ON SOME CONGRUENCES FOR ANDREWS' SINGULAR OVERPARTITIONS

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Abstract

Andrews' singular overpartitions can be enumerated by $\overline{C}_{k,i}(n)$, the number of overpartitions of n where only parts congruent to $\pm i \pmod{k}$ may be overlined, and no part is divisible by k. A number of authors have studied congruences satisfied by singular overpartitions. In particular, congruences for $\overline{C}_{3,1}(n)$ modulo $3, 8, 9, 18, 32, 36, 64, 72$ and 144 have been proved. In this article, we prove new congruences modulo $108, 192, 288$ and 432 for $\overline{C}_{3,1}(n)$.

1. Introduction

Recently, Andrews [2] defined and studied combinatorial objects associated with parameters k and i, which he termed as singular overpartitions. He proved that the number of such singular overpartitions is enumerated by $\overline{C}_{k,i}(n)$, the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. For $k \geq 3$ and $1 \leq i \leq \lfloor k/2 \rfloor$, the generating function for $\overline{C}_{k,i}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n) q^n = \frac{(q^k; q^k)_\infty (-q^i; q^k)_\infty (-q^{k-i}; q^k)_\infty}{(q; q)_\infty}, \tag{1}$$

where $(a;q)_\infty := \prod_{j=0}^{\infty}(1 - aq^j)$. To simplify notation, we shall use f_m to denote $(q^m;q^m)_\infty$.

Andrews paid particular interest to the case $(k,i) = (3,1)$, with the generating function

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(n)q^n = \frac{(q^3;q^3)_\infty(-q;q^3)_\infty(-q^2;q^3)_\infty}{(q;q)_\infty}$$

$$= \frac{f_2 f_3^2}{f_1^2 f_6}. \tag{2}$$

He noted that this is equivalent to $\overline{A}_3(n)$, the number of overpartitions of n into parts not divisible by 3. (See [10] and [15] for more information on overpartitions.) Andrews further proved two congruences satisfied by $\overline{C}_{3,1}(n)$.

Theorem 1 (Andrews [2]). *For every nonnegative integer n,*

$$\overline{C}_{3,1}(9n + 3) \equiv \overline{C}_{3,1}(9n + 6) \equiv 0 \pmod{3}.$$

Andrews' work attracted much interest and many articles proving new congruences satisfied by $\overline{C}_{k,i}(n)$ appeared in quick succession. In the following, we recall selected results for $\overline{C}_{3,1}(n)$. Chen, Hirschhorn and Sellers found several infinite families of congruences modulo 3 and 8. One of which is the following.

Theorem 2 (Chen, Hirschhorn and Sellers [7]). *For all nonnegative integers α and n,*
$$\overline{C}_{3,1}(2^\alpha(6n + 5)) \equiv 0 \pmod{8}.$$

Ahmed and Baruah [1] found several congruences modulo 18 and 36.

Theorem 3 (Ahmed and Baruah [1]). *For every nonnegative integer n,*

$$\overline{C}_{3,1}(48n + 12) \equiv 0 \pmod{18}, \tag{3}$$
$$\overline{C}_{3,1}(12n + 7) \equiv 0 \pmod{36}, \tag{4}$$
$$\overline{C}_{3,1}(12n + 11) \equiv 0 \pmod{36}, \tag{5}$$
$$\overline{C}_{3,1}(24n + 14) \equiv 0 \pmod{36}, \tag{6}$$
$$\overline{C}_{3,1}(24n + 22) \equiv 0 \pmod{36}. \tag{7}$$

Shen [18] studied congruences modulo 9 and two of her results are the following.

Theorem 4 (Shen [18]). *For every nonnegative integer n,*

$$\overline{C}_{3,1}(48n + 28) \equiv 0 \pmod{9}, \tag{8}$$
$$\overline{C}_{3,1}(48n + 44) \equiv 0 \pmod{9}. \tag{9}$$

Yao [20] considered congruences modulo 16, 32 and 64 and proved the following.

Theorem 5 (Yao [20]). *For every nonnegative integer n,*

$$\overline{C}_{3,1}(18n + 15) \equiv 0 \pmod{32}, \tag{10}$$

$$\overline{C}_{3,1}(24n + 23) \equiv 0 \pmod{32}, \tag{11}$$

$$\overline{C}_{3,1}(72n + 69) \equiv 0 \pmod{64}. \tag{12}$$

Mahadeva Naika and Gireesh [16] proved various congruences modulo $4, 6, 12, 24$ and 72.

