NEW CONGRUENCES FOR OVERPARTITIONS INTO ODD PARTS

Chiranjit Ray[1]
Dept. of Mathematics, Indian Institute of Technology Guwahati, Guwahati, India
chiranjitray.m@gmail.com

Rupam Barman
Dept. of Mathematics, Indian Institute of Technology Guwahati, Guwahati, India
rupam@iitg.ernet.in

Received: 1/20/17, Revised: 3/25/18, Accepted: 5/21/18, Published: 6/5/18

Abstract

Let $\overline{p}_o(n)$ denote the number of overpartitions of n into odd parts. In this article, we study congruences for $\overline{p}_o(n)$ modulo 8 and 16. Chen proved that $\overline{p}_o(n)$ satisfies the identity

$$\sum_{n=0}^{\infty}\overline{p}_o(16n+14)q^n = 112\frac{f_2^{27}}{f_1^{25}f_4^2} + 256q\frac{f_2^3f_4^{14}}{f_1^{17}},$$

where $f_k := \prod_{n=1}^{\infty}(1-q^{nk})$. We prove similar identities for $\overline{p}_o(16n+2)$, $\overline{p}_o(16n+6)$, and $\overline{p}_o(16n+10)$. Along the way, we find a new proof of the identity of Chen. We also derive infinite families of congruences modulo 8 and 16 for $\overline{p}_o(n)$. We use Ramanujan's theta function identities and some new p-dissections in our proofs.

1. Introduction and Statement of Results

Throughout this paper, for complex numbers a and q, $(a;q)_\infty$ stands for the q-shifted factorial

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}), \quad |q| < 1; \tag{1}$$

and f_k stands for $(q^k;q^k)_\infty$. In [7], Corteel and Lovejoy introduce the notion of overpartitions. Many interesting arithmetic properties of overpartitions are found

[1]Chiranjit Ray was supported by the Department of Atomic Energy, Government of India under NBHM Fellowship.

by many mathematicians, for example, see Mahlburg [14], Hirschhorn and Sellers [10], and Kim [12, 13]. An *overpartition* of a nonnegative integer n is a partition of n in which the first occurrence of a part may be over-lined. For example, the eight overpartitions of 3 are $3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1$, and $\overline{1}+1+1$. Let $\overline{p}(n)$ denote the number of overpartitions of n. The generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{f_2}{f_1^2}. \tag{2}$$

For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{3}$$

In Ramanujan's notation, the Jacobi triple product identity [4, Entry 19, p. 36] takes the shape

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{4}$$

The most important special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{5}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}, \tag{6}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1. \tag{7}$$

We now recall two definitions from [9, p. 225]. Let Π represent a pentagonal number (a number of the form $\frac{3n^2+n}{2}$) and Ω represent an octagonal number (a number of the form $3n^2+2n$). Let $\Pi(q) = \sum_{n=-\infty}^{\infty} q^{\frac{3n^2+n}{2}}$ and $\Omega(q) = \sum_{n=-\infty}^{\infty} q^{3n^2+2n}$. Then,

$$\Pi(q) = \frac{f_2 f_3^2}{f_1 f_6}. \tag{8}$$

Also,

$$\begin{aligned}\Omega(-q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+2n} = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n} \\ &= \prod_{n\geq 1} (1-q^{6n-5})(1-q^{6n-1})(1-q^{6n}) = \frac{f_1 f_6^2}{f_2 f_3}. \end{aligned} \tag{9}$$

In this article, we study overpartitions in which only odd parts are used. This function has arisen in a number of recent papers, but in contexts which are very different from overpartitions. For example, see Ardonne, Kedem and Stone [1], Bessenrodt [3], and Santos and Sills [15]. We denote by $\overline{p}_o(n)$ the number of overpartitions of n into odd parts. Hirschhorn and Sellers [11] obtain many interesting arithmetic properties of $\overline{p}_o(n)$. They observe that the generating function for $\overline{p}_o(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}_o(n) q^n = \prod_{n=1}^{\infty} \frac{1+q^{2n-1}}{1-q^{2n-1}} = \frac{f_2^3}{f_1^2 f_4}. \tag{10}$$

They establish a number of arithmetic results including several Ramanujan-like congruences satisfied by $\overline{p}_o(n)$, and some easily-stated characterizations of $\overline{p}_o(n)$ modulo small powers of 2. For example, the following two Ramanujan-like congruences can readily be seen from one of their main theorems:

$$\overline{p}_o(8n+5) \equiv 0 \pmod{8}, \tag{11}$$

$$\overline{p}_o(8n+7) \equiv 0 \pmod{16}. \tag{12}$$

They also prove that, for $n \geq 1$, $\overline{p}_o(n)$ is divisible by 4 if and only if n is neither a square nor twice a square. In [6, Theorem 1], Chen proves that

$$\sum_{n=0}^{\infty} \overline{p}_o(16n+14) q^n = 112 \frac{f_2^{27}}{f_1^{25} f_4^2} + 256q \frac{f_2^3 f_4^{14}}{f_1^{17}}, \tag{13}$$

from which it readily follows that $\overline{p}_o(16n+14) \equiv 0 \pmod{16}$. Using elementary theory of modular forms, he further proves infinitely many congruences for $\overline{p}_o(n)$ modulo 32 and 64. Let $t \geq 0$ be an integer and $p_1, p_2 \equiv 1 \pmod{8}$ be primes. Chen [6, Theorem 2] proves that

$$\overline{p}_o(p_1^{2t+1}(16n+14)) \equiv 0 \pmod{32}, \tag{14}$$

$$\overline{p}_o(p_1^{4t+3}(16n+14)) \equiv 0 \pmod{64}, \tag{15}$$

$$\overline{p}_o(p_1 p_2(16n+14)) \equiv 0 \pmod{64}. \tag{16}$$

The first two congruences are valid for all nonnegative integers n satisfying $8n \not\equiv -7 \pmod{p_1}$. The last congruence is valid for all nonnegative integers n satisfying $8n \not\equiv -7 \pmod{p_1}$ and $8n \not\equiv -7 \pmod{p_2}$.

In this article, we prove the following identities for $\overline{p}_o(n)$ similar to (13) for other values of n. Along the way, we also obtain (13).

