AN IMPROVEMENT OF AN INEQUALITY OF OCHEM AND RAO CONCERNING ODD PERFECT NUMBERS

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Received: 6/22/17, Revised: 10/29/17, Accepted: 4/14/18, Published: 5/25/18

Abstract

Let \( \Omega(n) \) denote the total number of prime divisors of \( n \) (counting multiplicity) and let \( \omega(n) \) denote the number of distinct prime divisors of \( n \). Various inequalities have been proved relating \( \omega(N) \) and \( \Omega(N) \) when \( N \) is an odd perfect number. We improve on these inequalities. In particular, we show that if \( (3, N) = 1 \), then \( \Omega(N) \geq \frac{2}{3} \omega(N) - \frac{7}{3} \), and if \( 3 \mid N \) then \( \Omega(N) \geq \frac{21}{8} \omega(N) - \frac{39}{8} \).

1. Introduction

Let \( \Omega(n) \) denote the total number of prime divisors of \( n \) (counting multiplicity) and let \( \omega(n) \) denote the number of distinct prime divisors of \( n \).

Let \( N \) be an odd perfect number. Ochem and Rao [2] have proved that \( N \) must satisfy

\[
\Omega(N) \geq \frac{18\omega(N) - 31}{7}
\]

and

\[
\Omega(N) \geq 2\omega(n) + 51.
\]

Note that Ochem and Rao’s second inequality is stronger than the first as long as \( \omega(N) \leq 96 \). Nielsen [1] has shown that \( \omega(n) \geq 10 \).

In this note, we improve Ochem and Rao’s first inequality. In particular we have:

Theorem 1. If \( N \) is an odd perfect number with \( 3 \nmid N \) then,

\[
\Omega(N) \geq \frac{8}{3} \omega(N) - \frac{7}{3}.
\]

If \( N \) is an odd perfect number with \( 3 \mid N \) then,

\[
\Omega(N) \geq \frac{21}{8} \omega(N) - \frac{39}{8}.
\]
Inequality 3 is always better than inequality 1, while inequality 4 becomes stronger than inequality 1 when $\omega(N) \geq 9$ and thus for all odd perfect numbers by Nielsen’s result.

Note that, if one only uses Ochem and Rao’s original system of inequalities but assumes that $3 \nmid N$, then one can improve the constant term of inequality 1 but one still has a linear coefficient of $18/7$. Thus, both cases here do represent non-trivial improvement.

2. Proof of the Main Results

Our method of proof is very similar to that of Ochem and Rao. They created a series of linear inequalities involving the number of different types of prime factors (both total and distinct) of $N$, and showed that the linear system in question forced a certain lower bound. We will use a similar method, but with additional inequalities.

Euler proved that $N$ must have the form $N = q^e m^2$ where $q$ is a prime such that $q \equiv e \equiv 1 \pmod{4}$, $(q, m) = 1$. Traditionally $q$ is called the special prime. Note that from Euler’s result one immediately has $\Omega(N) \geq 2\omega(N) - 1$. For the remainder of this paper, we will assume that $N$ is an odd perfect number with $q$, $e$ and $m$ given as above. We will write $\Omega(N)$ as just $\Omega$ and $\omega(N)$ as $\omega$.

The following Lemma is the primary insight that allows us to have a system that is tighter than that of Ochem and Rao.

**Lemma 1.** Let $a$ and $b$ be distinct odd primes and $p$ a prime such that $p \mid (a^2 + a + 1)$ and $p \mid (b^2 + b + 1)$. If $a \equiv b \equiv 2 \pmod{3}$, then $p \leq \frac{a+b+1}{3}$. If $a \equiv b \equiv 1 \pmod{3}$ then $p \leq \frac{a+b+1}{3}$.  

**Proof.** We will prove this when $a \equiv b \equiv 2 \pmod{3}$ (the $1 \pmod{3}$ proof is nearly identical). Without loss of generality, assume that $a > b$. Note that one must have $p \equiv 1 \pmod{3}$. We have

$$p \mid (a^2 + a + 1) - (b^2 + b + 1) = (a - b)(a + b + 1).$$

So either $p \mid (a - b)$ or $p \mid (a + b + 1)$. In the first case, we note that $6 \mid (a - b)$, so $6p \mid (a - b)$ and thus

$$p \leq \frac{a - b}{6} \leq \frac{a + b + 1}{5}.$$ 

In the second case, we have that $p \mid (a + b + 1)$ gives us $pk = a + b + 1$ for some $k$ with $k \equiv 5 \pmod{6}$ and so $k \geq 5$. Thus,

$$p \leq \frac{a + b + 1}{5}.$$ 

Thus, in both cases we have the desired inequality. $\square$
Note that we do not have a version of Lemma 1 when \( a \equiv 1 \pmod{3} \) and \( b \equiv 2 \pmod{3} \). In that case, one cannot do better \( p \leq a + b + 1 \). The specific example of case of \( a = 7, b = 11 \) and \( p = 19 \) shows that one can in fact have \( p = a + b + 1 \). In order to use our lemma we need an inequality strong enough that we can conclude that \( p < \max(a, b) \), and so \( p \leq a + b + 1 \). will be insufficient.

We will also need the following result from [2].

**Lemma 2 (Ochem and Rao).** Let \( p, q \) and \( r \) be positive integers. If \( p^2 + p + 1 = r \) and \( q^2 + q + 1 = 3r \), then \( p \) is not an odd prime.