Theorem 6 (Mahadeva Naika and Gireesh [16]). *For every nonnegative integer n,*

$$\overline{C}_{3,1}(24n + 14) \equiv 0 \pmod{72}, \tag{13}$$

$$\overline{C}_{3,1}(24n + 22) \equiv 0 \pmod{72}. \tag{14}$$

In their article, Mahadeva Naika and Gireesh also conjectured a congruence modulo 144, which was proved independently by two groups of researchers.

Theorem 7 (Kathiravan and Fathima [14], Barman and Ray [3]). *For every nonnegative integer n,*

$$\overline{C}_{3,1}(12n + 11) \equiv 0 \pmod{144}.$$

However, none of the three groups of researchers mentioned the existence of the following congruence.

Theorem 8. *For every nonnegative integer n,*

$$\overline{C}_{3,1}(24n + 23) \equiv 0 \pmod{288}.$$

In this article, we shall prove the above mentioned theorem and establish other new congruences for $\overline{C}_{3,1}(n)$. In fact, we can extend Theorem 8 by inserting an arbitrary number of pairs of primes of a certain type.

Theorem 9. *Let $p_1, \ldots, p_k$ be (not necessarily distinct) primes such that each $p_i \geq 5$ and has Legendre symbol $\left(\frac{-2}{p_i}\right) = -1$. Then for every nonnegative integer n,*

$$\overline{C}_{3,1}\left((24n + 23)\prod_{i=1}^{k} p_i^2\right) \equiv 0 \pmod{288}.$$

Our next result is an extension of one of the congruences from Theorem 5.

Theorem 10. *Let $p_1, \ldots, p_k$ be (not necessarily distinct) primes such that each $p_i \geq 5$ and has Legendre symbol $\left(\frac{-2}{p_i}\right) = -1$. Then for every nonnegative integer n,*

$$\overline{C}_{3,1}\left((72n + 69)\prod_{i=1}^{k} p_i^2\right) \equiv 0 \pmod{192}.$$

The final results of this article are the following generalizations of Theorem 4.

Theorem 11. *For every nonnegative integer n,*

$$\overline{C}_{3,1}(48n + 28) \equiv 0 \pmod{108}, \tag{15}$$
$$\overline{C}_{3,1}(48n + 44) \equiv 0 \pmod{432}. \tag{16}$$

Both of the above congruences can also be extended.

Theorem 12. *Let $p_1, \ldots, p_k$ be (not necessarily distinct) primes such that each $p_i \geq 5$ and has Legendre symbol $\left(\frac{-1}{p_i}\right) = -1$. Then for every nonnegative integer n,*

$$\overline{C}_{3,1}\left((48n + 28)\prod_{i=1}^{k} p_i^2\right) \equiv 0 \pmod{108}, \tag{17}$$

$$\overline{C}_{3,1}\left((48n + 44)\prod_{i=1}^{k} p_i^2\right) \equiv 0 \pmod{108}. \tag{18}$$

Our article is structured as follows. In the next section, we recall some preliminary results. In Section 3, we prove Theorem 8 and its generalization. In Section 4, we prove Theorem 10. The final section is devoted to establishing Theorem 11 and its corresponding generalization.

2. Preliminary Results

In this section, we first recall some results from the literature. The following is a 2-dissection due to Ramanujan.

Lemma 1.
$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}.$$

This identity can be obtained by summing Entries 25(v) and 25(vi) in Chapter 16 of Ramanujan's notebooks [5, p. 40]. An alternative proof is given in [19, Cor. 2.4].

Our next lemma is essentially the 2-dissection of one of the cubic theta functions and was proved in [12, Eq. (1.36)].

Lemma 2.

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}.$$

We also require the 3-dissection of the overpartition generating function that was proved by Hirschhorn and Sellers [13].

Lemma 3.

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}.$$

The following identities were proved by Baruah and Ojah, and appeared respectively as equation (4.37), Theorem 4.17 and Theorem 4.3 in [4].

Lemma 4.

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{19}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{20}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}. \tag{21}$$

We will use the following consequence of the binomial theorem throughout our article. For any positive integer k and prime p,

$$f_k^p \equiv f_{pk} \pmod{p}.$$

We now recall the special case of a result due to Newman [17].