Theorem 1. *We have*

$$\sum_{n=0}^{\infty} \overline{p}_o(4n) q^n = \frac{f_2^5 f_4^3}{f_1^6 f_8^2}, \tag{17}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(4n+1)q^n = 2\frac{f_4^7}{f_1^4f_2f_8^2}, \tag{18}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(4n+2)q^n = 2\frac{f_2^7f_8^2}{f_1^6f_4^3}, \tag{19}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(4n+3)q^n = 4\frac{f_2f_4f_8^2}{f_1^4}, \tag{20}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(16n+2)q^n = 2\frac{f_2^{45}}{f_1^{31}f_4^{14}}+224q\frac{f_2^{21}f_4^2}{f_1^{23}}, \tag{21}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(16n+6)q^n = 12\frac{f_2^{39}}{f_1^{29}f_4^{10}}+320q\frac{f_2^{15}f_4^6}{f_1^{21}}, \tag{22}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(16n+10)q^n = 40\frac{f_2^{33}}{f_1^{27}f_4^6}+384q\frac{f_2^9f_4^{10}}{f_1^{19}}. \tag{23}$$

We also find congruences modulo 8 and 16 for $\overline{p}_o(n)$ using Ramanujan's theta function identities and some dissections of theta functions. We prove the following congruences for $\overline{p}_o(n)$ modulo 8 and 16.

Theorem 2. *We have*

$$\sum_{n=0}^{\infty}\overline{p}_o(8n+3)q^n \equiv 4\frac{f_2^5}{f_1} \pmod{16}, \tag{24}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(16n+6)q^n \equiv 12\frac{f_2^5}{f_1} \pmod{16}, \tag{25}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(16n+10)q^n \equiv 8\frac{f_2^2f_4^3}{f_1} \pmod{16}, \tag{26}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(32n+4)q^n \equiv 6\frac{f_1f_2^5}{f_4^2} \pmod{8}, \tag{27}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(32n+12)q^n \equiv 4\frac{f_1^3f_4^2}{f_2} \pmod{8}, \tag{28}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(24n+1)q^n \equiv 2\frac{f_2}{f_1} \pmod{8}, \tag{29}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(24n+17)q^n \equiv 4\frac{f_1f_3^2f_6^2}{f_2} \pmod{8}, \tag{30}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(72n+9)q^n \equiv 6f_1f_2 \pmod{8}. \tag{31}$$

Theorem 3. *For nonnegative integers n and α we have*

$$\overline{p}_o\left(2^\alpha(32n+20)\right) \equiv 0 \pmod{8}, \tag{32}$$
$$\overline{p}_o\left(2^\alpha(32n+28)\right) \equiv 0 \pmod{8}. \tag{33}$$

We next prove certain infinite families of congruences for $\overline{p}_o(n)$ modulo 8 and 16 as stated in the following theorems. We establish new p-dissections of $\frac{f_2^5}{f_4^2}$ and $\Omega(-q)$, and use them to prove the congruences.

We recall that, for an odd prime p, the Legendre symbol is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a square modulo } p \text{ and } a \not\equiv 0 \pmod{p}; \\ -1 & \text{if } a \text{ is not a square modulo } p; \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

Theorem 4. *Let $p \geq 3$ be a prime, $n \geq 0$ and $\alpha \geq 1$. If $\left(\frac{-2}{p}\right) = -1$, then we have*

$$\overline{p}_o\left(8p^{2\alpha}n + (3p+8j)p^{2\alpha-1}\right) \equiv 0 \pmod{16}, \tag{34}$$
$$\overline{p}_o\left(16p^{2\alpha}n + (6p+16j)p^{2\alpha-1}\right) \equiv 0 \pmod{16}, \tag{35}$$
$$\overline{p}_o\left(72p^{2\alpha}n + (9p+72j)p^{2\alpha-1}\right) \equiv 0 \pmod{8}, \tag{36}$$
$$\overline{p}_o\left(24p^{2\alpha}n + (17p+24j)p^{2\alpha-1}\right) \equiv 0 \pmod{8}, \tag{37}$$
$$\overline{p}_o\left(32p^{2\alpha}n + (4p+32j)p^{2\alpha-1}\right) \equiv 0 \pmod{8}, \tag{38}$$
$$\overline{p}_o\left(32p^{2\alpha}n + (12p+32j)p^{2\alpha-1}\right) \equiv 0 \pmod{8}. \tag{39}$$

If $p \equiv 3 \pmod{4}$, then we have

$$\overline{p}_o\left(16p^{2\alpha}n + (10p+16j)p^{2\alpha-1}\right) \equiv 0 \pmod{16}, \tag{40}$$

where $j = 1, 2, \ldots, p-1$.

2. Preliminaries

In this section, we study certain p-dissection identities. We first state the following 4-dissection formula from [9, (1.9.4)] and [4, Entry 25, p. 40].

Lemma 1. *We have*

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{41}$$

That is,

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}. \tag{42}$$

We next recall the following 3-dissection formula from [9, (26.1.2)] and [4, Corollary (i), p. 49].

Lemma 2. *We have*

$$\psi(q) = \Pi(q^3) + 2q\psi(q^9). \tag{43}$$

That is,

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{44}$$

The following 4-dissection formulas are due to Hirschhorn and Sellers [10].

Lemma 3. *We have*

$$\frac{1}{\varphi(-q)} = \frac{1}{\varphi(-q^4)^4}\left(\varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3\right). \tag{45}$$

That is,

$$\frac{f_2}{f_1^2} = \frac{f_8^4}{f_4^8}\left(\frac{f_8^{15}}{f_4^6 f_{16}^6} + 2q\frac{f_8^9}{f_4^4 f_{16}^2} + 4q^2\frac{f_8^3 f_{16}^2}{f_4^2} + 8q^3\frac{f_{16}^6}{f_8^3}\right). \tag{46}$$

The following 3-dissection formulas are due to Hirschhorn and Sellers [10].

Lemma 4. *We have*

$$\frac{1}{\varphi(-q)} = \frac{\varphi(-q^9)}{\varphi(-q^3)^4}\left(\varphi(-q^9)^2 + 2q\varphi(-q^9)\Omega(-q^3) + 4q^2\Omega(-q^3)^2\right). \tag{47}$$

That is,

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^2}{f_3^8 f_{18}}\left(\frac{f_9^4}{f_{18}^2} + 2q\frac{f_3 f_9 f_{18}}{f_6} + 4q^2\frac{f_3^2 f_{18}^4}{f_6^2 f_9^2}\right). \tag{48}$$

We now recall p-dissections of $\psi(q)$, $f(-q)$ and $\psi(q^2)f(-q)^2$ which will be used to prove our main results.