Now, for the proof of the main result we will write

\[
S = \prod_{p \mid m, p \neq 3} p
\]

and

\[
T = \prod_{p^2 \mid m, p \neq 3} p.
\]

We will set \( S = S_1S_2S_3 \), where a prime \( p \) appears in \( S_1 \) for \( 1 \leq i \leq 2 \) if \( \sigma(p^2) \) is a product of \( i \) primes. \( S_5 \) will contain all the primes of \( S \) where \( \sigma(p^2) \) has at least 3 prime factors. We will write \( s = \omega(S) \) and write \( t = \omega(T) \). We define \( s_1, s_2 \) and \( s_3 \) similarly.

We will write \( S_{i,j} \) to be the set of primes in \( S_i \) which are \( j \pmod{3} \). We will as in the previous paragraph use lower case letters to denote the number of primes. That is, we set \( s_{i,j} = \omega(S_{i,j}) \) and will note that \( s_{1,1} = 0 \). Thus, we do not need to concern ourselves with this split for \( S_1 \); since all primes in \( S_1 \) are \( 2 \pmod{3} \) there is no need to split \( S_1 \) further.

We are now in a position to start writing down our inequalities for various terms. The special exponent is at least 1:

\[
1 \leq e. \tag{5}
\]

We have the following straightforward equation from the definitions of \( s_1, s_2, \) and \( s_3 \) as having to sum up to \( s \):

\[
s = s_1 + s_2 + s_3. \tag{6}
\]

Similarly, we have

\[
s_2 = s_{2,1} + s_{2,2}, \tag{7}
\]

and we have

\[
s_3 = s_{3,1} + s_{3,2}. \tag{8}
\]

We define \( f_4 \) as the number of prime divisors (counting multiplicity) in \( N \) which are not the special prime and are raised to at least the fourth power. From a simple counting argument we obtain
Lemma 3. We have

\[ s_1 + s_{2,2} \leq t + s_{2,1} + s_{3,1} + 1 \]  

and

\[ s_1 \leq t + s_{3,1} + 1. \]

Proof. The claim will be proven if we can show that each \( p \) in \( S_1S_{2,i} \) has to contribute at least one distinct prime (since then one of the primes may be the special prime and the other primes must all contribute to \( t \)). This is trivial for \( S_1 \) (and was used in Ochem and Rao’s result). We will show this by showing that no two prime divisors of \( S_1S_{2,i} \) can contribute the same largest prime of the primes they contribute. We have two cases we need to consider, both primes arising from \( S_{2,i} \), or one arising from \( S_{2,i} \) and one arising from \( S_1 \).

Case I: Assume we have two prime divisors of \( S_{2,2} \), \( a \) and \( b \) with \( a > b \), and assume they have some shared prime factor \( q \) which divides both \( \sigma(a^2) = a^2 + a + 1 \) and \( \sigma(b^2) = b^2 + b + 1 \). Since we have \( a \equiv b \equiv 2 \pmod{3} \), we may apply Lemma 1 to conclude that that

\[ q \leq \frac{a + b + 1}{5} \leq \frac{3a}{5}. \]

Thus,

\[ \frac{\sigma(a^2)}{q} > \frac{a^2}{3a/5} = 5/3a > (a^2 + a + 1)^{1/2}. \]

But \( \sigma(a^2) \) only has two prime factors and this shows that the shared contributed prime cannot be the largest prime contributed by \( a \). The case of two primes dividing \( S_{2,1} \) is similar.

Case II: Assume that we have a prime \( a \) dividing \( S_1 \) and a prime \( b \) dividing \( S_{2,i} \). If \( i = 2 \) then the same logic as above works. So assume that \( i = 1 \), and \( b \equiv 1 \pmod{3} \). Thus we have a prime \( p \) such that \( a^2 + a + 1 = p \) and \( b^2 + b + 1 = 3p \) for some prime \( p \). This is precisely the situation ruled out by Ochem and Rao’s Lemma.

Next we have

\[ s_{2,1} + s_{3,1} \leq f_3, \]

since if \( x \equiv 1 \pmod{3} \), then \( x^2 + x + 1 \equiv 0 \pmod{3} \).

We also have, by counting all the \( 1 \pmod{3} \) primes which are contributed by primes in \( S \),

\[ s_1 + 2s_{2,2} + 3s_{3,2} + s_{2,1} + 2s_{3,1} \leq f_4 + e + 2s_{2,1} + 2s_{3,1}. \]
This simplifies to
\[ s_1 + 2s_{2,2} + 3s_{3,2} \leq f_4 + e + s_{2,1}. \] (13)

And, of course, we have
\[ 4t \leq f_4. \] (14)

If \( 3 \nmid N \), then our system of equations and inequalities also includes the additional constraint \( f_3 = 0 \) as well as
\[ \omega = s + t + 1. \] (15)

Since \( f_3 = 0 \) we have \( s_{2,1} = s_{3,1} = 0 \), and after zeroing those variables we obtain inequality 3 by taking \( 7/9 \times (5) + 2/3 \times (6) + 2/3 \times (7) + 2/3 \times (8) + 1 \times (9) + 4/9 \times (10) + 2/9 \times (13) + 7/9 \times (14) + 8/3 \times (15) \) where the bold numbers represent the corresponding numbered equation or inequality.

If \( 3 \mid N \) we have \( 2 \leq f_3 \) and
\[ \omega = s + t + 2. \] (16)

Similar to the previous case, the linear combination \( 3/4 \times (5) + 5/8 \times (6) + 5/8 \times (7) + 5/8 \times (8) + 1 \times (9) + 1/8 \times (10) + 1/4 \times (11) + 1 \times (12) + 1/4 \times (13) + 3/4 \times (14) + 21/8 \times (16) \) yields inequality 4.

**Acknowledgments.** Pascal Ochem greatly assisted in early drafts of this paper both in exposition and in clarifying the results. Maria Stadnik also made helpful suggestions which contributed to the exposition.

**References**