Lemma 5. *Let p be a prime satisfying $p \equiv 3 \pmod 4$. If*

$$\sum_{n=0}^{\infty} H(n)q^n = f_3^6,$$

then for every nonnegative integer n,

$$H\left(p^2 n + \frac{3(p^2 - 1)}{4}\right) = p^2 H(n).$$

An elementary proof of this result, depending on only the series expansion of f_1^3 can be found in [9]. We also require another two analogues of Newman's result.

Lemma 6. *Let p be a prime satisfying, $p > 3$ and $p \equiv 3 \pmod 4$. If*

$$\sum_{n=0}^{\infty} G(n)q^n = \left(\frac{f_6^2}{f_3}\right)^2,$$

then for every nonnegative integer n,

$$G\left(p^2 n + \frac{3(p^2-1)}{4}\right) = G(n).$$

Proof. Using the series expansion for $\frac{f_6^2}{f_3}$ [5, p. 36], we have

$$\sum_{n=0}^{\infty} G(n)q^n = \left(\sum_{i=-\infty}^{\infty} q^{\frac{3(4i+1)^2-3}{8}}\right)^2$$

$$= \sum_{\substack{j\equiv 1 \pmod 4 \\ k\equiv 1 \pmod 4}} q^{((3j)^2+(3k)^2-18)/24}. \tag{22}$$

Suppose $p > 3$ is a prime satisfying $p \equiv 3 \pmod 4$, and

$$pm + \frac{3(p^2-1)}{4} = \frac{(3j)^2 + (3k)^2 - 18}{24},$$

which simplifies to

$$(3j)^2 + (3k)^2 = 24pm + 18p^2 \equiv 0 \pmod p. \tag{23}$$

Since p cannot be represented as a sum of two squares, we conclude that $p \mid j$ and $p \mid k$. We can thus find j' and k' such that

$$j = -j'p \quad \text{and} \quad k = -k'p.$$

Now equation (23) becomes

$$(3j')^2 + (3k')^2 = \frac{24m}{p} + 18.$$

Thus

$$G\left(pm + \frac{3(p^2-1)}{4}\right) = \sum_{\substack{j'\equiv 1 \pmod 4 \\ k'\equiv 1 \pmod 4 \\ (3j')^2+(3k')^2=\frac{24m}{p}+18}} 1$$

$$= G\left(\frac{m}{p}\right). \tag{24}$$

$\square$

Our second analogue of Lemma 5 is the following.

Lemma 7. *Let p be a prime satisfying $p \geq 5$ and with Legendre symbol $\left(\frac{-2}{p}\right) = -1$. If*

$$\sum_{n=0}^{\infty} F(n)q^n = f_4 f_6^3,$$

then for every nonnegative integer n,

$$F\left(p^2 n + \frac{11(p^2 - 1)}{12}\right) = \epsilon p F(n),$$

where

$$\epsilon = \begin{cases} 1 & \text{if } p \equiv 7, 13 \pmod{24}, \\ -1 & \text{if } p \equiv 5, 23 \pmod{24}. \end{cases}$$

Proof. Using the series expansions for f_4 and f_6^3 [5, p. 36 and p. 39], we have

$$\sum_{n=0}^{\infty} F(n)q^n = \left(\sum_{i=-\infty}^{\infty} (-1)^i q^{\frac{(6i+1)^2 - 1}{6}}\right) \left(\sum_{i=-\infty}^{\infty} (4i+1) q^{\frac{3(4i+1)^2 - 3}{4}}\right)$$

$$= \sum_{\substack{j \equiv 1 \pmod 4 \\ k \equiv 1 \pmod 6}} (-1)^{(k-1)/6} j q^{((3j)^2 + 2k^2 - 11)/12}. \tag{25}$$

Suppose $p \geq 5$ is a prime such that

$$pm + \frac{11(p^2 - 1)}{12} = \frac{(3j)^2 + 2k^2 - 11}{12},$$

which simplifies to

$$(3j)^2 + 2k^2 = 12pm + 11p^2 \equiv 0 \pmod{p}. \tag{26}$$

However, since -2 is a quadratic non-residue modulo p, we conclude that $p \mid j$ and $p \mid k$. We can thus find j' and k' such that

$$j = \epsilon_j j' p \quad \text{and} \quad k = \epsilon_k k' p$$

where

$$\epsilon_j = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ -1 & \text{if } p \equiv 3 \pmod 4; \end{cases} \quad \text{and} \quad \epsilon_k = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 6, \\ -1 & \text{if } p \equiv 5 \pmod 6. \end{cases}$$