Lemma 5. *[8, Theorem 2.1] For any odd prime p, we have*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{49}$$

Furthermore, for $0 \le k \le \frac{p-3}{2}$, $\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}$.

Before we state the next lemma, we define that for any prime $p \ge 5$,

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6} & \text{if } p \equiv 1 \pmod{6}; \\ \frac{-p-1}{6} & \text{if } p \equiv -1 \pmod{6}. \end{cases}$$

Lemma 6. *[8, Theorem 2.2] For any prime $p \geq 5$, we have*

$$f(-q) = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}). \quad (50)$$

Furthermore, if $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$ and $k \neq \frac{\pm p-1}{6}$, then $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}$.

Lemma 7. *[2, Lemma 2.4] If $p \geq 5$ is a prime and*

$$\frac{\pm p-1}{3} := \begin{cases} \frac{p-1}{3} & \text{if } p \equiv 1 \pmod{3} \\ \frac{-p-1}{3} & \text{if } p \equiv -1 \pmod{3}, \end{cases}$$

then

$$\psi(q^2)f(-q)^2 = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{3}}}^{\frac{p-1}{2}} q^{3k^2+2k} \sum_{n=-\infty}^{\infty} (3pn+3k+1) q^{pn(3pn+6k+2)} \pm pq^{\frac{p^2-1}{3}} \psi(q^{2p^2}) f(-q^{p^2})^2. \quad (51)$$

Furthermore, if $k \neq \frac{\pm p-1}{3}$ and $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$, then $3k^2+2k \not\equiv \frac{p^2-1}{3} \pmod{p}$.

The following lemma readily follows from [2, Lemma 2.3] by putting q^2 in place of q. The lemma gives a p-dissection of $f(-q^2)^3$.

Lemma 8. *For any prime $p \geq 3$, we have*

$$f(-q^2)^3 = \frac{1}{2} \sum_{\substack{k=0 \\ k \neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{k^2+k} \sum_{n=-\infty}^{\infty} (-1)^n (2pn+2k+1) q^{pn(pn+2k+1)} + (-1)^{\frac{p-1}{2}} pq^{\frac{p^2-1}{4}} f(-q^{2p^2})^3. \quad (52)$$

Furthermore, if $0 \leq k \leq p-1$ and $k \neq \frac{p-1}{2}$, then $k^2+k \not\equiv \frac{p^2-1}{4} \pmod{p}$.

In the following two lemmas, we deduce new p-dissections of $\frac{f_2^5}{f_4^2}$ and $\Omega(-q)$, respectively.

Lemma 9. *For a prime $p \geq 5$,*

$$\frac{f_2^5}{f_4^2} = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{3k^2+k} \sum_{n=-\infty}^{\infty} (6pn+6k+1) q^{pn(3pn+6k+1)} \pm pq^{\frac{p^2-1}{12}} \frac{f_{2p^2}^5}{f_{4p^2}^2}. \quad (53)$$

In addition, if $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$ and $k \neq \frac{\pm p-1}{6}$, then $3k^2+k \not\equiv \frac{p^2-1}{12} \pmod{p}$.

Proof. Due to Hirschhorn [9, (10.7.3)] and Berndt [5, (1.3.60)], we have

$$\begin{aligned}
\frac{f_2^5}{f_4^2} &= \sum_{n=-\infty}^{\infty}(6n+1)q^{3n^2+n}\\
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} \sum_{n=-\infty}^{\infty}[6(pn+k)+1]q^{3(pn+k)^2+(pn+k)}\\
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} q^{3k^2+k}\sum_{n=-\infty}^{\infty}(6pn+6k+1)q^{pn(3pn+6k+1)}\\
&= \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{3k^2+k}\sum_{n=-\infty}^{\infty}(6pn+6k+1)q^{pn(3pn+6k+1)}\\
&\pm q^{\frac{p^2-1}{12}}\sum_{n=-\infty}^{\infty}p(6n+1)q^{p^2(3n^2+n)}\\
&= \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{3k^2+k}\sum_{n=-\infty}^{\infty}(6pn+6k+1)q^{pn(3pn+6k+1)} \pm pq^{\frac{p^2-1}{12}}\frac{f_{2p^2}^5}{f_{4p^2}^2}.
\end{aligned}$$

We observe that, for $-\frac{p-1}{2}\le k\le\frac{p-1}{2}$, if $3k^2+k\equiv\frac{p^2-1}{12} \pmod p$, then we have $(6k+1)^2\equiv 0 \pmod p$, which yields $k=\frac{\pm p-1}{6}$. □

Lemma 10. *If $p\geq 5$ is a prime and*

$$\frac{\pm p+1}{3} := \begin{cases} \frac{-p+1}{3} & \text{if } p\equiv 1 \pmod 3\\ \frac{p+1}{3} & \text{if } p\equiv -1 \pmod 3,\end{cases}$$

then

$$\Omega(-q) = \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p+1}{3}}}^{\frac{p-1}{2}} (-1)^k q^{3k^2-2k}\sum_{n=-\infty}^{\infty}(-1)^n q^{pn(3pn+6k-2)} + (-1)^{\frac{\pm p+1}{3}}q^{\frac{p^2-1}{3}}\Omega(-q^{p^2}). \tag{54}$$

Furthermore, if $-\frac{p-1}{2}\le k\le\frac{p-1}{2}$ and $k\neq\frac{\pm p+1}{3}$, then $3k^2-2k\not\equiv\frac{p^2-1}{3} \pmod p$.

