_1 further.

We are now in a position to start writing down our inequalities for various terms. The special exponent is at least 1:

$$1 \leq e. \tag{5}$$

We have the following straightforward equation from the definitions of s_1, s_2, and s_3 as having to sum up to s:

$$s = s_1 + s_2 + s_3. \tag{6}$$

Similarly, we have

$$s_2 = s_{2,1} + s_{2,2}, \tag{7}$$

and we have

$$s_3 = s_{3,1} + s_{3,2}. \tag{8}$$

We define f_4 as the number of prime divisors (counting multiplicity) in N which are not the special prime and are raised to at least the fourth power. From a simple counting argument we obtain

$$e + f_3 + 2s + f_4 \leq \Omega. \tag{9}$$

Lemma 3. *We have*

$$s_1 + s_{2,2} \leq t + s_{2,1} + s_{3,1} + 1 \tag{10}$$

and

$$s_1 \leq t + s_{3,1} + 1. \tag{11}$$

Proof. The claim will be proven if we can show that each p in $S_1 S_{2,i}$ has to contribute at least one distinct prime (since then one of the primes may be the special prime and the other primes must all contribute to t). This is trivial for S_1 (and was used in Ochem and Rao's result). We will show this by showing that no two prime divisors of $S_1 S_{2,i}$ can contribute the same largest prime of the primes they contribute. We have two cases we need to consider, both primes arising from $S_{2,i}$, or one arising from $S_{2,i}$ and one arising from S_1.

`Case I:` Assume we have two prime divisors of $S_{2,2}$, a and b with $a > b$, and assume they have some shared prime factor q which divides both $\sigma(a^2) = a^2 + a + 1$ and $\sigma(b^2) = b^2 + b + 1$. Since we have $a \equiv b \equiv 2 \pmod 3$, we may apply Lemma 1 to conclude that that

$$q \leq \frac{a+b+1}{5} \leq \frac{3a}{5}.$$

Thus,

$$\frac{\sigma(a^2)}{q} > \frac{a^2}{3a/5} = 5/3a > (a^2 + a + 1)^{1/2}.$$

But $\sigma(a^2)$ only has two prime factors and this shows that the shared contributed prime cannot be the largest prime contributed by a. The case of two primes dividing $S_{2,1}$ is similar.

`Case II:` Assume that we have a prime a dividing S_1 and a prime b dividing $S_{2,i}$. If $i = 2$ then the same logic as above works. So assume that $i = 1$, and $b \equiv 1$ (mod 3). Thus we have a prime p such that $a^2 + a + 1 = p$ and $b^2 + b + 1 = 3p$ for some prime p. This is precisely the situation ruled out by Ochem and Rao's Lemma. $\square$

Next we have

$$s_{2,1} + s_{3,1} \leq f_3, \tag{12}$$

since if $x \equiv 1 \pmod 3$, then $x^2 + x + 1 \equiv 0 \pmod 3$.

We also have, by counting all the 1 (mod 3) primes which are contributed by primes in S,

$$s_1 + 2s_{2,2} + 3s_{3,2} + s_{2,1} + 2s_{3,1} \leq f_4 + e + 2s_{2,1} + 2s_{3,1}.$$

This simplifies to

$$s_1 + 2s_{2,2} + 3s_{3,2} \leq f_4 + e + s_{2,1}. \tag{13}$$

And, of course, we have

$$4t \leq f_4. \tag{14}$$

If $3 \nmid N$, then our system of equations and inequalities also includes the additional constraint $f_3 = 0$ as well as

$$\omega = s + t + 1. \tag{15}$$

Since $f_3 = 0$ we have $s_{2,1} = s_{3,1} = 0$, and after zeroing those variables we obtain inequality 3 by taking $7/9 \times (\mathbf{5}) + 2/3 \times (\mathbf{6}) + 2/3 \times (\mathbf{7}) + 2/3 \times (\mathbf{8}) + 1 \times (\mathbf{9}) + 4/9 \times (\mathbf{10}) + 2/9 \times (\mathbf{13}) + 7/9 \times (\mathbf{14}) + 8/3 \times (\mathbf{15})$ where the bold numbers represent the corresponding numbered equation or inequality.

If $3 \mid N$ we have $2 \leq f_3$ and

$$\omega = s + t + 2. \tag{16}$$

Similar to the previous case, the linear combination $3/4 \times (\mathbf{5}) + 5/8 \times (\mathbf{6}) + 5/8 \times (\mathbf{7}) + 5/8 \times (\mathbf{8}) + 1 \times (\mathbf{9}) + 1/8 \times (\mathbf{10}) + 1/4 \times (\mathbf{11}) + 1 \times (\mathbf{12}) + 1/4 \times (\mathbf{13}) + 3/4 \times (\mathbf{14}) + 21/8 \times (\mathbf{16})$ yields inequality 4.

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References

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[2] P. Ochem and M. Rao, On the number of prime factors of an odd perfect number, *Math. Comp.* **83** (2014), 2435-2439.