Furthermore equation (26) becomes

$$(3j')^2 + 2(k')^2 = \frac{12m}{p} + 11.$$

Thus

$$F\left(pm + \frac{11(p^2 - 1)}{12}\right) = \sum_{\substack{j' \equiv 1 \pmod 4 \\ k' \equiv 1 \pmod 6 \\ (3j')^2 + 2(k')^2 = \frac{12m}{p} + 11}} (-1)^{(\epsilon_k k' p - 1)/6} \epsilon_j j' p$$

$$= \epsilon p F\left(\frac{m}{p}\right), \tag{27}$$

where ϵ is defined in the statement of the Lemma. $\square$

3. Proofs of Theorems 8 and 9

As mentioned in the introduction, Theorem 7 was proved by Kathiravan and Fathima, and also independently by Barman and Ray. The two proofs are almost identical and begin with the following result [16, Eq. (3.19)],

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(4n + 3)q^n = 6\frac{f_2^3 f_6^3}{f_1^6} = 6f_6^3\left(\frac{f_2}{f_1^2}\right)^3. \tag{28}$$

Substituting Lemma 3 gives

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(4n + 3)q^n = 6\frac{f_6^{15} f_9^{18}}{f_3^{24} f_{18}^9} + 36q\frac{f_6^{14} f_9^{15}}{f_3^{23} f_{18}^6} + 144q^2\frac{f_6^{13} f_9^{12}}{f_3^{22} f_3^3} + 336q^3\frac{f_6^{12} f_9^9}{f_3^{21}}$$

$$+ 576q^4\frac{f_6^{11} f_9^6 f_{18}^3}{f_3^{20}} + 576q^5\frac{f_6^{10} f_9^3 f_{18}^6}{f_3^{19}} + 384q^6\frac{f_6^9 f_{18}^9}{f_3^{18}}. \tag{29}$$

Extracting coefficients of q^{3n+2}, we get

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(12n + 11)q^n = 144\frac{f_2^{13} f_3^{12}}{f_1^{22} f_6^3} + 576q\frac{f_2^{10} f_3^3 f_6^6}{f_1^{19}}, \tag{30}$$

which appeared in both [14] and [3]. Next using the binomial theorem, the above equation can be written as

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(12n + 11)q^n \equiv 144f_4 f_6^3 \pmod{288}. \tag{31}$$

We first observe that extracting coefficients of q^{2n+1} completes the proof of Theorem 8. In fact, if we are only interested in proving Theorem 8, we can simply combine the two congruences (5) and (11). However, Lemma 7 allows us to generalize equation (31). We have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(12n + 11)q^n \equiv 144\sum_{n=0}^{\infty} F(n)q^n \pmod{288}. \tag{32}$$

Now let p be any prime satisfying the conditions in Lemma 7, then

$$\overline{C}_{3,1}(12p^2n + 11p^2) \equiv 144\epsilon pF(n) \equiv 144F(n) \pmod{288}. \tag{33}$$

Continuing the process by induction, we can insert any number of copies of p_i^2 into the congruence. Finally extracting terms involving q^{2n+1} completes the proof of Theorem 9.

4. Proof of Theorem 10

We first observe that combining the second congruence from Theorem 1 with congruences (10) and (12) respectively proves the following.

Corollary 1. *For every nonnegative integer n,*

$$\overline{C}_{3,1}(18n + 15) \equiv 0 \pmod{96}, \tag{34}$$

$$\overline{C}_{3,1}(72n + 69) \equiv 0 \pmod{192}. \tag{35}$$

However, we cannot use the latter congruence as a basis to prove Theorem 10. Our starting point is the following identity that was proved in [20, Equation (3.13)]

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(9n + 6)q^n = 24\frac{f_2^7 f_3^{12}}{f_1^{16} f_6^3} + 96q\frac{f_2^4 f_3^3 f_6^6}{f_1^{13}}. \tag{36}$$

Using Lemma 1 and identity (19), we find that

$$24\frac{f_2^7}{f_6^3}\left(\frac{f_3^2}{f_1^2}\right)^6\left(\frac{1}{f_1^4}\right) \equiv 24\frac{f_4^{38} f_6^3 f_{12}^{12}}{f_2^{37} f_8^{10} f_{24}^6} + 96q\frac{f_4^{26} f_6^3 f_{12}^{12}}{f_2^{33} f_8^2 f_{24}^6} + 96q\frac{f_4^{35} f_6^4 f_{12}^9}{f_2^{36} f_8^8 f_{24}^4}$$
$$+ 96q^2\frac{f_4^{32} f_6^5 f_{12}^6}{f_2^{35} f_8^6 f_{24}^2} \pmod{192}, \tag{37}$$