Proof. From (9), we have

$$
\begin{aligned}
\Omega(-q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n} \\
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} \sum_{n=-\infty}^{\infty} (-1)^{pn+k} q^{3(pn+k)^2-2(pn+k)} \\
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{3k^2-2k} \sum_{n=-\infty}^{\infty} (-1)^n q^{pn(3pn+6k-2)} \\
&= \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq\frac{\pm p+1}{3}}}^{\frac{p-1}{2}} (-1)^k q^{3k^2-2k} \sum_{n=-\infty}^{\infty} (-1)^n q^{pn(3pn+6k-2)} \\
&+ (-1)^{\frac{\pm p+1}{3}} q^{\frac{p^2-1}{3}} \sum_{n=-\infty}^{\infty} (-1)^n q^{p^2(3n^2-2n)} \\
&= \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq\frac{\pm p+1}{3}}}^{\frac{p-1}{2}} (-1)^k q^{3k^2-2k} \sum_{n=-\infty}^{\infty} (-1)^n q^{pn(3pn+6k-2)} \\
&+ (-1)^{\frac{\pm p+1}{3}} q^{\frac{p^2-1}{3}} \Omega(-q^{p^2}).
\end{aligned}
$$

Note that, if $3k^2 - 2k \equiv \frac{p^2-1}{3} \pmod{p}$, then $k = \frac{\pm p+1}{3}$. □

3. Proofs of Theorem 1, Theorem 2 and Theorem 3

In this section, we prove Theorems 1, 2 and 3.

Proof of Theorem 1. From (10) and (41), we find that

$$\sum_{n=0}^{\infty} \overline{p}_o(n)q^n = \frac{f_2^3}{f_1^2 f_4} \tag{55}$$

$$= \frac{f_2^3}{f_4}\left(\frac{1}{f_1^2}\right) \tag{56}$$

$$= \frac{f_2^3}{f_4}\left(\frac{f_4^2}{f_2^5}\varphi(q)\right) \tag{57}$$

$$= \frac{f_4}{f_2^2}\varphi(q) \tag{58}$$

$$= \frac{\varphi(q)}{\varphi(-q^2)} \tag{59}$$

$$= \frac{\varphi(q)\varphi(q^2)}{\varphi(q^2)\varphi(-q^2)} \tag{60}$$

$$= \frac{\varphi(q)\varphi(q^2)}{\varphi(-q^4)^2} \tag{61}$$

$$= \frac{\left(\varphi(q^4)+2q\psi(q^8)\right)\left(\varphi(q^8)+2q^2\psi(q^{16})\right)}{\varphi(-q^4)^2}, \tag{62}$$

from which it follows that

$$\sum_{n=0}^{\infty}\overline{p}_o(4n)q^n = \frac{\varphi(q)\varphi(q^2)}{\varphi(-q)^2} = \frac{f_2^5}{f_1^2f_4^2}\frac{f_4^5}{f_2^2f_8^2}\frac{f_2^2}{f_1^4} = \frac{f_2^5f_4^3}{f_1^6f_8^2}, \tag{63}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(4n+1)q^n = 2\frac{\psi(q^2)\varphi(q^2)}{\varphi(-q)^2} = 2\frac{f_4^2}{f_2}\frac{f_4^5}{f_2^2f_8^2}\frac{f_2^2}{f_1^4} = 2\frac{f_4^7}{f_1^4f_2f_8^2}, \tag{64}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(4n+2)q^n = 2\frac{\varphi(q)\psi(q^4)}{\varphi(-q)^2} = 2\frac{f_2^5}{f_1^2f_4^2}\frac{f_8^2}{f_4}\frac{f_2^2}{f_1^4} = 2\frac{f_2^7f_8^2}{f_1^6f_4^3}, \tag{65}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(4n+3)q^n = 4\frac{\psi(q^2)\psi(q^4)}{\varphi(-q)^2} = 4\frac{f_4^2}{f_2}\frac{f_8^2}{f_4}\frac{f_2^2}{f_1^4} = 4\frac{f_2f_4f_8^2}{f_1^4}. \tag{66}$$

This completes the proofs of (17), (18), (19) and (20). We note that the identity (20) is also found by Hirschhorn and Sellers [10, Theorem 2.12].

Using (41) and (45) in (19), we deduce that

$$\sum_{n=0}^{\infty}\overline{p}_o(4n+2)q^n \tag{67}$$

$$= 2\frac{f_2^7f_8^2}{f_1^6f_4^3} \tag{68}$$

$$= 2\left(\frac{f_8^2}{f_4}\right)\left(\frac{f_2^5}{f_1^2f_4^2}\right)\left(\frac{f_2^2}{f_1^4}\right) \tag{69}$$

$$= 2\psi(q^4)\varphi(q)\frac{1}{\varphi(-q)^2} \tag{70}$$

$$= 2\frac{\psi(q^4)}{\varphi(-q^4)^8}\left(\varphi(q^4)+2q\psi(q^8)\right)$$
$$\times\left(\varphi(q^4)^3+2q\varphi(q^4)^2\psi(q^8)+4q^2\varphi(q^4)\psi(q^8)^2+8q^3\psi(q^8)^3\right)^2 \tag{71}$$

$$= 2\frac{\psi(q^4)}{\varphi(-q^4)^8}\left(\varphi(q^4)+2q\psi(q^8)\right)(\varphi(q^4)^6+4q\varphi(q^4)^5\psi(q^8)+12q^2\varphi(q^4)^4\psi(q^8)^2$$
$$+32q^3\varphi(q^4)^3\psi(q^8)^3+48q^4\varphi(q^4)^2\psi(q^8)^4+64q^5\varphi(q^4)\psi(q^8)^5+64q^6\psi(q^8)^6) \tag{72}$$

$$= 2\frac{\psi(q^4)}{\varphi(-q^4)^8}(\varphi(q^4)^7 + 6q\varphi(q^4)^6\psi(q^8) + 20q^2\varphi(q^4)^5\psi(q^8)^2 + 56q^3\varphi(q^4)^4\psi(q^8)^3$$
$$+ 112q^4\varphi(q^4)^3\psi(q^8)^4 + 160q^5\varphi(q^4)^2\psi(q^8)^5 + 192q^6\varphi(q^4)\psi(q^8)^6 + 128q^7\psi(q^8)^7). \tag{73}$$