Likewise expanding with Lemma 2 gives

$$96qf_2^4 f_6^6\left(\frac{f_3^3}{f_1}\right)\left(\frac{1}{f_1^4}\right)^3 \equiv 96q\frac{f_4^{45} f_6^8}{f_2^{40} f_8^{12} f_{12}} + 96q^2\frac{f_6^6 f_{12}^3 f_4^{41}}{f_2^{38} f_8^{12}} \pmod{192}. \tag{38}$$

Combining the two previous equations and simplifying with the binomial theorem, we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(18n + 15)q^n \equiv 96\left(\frac{f_2^{26} f_3^3 f_6^{12}}{f_1^{33} f_4^2 f_{12}^6} + \frac{f_2^{35} f_3^4 f_6^9}{f_1^{36} f_4^8 f_{12}^4} + \frac{f_2^{45} f_3^8}{f_1^{40} f_4^{12} f_6}\right)$$
$$\equiv 96f_2^{22}\left(\frac{f_3^3}{f_1}\right)\left(\frac{1}{f_1^4}\right)^8 \pmod{192} \tag{39}$$

which implies congruence (34).

We shall now 2-dissect each of the products of the right side of congruence (39). We use Lemmas 1 and 2 to compute that

$$f_2^{22}\left(\frac{f_3^3}{f_1}\right)\left(\frac{1}{f_1^4}\right)^8 \equiv \frac{f_4^{115}f_6^2}{f_2^{92}f_8^{32}f_{12}} + q\frac{f_4^{111}f_{12}^3}{f_2^{90}f_8^{32}} \pmod{2}. \tag{40}$$

This means

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(36n+33)q^n \equiv 96\frac{f_2^{111}f_6^3}{f_1^{90}f_4^{32}} \pmod{192}$$

$$\equiv 96f_4f_6^3 \pmod{192}$$

$$\equiv 96\sum_{n=0}^{\infty}F(n)q^n \pmod{192}. \tag{41}$$

As in the previous section, if p is any prime satisfying the conditions in Lemma 7, then

$$\overline{C}_{3,1}(36p^2n+33p^2) \equiv 96F(n) \pmod{192}. \tag{42}$$

We can continue to insert any number of copies of p_i^2 before finally extracting coefficients of q^{2n+1} to prove Theorem 10.

5. Proof of Theorems 11 and 12

The proof of Theorem 11 is more intricate. In order to simplify our proof, we introduce the following three functions defined by their generating functions:

$$\sum_{n=0}^{\infty}A(n)q^n := \frac{f_2^{19}f_3^6f_{12}^2}{f_1^{18}f_4^2f_6^7}, \tag{43}$$

$$\sum_{n=0}^{\infty}B(n)q^n := \frac{f_2^9f_3^2f_4^2f_6^3}{f_1^{14}f_{12}^2}, \tag{44}$$

$$\sum_{n=0}^{\infty}C(n)q^n := q\frac{f_2^4f_3^4f_4f_6^2f_{12}}{f_1^{12}}. \tag{45}$$

We now prove several lemmas involving these functions.

Lemma 8. *For every nonnegative integer n,*

$$A(6n+3)+9B(6n+3) \equiv 0 \pmod{27},$$
$$A(6n+5)+9B(6n+5) \equiv 0 \pmod{27}.$$

Proof. We can rewrite

$$\sum_{n=0}^{\infty} A(n)q^n = \frac{f_2^{19} f_{12}^2}{f_4^2 f_6^7} \left(\frac{f_3}{f_1^3}\right)^6, \tag{46}$$

$$\sum_{n=0}^{\infty} B(n)q^n = \frac{f_2^9 f_4^2 f_6^3}{f_{12}^2} \left(\frac{f_3}{f_1^3}\right)^4 \left(\frac{1}{f_1^2 f_3^2}\right). \tag{47}$$

Using identities (20) and (21), we can extract the coefficients of q^{2n+1} to get

$$\sum_{n=0}^{\infty} A(2n+1)q^n \equiv 9 \sum_{n=0}^{\infty} B(2n+1)q^n \equiv 18 \frac{f_2^{30} f_3^9}{f_1^{33} f_6^6} \pmod{27}. \tag{48}$$

By the binomial theorem, we have

$$18 \frac{f_2^{30} f_3^9}{f_1^{33} f_6^6} \equiv 18 \frac{f_6^4}{f_3^2} \pmod{27}.$$

Extracting the coefficients of q^{3n+1} and q^{3n+2} separately proves the lemma. □

Recall that the theta function $\varphi(q)$ has the following infinite product representation:

$$\varphi(q) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}. \tag{49}$$

Ramanujan proved many identities involving combinations of $\varphi(q)^2$ and $\varphi(q^3)^2$.

Lemma 9. *The following hold:*

$$\varphi(q^3)^2 + \varphi(q)^2 = \frac{f_6^{10}}{f_3^4 f_{12}^4} + \frac{f_2^{10}}{f_1^4 f_4^4} = 2 \frac{f_2^3 f_3^2 f_6}{f_1^2 f_4 f_{12}}, \tag{50}$$

$$3\varphi(q^3)^2 - \varphi(q)^2 = 3 \frac{f_6^{10}}{f_3^4 f_{12}^4} - \frac{f_2^{10}}{f_1^4 f_4^4} = 2 \frac{f_1^2 f_2 f_6^3}{f_3^2 f_4 f_{12}}, \tag{51}$$

$$3\varphi(q^3)^2 + \varphi(q)^2 = 3 \frac{f_6^{10}}{f_3^4 f_{12}^4} + \frac{f_2^{10}}{f_1^4 f_4^4} = 4 \frac{f_2 f_4^2 f_6^3}{f_1 f_3 f_{12}^2}. \tag{52}$$

Identity (50) can be found in [5, p. 351, line 3] while identity (51) is in [5, p. 232, line 20]. Identity (52) can be deduced by combining Entry 44 in Chapter 25 of Ramanujan's notebooks [6, p. 197] and identity (50). Alternative proofs of Lemma 9 can also be found in Chapter 34 of [11] or derived from [8, Theorem 3.25]. Identity (50) is also equivalent to the following.

Corollary 2. *We have*

$$27q \frac{f_2^{11} f_3^2 f_6 f_{12}^2}{f_1^{14} f_4^2} + 27q \frac{f_2 f_4^2 f_6^{11}}{f_1^{10} f_3^2 f_{12}^2} = 54 \sum_{n=0}^{\infty} C(n)q^n.$$

Lemma 10. *For every nonnegative integer n,*

$$A(2n+1) + 9B(2n+1) \equiv 0 \pmod{16}.$$

Proof. We consider the generating functions of $A(n)$ and $B(n)$,

$$
\begin{aligned}
\sum_{n=0}^{\infty} A(n)q^n + 9\sum_{n=0}^{\infty} B(n)q^n &= \frac{f_2^{19} f_3^6 f_{12}^2}{f_1^{18} f_4^2 f_6^7} + 9\frac{f_2^9 f_3^2 f_4^2 f_6^3}{f_1^{14} f_{12}^2} \\
&= \left(\varphi(q)^2 + 9\varphi(q^3)^2\right) \frac{f_2^9 f_3^6 f_4^2 f_{12}^2}{f_1^{14} f_6^7} \\
&= \left(6\varphi(q^3)^2 + 2\varphi(q)^2 + 3\varphi(q^3)^2 - \varphi(q)^2\right) \frac{f_2^9 f_3^6 f_4^2 f_{12}^2}{f_1^{14} f_6^7} \\
&= \left(8\frac{f_2 f_4^2 f_6^3}{f_1 f_3 f_{12}^2} + 2\frac{f_1^2 f_2 f_6^3}{f_3^2 f_4 f_{12}}\right) \frac{f_2^9 f_3^6 f_4^2 f_{12}^2}{f_1^{14} f_6^7}, \quad (53)
\end{aligned}
$$

where Lemma 9 was used. Now using the binomial theorem, we have

$$
\sum_{n=0}^{\infty} A(n)q^n + 9\sum_{n=0}^{\infty} B(n)q^n \equiv 8\left(\frac{1}{f_1 f_3}\right)^3 f_2^{12} + 2\left(\frac{f_3}{f_1^3}\right)^4 \frac{f_2^{10} f_4 f_{12}}{f_6^4} \pmod{16}.
$$