Extracting the terms containing q^{4n+i} for $i = 0, 1, 2$, respectively, we obtain

$$\sum_{n=0}^{\infty}\overline{p}_o(16n+2)q^n$$
$$= 2\frac{\varphi(q)^7\psi(q)}{\varphi(-q)^8} + 224q\frac{\varphi(q)^3\psi(q)\psi(q^2)^4}{\varphi(-q)^8} = 2\frac{f_2^{45}}{f_1^{31}f_4^{14}} + 224q\frac{f_2^{21}f_4^2}{f_1^{23}}, \tag{74}$$
$$\sum_{n=0}^{\infty}\overline{p}_o(16n+6)q^n$$
$$= 12\frac{\varphi(q)^6\psi(q)\psi(q^2)}{\varphi(-q)^8} + 320q\frac{\varphi(q)^2\psi(q)\psi(q^2)^5}{\varphi(-q)^8} = 12\frac{f_2^{39}}{f_1^{29}f_4^{10}} + 320q\frac{f_2^{15}f_4^6}{f_1^{21}}, \tag{75}$$
$$\sum_{n=0}^{\infty}\overline{p}_o(16n+10)q^n$$
$$= 40\frac{\varphi(q)^5\psi(q)\psi(q^2)^2}{\varphi(-q)^8} + 384q\frac{\varphi(q)\psi(q)\psi(q^2)^6}{\varphi(-q)^8} = 40\frac{f_2^{33}}{f_1^{27}f_4^6} + 384q\frac{f_2^9f_4^{10}}{f_1^{19}}. \tag{76}$$

This completes the proofs of (21), (22) and (23), respectively. □

Remark 1. If we extract the coefficients of q^{4n+3} from (73), we readily obtain (13).

We now prove Theorem 2.

Proof of Theorem 2. From the binomial theorem, we have

$$f_1^4 \equiv f_2^2 \pmod 4. \tag{77}$$

Now, applying the above congruence in (20), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_o(4n+3)q^n = 4\frac{f_2f_4f_8^2}{f_1^4} \tag{78}$$
$$= 4\left(\frac{f_2}{f_1^4}\right)\left(f_4f_8^2\right) \tag{79}$$
$$\equiv 4\left(\frac{f_2}{f_2^2}\right)\left(f_4f_4^4\right) \tag{80}$$
$$= 4\frac{f_4^5}{f_2} \pmod{16}. \tag{81}$$

Extracting the terms containing q^{2n}, we readily deduce (24).

Using the binomial theorem, from (22) and (23), we obtain, respectively, modulo 16,

$$\sum_{n=0}^{\infty} \overline{p}_o(16n+6)q^n \equiv 12\frac{f_2^{39}}{f_1^{29}f_4^{10}} = 12\left(\frac{f_2^{14}}{f_1^{28}}\right)\frac{f_2^5}{f_1}\left(\frac{f_2^{20}}{f_4^{10}}\right) \equiv 12\frac{f_2^5}{f_1}, \tag{82}$$

$$\sum_{n=0}^{\infty} \overline{p}_o(16n+10)q^n \equiv 8\frac{f_2^{33}}{f_1^{27}f_4^{6}} = 8\left(\frac{f_2^{13}}{f_1^{26}}\right)\frac{f_2^2f_4^3}{f_1}\left(\frac{f_2^{18}}{f_4^{9}}\right) \equiv 8\frac{f_2^2f_4^3}{f_1}. \tag{83}$$

This completes the proofs of (25) and (26).

From (17) and (41), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_o(4n)q^n = \frac{f_2^5f_4^3}{f_1^6f_8^2} \tag{84}$$

$$= \frac{f_2^5f_4^3}{f_8^2}\left(\frac{1}{f_1^2}\right)^3 \tag{85}$$

$$= \frac{f_2^5f_4^3}{f_8^2}\left(\frac{f_4^2}{f_2^5}\varphi(q)\right)^3 \tag{86}$$

$$= \frac{f_4^9}{f_2^{10}f_8^2}\varphi(q)^3 \tag{87}$$

$$= \frac{f_4^9}{f_2^{10}f_8^2}\left(\varphi(q^4)+2q\psi(q^8)\right)^3 \tag{88}$$

$$= \frac{f_4^9}{f_2^{10}f_8^2}\left(\varphi(q^4)^3+6q\varphi(q^4)^2\psi(q^8)+12q^2\varphi(q^4)\psi(q^8)^2+8q^3\psi(q^8)^3\right). \tag{89}$$

Extracting the terms containing q^{2n+1}, and then using the binomial theorem, we obtain, modulo 8,

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+4)q^n = \frac{f_2^9}{f_1^{10}f_4^2}\left(6\varphi(q^2)^2\psi(q^4)+8q\psi(q^4)^3\right) \tag{90}$$

$$= \frac{f_2^9}{f_1^{10}f_4^2}\left(6\frac{f_4^{10}}{f_2^4f_8^4}\frac{f_8^2}{f_4}+8q\frac{f_8^6}{f_2^3}\right) \tag{91}$$

$$\equiv 6\frac{f_2^5f_4^7}{f_1^{10}f_8^2} \tag{92}$$

$$= 6\left(\frac{f_4^2}{f_1^8}\right)\frac{f_2^5f_4}{f_1^2}\left(\frac{f_4^4}{f_8^2}\right) \tag{93}$$

$$\equiv 6\frac{f_2^5f_4}{f_1^2}. \tag{94}$$

Applying (42) in (94) yields

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+4)q^n \equiv 6f_4\left(\frac{f_8^5}{f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_8}\right) \pmod 8. \tag{95}$$

Extracting the terms containing q^{4n} and q^{4n+1}, respectively, we have, modulo 8,

$$\sum_{n=0}^{\infty} \overline{p}_o(32n+4)q^n \equiv 6\frac{f_1f_2^5}{f_4^2} = 6f_1\varphi(-q^2)^2f(-q^2), \tag{96}$$

$$\sum_{n=0}^{\infty} \overline{p}_o(32n+12)q^n \equiv 4\frac{f_1^3f_4^2}{f_2} = 4f_1\psi(q^2)f(-q)^2. \tag{97}$$

This completes the proofs of (27) and (28).

Applying the congruences $f_1^4 \equiv f_2^2 \pmod 4$ and $f_8^2 \equiv f_4^4 \pmod 4$ in (18) yield

$$\sum_{n=0}^{\infty} \overline{p}_o(4n+1)q^n = 2\frac{f_4^7}{f_1^4f_2f_8^2} \tag{98}$$

$$= 2\left(\frac{f_2^2}{f_1^4}\right)\frac{f_4^3}{f_2^3}\left(\frac{f_4^4}{f_8^2}\right) \tag{99}$$

$$\equiv 2\frac{f_4^3}{f_2^3} \pmod 8. \tag{100}$$

Extracting the terms containing q^{2n}, we obtain, modulo 8,

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+1)q^n \equiv 2\frac{f_2^3}{f_1^3} = 2\frac{\psi(q)}{\varphi(-q)}. \tag{101}$$

Now, (43) and (47) yield

$$\begin{aligned}\frac{\psi(q)}{\varphi(-q)} &= \frac{\varphi(-q^9)}{\varphi(-q^3)^4}\big(\Pi(q^3)+2q\psi(q^9)\big)\big(\varphi(-q^9)^2+2q\varphi(-q^9)\Omega(-q^3)+4q^2\Omega(-q^3)^2\big)\\ &= \frac{\varphi(-q^9)^3}{\varphi(-q^3)^4}\Pi(q^3)+3q\frac{\varphi(-q^9)^3}{\varphi(-q^3)^4}\psi(q^9)+6q^2\frac{\varphi(-q^9)^2}{\varphi(-q^3)^4}\Omega(-q^3)\psi(q^9)\\ &+4q^3\frac{\phi(-q^9)}{\phi(-q^3)^4}\Omega(-q^3)^2\psi(q^9).\end{aligned} \tag{102}$$

From (101) and (102), and then extracting the terms containing q^{3n+i} for $i = 0, 1, 2,$

respectively, we find, modulo 8,

$$\sum_{n=0}^{\infty}\overline{p}_o(24n+1)q^n \equiv 2\frac{\varphi(-q^3)^3}{\varphi(-q)^4}\Pi(q) = 2\frac{f_2^5 f_3^8}{f_1^9 f_6^4} = 2\left(\frac{f_2^4}{f_1^8}\right)\frac{f_2}{f_1}\left(\frac{f_3^8}{f_6^4}\right) \equiv 2\frac{f_2}{f_1}, \tag{103}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(24n+9)q^n = 6\frac{\varphi(-q^3)^3}{\varphi(-q)^4}\psi(q^3) = 6\frac{f_2^4 f_3^5}{f_1^8 f_6} \equiv 6\left(\frac{f_2^4}{f_1^8}\right) f_3 f_6 \left(\frac{f_3^4}{f_6^2}\right) = 6 f_3 f_6, \tag{104}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(24n+17)q^n = 12\frac{\varphi(-q^3)^2}{\varphi(-q)^4}\Omega(-q)\psi(q^3) \equiv 4\left(\frac{f_2^4}{f_1^8}\right)\frac{f_1 f_3^2 f_6^2}{f_2} \equiv 4\frac{f_1 f_3^2 f_6^2}{f_2}. \tag{105}$$

This completes the proofs of (29) and (30). Extracting the terms containing q^{3n} from (104) and using (7), we readily obtain (31). This complete the proof of Theorem 2. □

If we extract the terms containing q^{3n+1} and q^{3n+2} from (104), the following two Ramanujan-like congruences can readily be obtained.

Corollary 1. *For any $n \geq 0$,*

$$\overline{p}_o(72n+33) \equiv 0 \pmod 8, \tag{106}$$

$$\overline{p}_o(72n+57) \equiv 0 \pmod 8. \tag{107}$$

We now prove Theorem 3.

Proof of Theorem 3. Extracting the terms containing q^{4n+2} and q^{4n+3} from (95), we have

$$\sum_{n=0}^{\infty}\overline{p}_o(32n+20)q^n \equiv 0 \pmod 8, \tag{108}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(32n+28)q^n \equiv 0 \pmod 8. \tag{109}$$

From [11, Corollary 2.10], we have that $\overline{p}_o(2n) \equiv 0 \pmod 8$ if $\overline{p}_o(n) \equiv 0 \pmod 8$. This proves (32) and (33) for any $\alpha \geq 0$. □

4. Infinite Families of Congruences for $\overline{p}_o(n)$

In this section, we prove Theorem 4. Before we prove Theorem 4, we first prove the following result.

Theorem 5. *Let $p \geq 3$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then, for all nonnegative integers n and α, we have*

$$\sum_{n=0}^{\infty} \overline{p}_o(8p^{2\alpha}n + 3p^{2\alpha})q^n \equiv 4f(-q^2)^3\psi(q) \pmod{16}. \tag{110}$$

Proof. From (24), we have that (110) is true for $\alpha = 0$. We now use induction on α to complete the proof. Observe that (110) can also be written as

$$\sum_{n=0}^{\infty} \overline{p}_o\left(8\left(p^{2\alpha}n + 3\frac{p^{2\alpha}-1}{8}\right) + 3\right) q^n \equiv 4f(-q^2)^3\psi(q) \pmod{16}. \tag{111}$$

We suppose that (111) holds for some $\alpha > 0$. Substituting (49) and (52) into (111), we have, modulo 16

$$\begin{aligned} &\sum_{n=0}^{\infty} \overline{p}_o\left(8\left(p^{2\alpha}n + 3\frac{p^{2\alpha}-1}{8}\right) + 3\right) q^n \\ &\equiv 4\left[\frac{1}{2}\sum_{\substack{k=0 \\ k\neq\frac{p-1}{2}}}^{p-1} (-1)^k q^{k^2+k} \sum_{n=-\infty}^{\infty} (-1)^n(2pn+2k+1)q^{pn(pn+2k+1)}\right. \\ &\left. +(-1)^{\frac{p-1}{2}} p q^{\frac{p^2-1}{4}} f(-q^{2p^2})^3\right] \\ &\times \left[\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(\frac{p^2+(2m+1)p}{2}, \frac{p^2-(2m+1)p}{2}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right]. \end{aligned} \tag{112}$$

For a prime $p \geq 3$ and $0 \leq k \leq p-1$, $0 \leq m \leq \frac{p-1}{2}$, we consider

$$(k^2+k) + \frac{m^2+m}{2} \equiv 3\frac{p^2-1}{8} \pmod{p} \tag{113}$$

which is equivalent to

$$2(2k+1)^2 + (2m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-2}{p}\right) = -1$, we have $k = m = \frac{p-1}{2}$ is the only solution of (113). Therefore, extracting the terms containing $q^{pn+3\frac{(p^2-1)}{8}}$ from both sides of (112), and then replacing q^p by q, we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_o\left(8\left(p^{2\alpha+1}n + 3\frac{p^{2\alpha+2}-1}{8}\right) + 3\right) q^n \equiv 4f(-q^{2p})^3\psi(q^p) \pmod{16}. \tag{114}$$

Similarly, extracting the terms containing q^{pn} from both sides of (114), and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{p}_o\left(8\left(p^{2(\alpha+1)}n+3\frac{p^{2(\alpha+1)}-1}{8}\right)+3\right)q^n \equiv 4f(-q^2)^3\psi(q) \pmod{16}, \quad (115)$$

proving the result for $\alpha+1$. This completes the proof of the theorem. $\square$

Proofs of (34) *and* (35). From (114), it follows that

$$\overline{p}_o(8p^{2\alpha+1}(pn+j)+3p^{2\alpha+2}) \equiv 0 \pmod{16}, \quad (116)$$

where $j=1,2,\ldots,p-1$. This completes the proof of (34) for $\alpha \geq 1$.