Using identities (20) and (21), we can extract the coefficients of q^{2n+1} to get

$$
\begin{aligned}
&\sum_{n=0}^{\infty} A(2n+1)q^n + 9\sum_{n=0}^{\infty} B(2n+1)q^n \\
&\equiv 8\left(3\frac{f_1^4 f_2^3 f_4^2 f_6^9}{f_3^{10} f_{12}^2} + q\frac{f_2^{15} f_{12}^6}{f_3^6 f_4^6 f_6^3}\right) \\
&\quad + 2\left(12\frac{f_2^{21} f_6^6}{f_1^{24} f_6^3} + 108q\frac{f_2^{13} f_3^2 f_6^5}{f_1^{20}}\right) \pmod{16} \\
&\equiv 8\left(f_1^{18} + qf_1^6 f_3^{12}\right) + 8\left(f_1^{18} + qf_1^6 f_3^{12}\right) \pmod{16} \\
&\equiv 0 \pmod{16}. \quad (54)
\end{aligned}
$$

The proof follows. $\square$

Lemma 11. *For every nonnegative integer n,*

$$C(3n) \equiv 0 \pmod 2, \quad (55)$$
$$C(6n+5) \equiv 0 \pmod 8. \quad (56)$$

Proof. We write the generating function $C(n)$ as

$$\sum_{n=0}^{\infty} C(n)q^n = qf_3^4 f_6^2 f_{12} \left(\frac{f_2}{f_1^2}\right)^6 \left(\frac{f_4}{f_2^2}\right). \quad (57)$$

Applying Lemma 3 and a second time with q^2 in place of q, we can 3-dissect the above identity. Equating coefficients of q^{3n} proves equation (55). Next, we can extract coefficients of q^{3n+2} to get

$$\sum_{n=0}^{\infty} C(3n+2)q^n \equiv 4\frac{f_2^{17} f_3^{33} f_4^5}{f_1^{43} f_6^9 f_{12}^3} + 4q\frac{f_2^{20} f_3^{36} f_4^3 f_{12}^3}{f_1^{44} f_6^{18}} \pmod 8$$

$$\equiv 4f_2^6 f_6^2 \left(\frac{1}{f_1 f_3}\right) + 4qf_2^4 f_6^6 \pmod 8. \tag{58}$$

Using identity (21) and equating coefficients of q^{2n+1}, yields

$$\sum_{n=0}^{\infty} C(6n+5)q^n \equiv 4\frac{f_1^2 f_2^5 f_{12}^2}{f_4^2 f_6} + 4f_1^4 f_3^6 \pmod 8$$

$$\equiv 0 \pmod 8. \tag{59}$$

where the last congruence results from the binomial theorem. $\square$

We are now ready for the proof of Theorem 11.

Proof. From [16, Eq.(3.14)],

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(2n)q^n = \frac{f_4^{13} f_6^2}{f_2^{10} f_8^4 f_{12}} + 4q\frac{f_4 f_6^2 f_8^4}{f_2^6 f_{12}}. \tag{60}$$

Equating the coefficients of q^{2n} we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(4n)q^n = \frac{f_2^{13} f_3^2}{f_1^{10} f_4^4 f_6} = \frac{f_2^{13}}{f_4^4 f_6} \left(\frac{f_3}{f_1^3}\right)^3 \left(\frac{1}{f_1 f_3}\right). \tag{61}$$

Using identities (20) and (21), we can equate coefficients of q^{2n+1} to get

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(8n+4)q^n = \frac{f_2^{19} f_3^6 f_{12}^2}{f_1^{18} f_4^2 f_6^7} + 9\frac{f_2^9 f_3^2 f_4^2 f_6^3}{f_1^{14} f_{12}^2} + 27q\frac{f_2^{11} f_3^2 f_6 f_{12}^2}{f_1^{14} f_4^2} + 27q\frac{f_2 f_4^2 f_6^{11}}{f_1^{10} f_3^2 f_{12}^2}$$

$$= \sum_{n=0}^{\infty} A(n)q^n + 9\sum_{n=0}^{\infty} B(n)q^n + 54\sum_{n=0}^{\infty} C(n)q^n, \tag{62}$$

where Corollary 2 was used in the final equality. By Lemma 8, we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(48n+28)q^n \equiv \sum_{n=0}^{\infty} A(6n+3)q^n + 9\sum_{n=0}^{\infty} B(6n+3)q^n \pmod{27}$$

$$\equiv 0 \pmod{27}, \tag{63}$$

as well as

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(48n+44)q^n \equiv \sum_{n=0}^{\infty} A(6n+5)q^n + 9\sum_{n=0}^{\infty} B(6n+5)q^n \pmod{27}$$
$$\equiv 0 \pmod{27}. \tag{64}$$