The proof of (35) proceeds along similar lines to the proof of (34). Therefore, we omit the details for reasons of brevity. $\square$

We now prove two results which will be used to prove (36) and (37).

Theorem 6. *Let $p \geq 3$ be a prime such that $\left(\frac{-2}{p}\right)=-1$. Then, for all nonnegative integers n and α, we have*

$$\sum_{n=0}^{\infty}\overline{p}_o(72p^{2\alpha}n+9p^{2\alpha})q^n \equiv 6f(-q)f(-q^2) \pmod{8}. \quad (117)$$

Proof. Clearly, (117) is true when $\alpha=0$ due to (31). We now use induction on α to complete the proof.

For a prime $p \geq 5$ and $-\frac{p-1}{2} \leq k,m \leq \frac{p-1}{2}$, we consider the congruence

$$\frac{3k^2+k}{2}+2\frac{3m^2+m}{2} \equiv 3\frac{p^2-1}{24} \pmod{p}, \quad (118)$$

which is equivalent to

$$(6k+1)^2+2(6m+1)^2 \equiv 0 \pmod{p}. \quad (119)$$

Since $\left(\frac{-2}{p}\right)=-1$, therefore $k=m=\frac{\pm p-1}{6}$ is the only solution of (118). By Lemma 6 and proceeding similarly as shown in the proof of Theorem 4, we deduce the following congruence

$$\sum_{n=0}^{\infty}\overline{p}_o\left(72\left(p^{2\alpha+1}n+9\frac{p^{2\alpha+2}-1}{72}\right)+9\right)q^n \equiv 6f(-q^p)f(-q^{2p}) \pmod{8}. \quad (120)$$

We next extract the terms containing q^{pn} from both sides of the above congruence, and observe that (117) is true when α is replaced by $\alpha+1$. This completes the proof of the result. $\square$

Theorem 7. *Let $p \geq 3$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then, for all nonnegative integers n and α, we have*

$$\sum_{n=0}^{\infty} \overline{p}_o\left(24\left(p^{2\alpha}n + 17\frac{p^{2\alpha}-1}{24}\right)+17\right)q^n \equiv 4f(-q^3)^3\Omega(-q) \pmod 8. \tag{121}$$

Proof. From (30) we can see that (121) is true when $\alpha = 0$. Suppose that (121) holds for some $\alpha > 0$. Substituting (52) and (54) into (121), we have, modulo 8

$$\sum_{n=0}^{\infty} \overline{p}_o\left(24\left(p^{2\alpha}n + 17\frac{p^{2\alpha}-1}{24}\right)+17\right)q^n \tag{122}$$
$$= \left[\frac{1}{2}\sum_{\substack{m=0\\ m\neq \frac{p-1}{2}}}^{p-1}(-1)^m q^{3\frac{m^2+m}{2}}\sum_{n=-\infty}^{\infty}(-1)^n(2pn+2m+1)q^{\frac{3pn(pn+2m+1)}{2}}\right.$$
$$\left.+(-1)^{\frac{p-1}{2}}pq^{3\frac{p^2-1}{8}}f(-q^{3p^2})^3\right]$$
$$\times\left[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p+1}{3}}}^{\frac{p-1}{2}}(-1)^k q^{3k^2-2k}\sum_{n=-\infty}^{\infty}(-1)^n q^{pn(3pn+6k-2)} + (-1)^{\frac{\pm p+1}{3}}q^{\frac{p^2-1}{3}}\Omega(-q^{p^2})\right].$$

For a prime $p \geq 5$, $0 \leq m \leq p-1$ and $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$, we consider

$$\frac{3m(m+1)}{2} + 3k^2 - 2k \equiv 17\frac{p^2-1}{24} \pmod p, \tag{123}$$

which is equivalent to $(6m+3)^2 + 2(6k-2)^2 \equiv 0 \pmod p$. Since $\left(\frac{-2}{p}\right) = -1$, we have $m = \frac{p-1}{2}$ and $k = \frac{\pm p+1}{3}$ is the only solution of (123). Therefore, extracting the terms containing $q^{pn+17\frac{p^2-1}{24}}$ from both sides of (122), and then replacing q^p by q, we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_o\left(24\left(p^{2\alpha+1}n + 17\frac{p^{2\alpha+2}-1}{24}\right)+17\right)q^n \equiv 4f(-q^{3p})^3\Omega(-q^p) \pmod 8. \tag{124}$$

Similarly, extracting the terms containing q^{pn} from both sides of (124), and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_o\left(24\left(p^{2(\alpha+1)}n + 17\frac{p^{2(\alpha+1)}-1}{24}\right)+17\right)q^n \equiv 4f(-q^3)^3\Omega(-q) \pmod 8. \tag{125}$$

This completes the proof of the result. $\square$

Proofs of (36) *and* (37). By extracting the terms containing q^{pn+j} from (120), where $j = 1, 2, \ldots, p-1$, it follows that

$$\overline{p}_o\left(72p^{2\alpha}n + (9p + 72j)p^{2\alpha-1}\right) \equiv 0 \pmod{8}. \tag{126}$$

This completes the proof of (36) for $\alpha \geq 1$.

From (124), it follows that

$$\overline{p}_o(24p^{2\alpha+1}(pn + j) + 17p^{2\alpha+2}) \equiv 0 \pmod{8}, \tag{127}$$

where $j = 1, 2, \ldots, p-1$. This completes the proof of (37) for $\alpha \geq 1$. $\square$

Proofs of (38) *and* (39). We now substitute the p-dissection identities, namely (50), (51) and (53) into (27) and (28). For $p \geq 3$, $-\frac{p-1}{2} \leq k, m \leq \frac{p-1}{2}$ and $\left(\frac{-2}{p}\right) = -1$, the congruences

$$\frac{3k^2 + k}{2} + 3m^2 + m \equiv 3\frac{p^2 - 1}{24} \pmod{p}, \tag{128}$$

$$\frac{3k^2 + k}{2} + 3m^2 + 2m \equiv 9\frac{p^2 - 1}{24} \pmod{p} \tag{129}$$

have only the solutions $k = m = \frac{\pm p-1}{6}$, and $k = \frac{\pm p-1}{6}$, $m = \frac{\pm p-1}{3}$, respectively. Proceeding similarly as shown in the proof of (37), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_o\left(32\left(p^{2\alpha+1}n + 3\frac{p^{2\alpha+2} - 1}{24}\right) + 4\right) q^n \equiv 6\frac{f_p f_{2p}^5}{f_{4p}^2} \pmod{8}, \tag{130}$$

$$\sum_{n=0}^{\infty} \overline{p}_o\left(32\left(p^{2\alpha+1}n + 9\frac{p^{2\alpha+2} - 1}{24}\right) + 12\right) q^n \equiv 4\frac{f_p^3 f_{4p}^2}{f_{2p}} \pmod{8}. \tag{131}$$

Now, from (130) and (131), it follows that

$$\overline{p}_o(32p^{2\alpha+1}(pn + j) + 4p^{2\alpha+2}) \equiv 0 \pmod{8}, \tag{132}$$

$$\overline{p}_o(32p^{2\alpha+1}(pn + j) + 12p^{2\alpha+2}) \equiv 0 \pmod{8}, \tag{133}$$

where $j = 1, 2, \ldots, p-1$. This completes the proofs of (38) and (39) for $\alpha \geq 1$. $\square$

Before we prove (40), we first prove the following result.

Theorem 8. *Let $p \geq 3$ be a prime such that $p \equiv 3 \pmod{4}$. Then, for all nonnegative integers n and α, we have*

$$\sum_{n=0}^{\infty} \overline{p}_o\left(16\left(p^{2\alpha}n + 5\frac{p^{2\alpha} - 1}{8}\right) + 10\right) q^n \equiv 8f(-q^4)^3\psi(q) \pmod{16}. \tag{134}$$

Proof. We use induction on α to proof the theorem. Clearly, (134) is true when $\alpha = 0$ due to (26). Suppose that (134) holds for some $\alpha > 0$. For a prime $p \geq 5$ and $0 \leq k \leq p-1$, $0 \leq m \leq \frac{p-1}{2}$, the equation

$$2(k^2+k) + \frac{m^2+m}{2} \equiv 5\frac{p^2-1}{8} \pmod{p}, \tag{135}$$

which is equivalent to $4(2k+1)^2 + (2m+1)^2 \equiv 0 \pmod{p}$, has the only solution $k = m = \frac{p-1}{2}$ as $p \equiv 3 \pmod{4}$. Applying (49) and (52) in (134), and then proceeding similarly as shown in the proof of Theorem 4, we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_o\left(16\left(p^{2\alpha+1}n + 5\frac{p^{2\alpha+2}-1}{8}\right) + 10\right) q^n \equiv 4f(-q^{4p})^3\psi(q^p) \pmod{16}. \tag{136}$$

Extracting the terms containing q^{pn} from both sides of (136), we find that

$$\sum_{n=0}^{\infty} \overline{p}_o\left(16\left(p^{2(\alpha+1)}n + 5\frac{p^{2(\alpha+1)}-1}{8}\right) + 10\right) q^n \equiv 4f(-q^4)^3\psi(q) \pmod{16}, \tag{137}$$

completing the proof of (134). ☐

Proof of (40). From (136), it follows that

$$\overline{p}_o(16p^{2\alpha+1}(pn+j) + 10p^{2\alpha+2}) \equiv 0 \pmod{16}, \tag{138}$$

where $j = 1, 2, \ldots, p-1$. This completes the proof of (40) for $\alpha \geq 1$. ☐

Acknowledgment. We appreciate the careful review, comments and corrections made by the referee which helped to correct and improve the paper. We are very grateful to Professor Michael Hirschhorn for careful reading of a draft of the manuscript and for his valuable suggestions which have substantially improved our paper.

References

[1] E. Ardonne, R. Kedem and M. Stone, Filling the Bose sea: symmetric quantum Hall edge states and affine characters, *J. Phys. A* **38** (2005), 617-636.

[2] Z. Ahmed and N. D. Baruah, New congruences for ℓ-regular partitions for $\ell \in \{5, 6, 7, 49\}$, *Ramanujan J.* **40** (2016), 649-668.

[3] C. Bessenrodt, On pairs of partitions with steadily decreasing parts, *J. Combin. Theory Ser. A* **99** (2002), 162-174.

[4] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[5] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Student Mathematical Library, AMS, vol. 34, 2006.

[6] S. C. Chen, On the number of overpartitions into odd parts, *Discrete Math.* **325** (2014), 32-37.

[7] S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* **356** (2004), 1623-1635.

[8] S. P. Cui and N. S. S. Gu, Arithmetic properties of ℓ-regular partitions, *Adv. Appl. Math.* **51** (2013), 507-523.

[9] M. D. Hirschhorn, *The Power of q: A Personal Journey*, Developments in Mathematics, vol. 49, Springer, 2017.

[10] M. D. Hirschhorn and J. A. Sellers, Arithmetic relations for overpartitions, *J. Combin. Math. Combin. Comput.* **53** (2005), 65-73.

[11] M. D. Hirschhorn and J. A. Sellers, Arithmetic Properties of Overpartitions into Odd Parts, *Annals of Combinatorics* **10** (2006), 353-367.

[12] B. Kim, A short note on the overpartition function, *Discrete Math.* **309** (2009), 2528-2532.

[13] B. Kim, Overpartition pairs modulo powers of 2, *Discrete Math.* **311** (2011), 835-840.

[14] K. Mahlburg, The overpartition function modulo small powers of 2, *Discrete Math.* **286** (2004), 263-267.

[15] J. P. O. Santos and A. V. Sills, q-Pell sequences and two identities of V. A. Lebesgue, *Discrete Math.* **257** (2002), 125-142.