On the other hand, Lemmas 10 and 11 give

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(48n+28)q^n \equiv \sum_{n=0}^{\infty} A(6n+3)q^n + 9\sum_{n=0}^{\infty} B(6n+3)q^n$$
$$+ 2\sum_{n=0}^{\infty} C(6n+3)q^n \pmod{4}$$
$$\equiv 0 \pmod{4}. \tag{65}$$

Similarly,

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(48n+44)q^n \equiv \sum_{n=0}^{\infty} A(6n+5)q^n + 9\sum_{n=0}^{\infty} B(6n+5)q^n$$
$$+ 6\sum_{n=0}^{\infty} C(6n+5)q^n \pmod{16}$$
$$\equiv 0 \pmod{16}. \tag{66}$$

Since 27 is relatively prime to 4 and 16, this completes the proof of Theorem 11. $\square$

Finally, we prove Theorem 12.

Proof. Recall that in the proof of Lemma 8, we have

$$\sum_{n=0}^{\infty} A(2n+1)q^n \equiv 9\sum_{n=0}^{\infty} B(2n+1)q^n \equiv 18\frac{f_6^4}{f_3^2} \pmod{27}. \tag{67}$$

Thus, we can deduce from equation (62) that

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(16n+12)q^n \equiv \sum_{n=0}^{\infty} A(2n+1)q^n + 9\sum_{n=0}^{\infty} B(2n+1)q^n \pmod{27}$$
$$\equiv 9\left(\frac{f_6^2}{f_3}\right)^2 \pmod{27}, \tag{68}$$

Lemma 6 allows us to generalize congruence (68). We have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(16n+12)q^n \equiv 9\sum_{n=0}^{\infty} G(n)q^n \pmod{27}. \tag{69}$$

Now let $p > 3$ be a prime satisfying $p \equiv 3 \pmod 4$, then

$$\overline{C}_{3,1}(16p^2 n + 12p^2) \equiv 9G(n) \pmod{27}. \tag{70}$$

Continuing the process by induction, we can insert any number of copies of p_i^2 into the congruence. Extracting terms involving q^{3n+1} and q^{3n+2} separately, we get

$$\overline{C}_{3,1}\left((48n + 28)\prod_{i=1}^{k} p_i^2 \right) \equiv 0 \pmod{27}, \tag{71}$$

$$\overline{C}_{3,1}\left((48n + 44)\prod_{i=1}^{k} p_i^2 \right) \equiv 0 \pmod{27}. \tag{72}$$

On the other hand, we have

$$\sum_{n=0}^{\infty} C(n)q^n = q\left(\frac{f_3^2}{f_1^2} \right)^2 \left(\frac{1}{f_1^4} \right)^2 f_2^4 f_4 f_6^2 f_{12}. \tag{73}$$

Using Lemma 1 and identity (19), we can extract the coefficients of q^{2n+1} to get

$$\sum_{n=0}^{\infty} C(2n + 1)q^n \equiv \frac{f_2^{37} f_3^4 f_6^5}{f_1^{34} f_4^{10} f_{12}^2} \pmod 2$$

$$\equiv \left(f_3^3 \right)^2 \pmod 2. \tag{74}$$

From equation (62), Lemma 10 and congruence (74), we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(16n + 12)q^n \equiv 2\sum_{n=0}^{\infty} C(2n + 1)q^n \pmod 4$$

$$\equiv 2\left(f_3^3 \right)^2 \pmod 4. \tag{75}$$

Lemma 5 allows us to generalize congruence (75). We have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(16n + 12)q^n \equiv 2\sum_{n=0}^{\infty} H(n)q^n \pmod 4.$$

Now let $p > 3$ be a prime satisfying $p \equiv 3 \pmod 4$, then

$$\overline{C}_{3,1}(16p^2 n + 12p^2) \equiv 2p^2 H(n) \equiv 2H(n) \pmod 4.$$

Continuing the process by induction, we can insert the same sequence of primes p_i^2 as congruences (71) and (72). Extracting terms involving q^{3n+1} and q^{3n+2} separately, we get

$$\overline{C}_{3,1}\left((48n + 28)\prod_{i=1}^{k} p_i^2 \right) \equiv 0 \pmod 4, \tag{76}$$

$$\overline{C}_{3,1}\left((48n + 44)\prod_{i=1}^{k} p_i^2 \right) \equiv 0 \pmod 4. \tag{77}$$

Since 27 and 4 are relatively prime, this completes the proof of Theorem 12.   □

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