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ARITHMETIC OF THE FABIUS FUNCTION

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Abstract
The Fabius function was defined in 1935 by Jessen and Wintner and has been independently defined at least six times since. We attempt to unify notations related to the Fabius function. The Fabius function $F(x)$ takes rational values at dyadic points. We study the arithmetic of these rational numbers. In particular, we define two sequences of natural numbers that determine these rational numbers. Using these sequences we solve a conjecture raised in MathOverflow by Vladimir Reshetnikov. We determine the dyadic valuation of $F(2^{-n})$, showing that $\nu_2(F(2^{-n})) = -\binom{n}{2} - \nu_2(n!) - 1$. We give the proof of a formula that allows an efficient computation of exact or approximate values of $F(x)$.

1. Introduction

The Fabius function is a natural object. It is not surprising that it has been independently defined several times. As far as we know it was defined independently in the following cases:

1. In 1935 by B. Jessen and A. Wintner [7, Ex. 5, p. 62] (English). They showed in five lines that it is infinitely differentiable.
2. In 1966 by J. Fabius [4] (English); considered as the distribution function of a random variable.
3. In 1971 by V. A. Rvachëv [12] (Ukrainian); defined as a solution to the equation $y'(x) = 2(y(2x+1) - y(2x-1))$.
4. In 1981 by G. Kh. Kirov and G. A. Totkov [8] (Bulgarian).
5. In 1982 by J. Arias de Reyna [2] (Spanish); defined in the same way as V. A. Rvachëv.
6. In 1985 by R. Schnabl [17] (German).

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Today there are many papers dealing with the Fabius function, due mainly to V. A. Rvachëv [12]–[16] (a summary of his results can be found in [16]). In the paper [2] the function is defined and its main properties are proven anew. Recently [2] has been translated to English and posted in arXiv:1702.05442.

1.1. Definitions and First Properties of Rvachëv and Fabius Functions

There are two closely related functions: Fabius function $F(x)$ and Rvachëv function $\operatorname{up}(x)$. Rvachëv defined his function $\operatorname{up}\colon \mathbb{R} \to \mathbb{R}$ as the unique solution of the equation $y'(x) = 2(y(2x+1) - y(2x-1))$ satisfying $y(0) = 1$. This functional differential equation was also our starting point in [2]. We were searching for a bump function $y(x)$ with support in $[-1,1]$ so that the graph of its derivative in the interval $[-1,0]$ is the graph of $y(x)$ tansformed by an affine map, and the same happens with the graph of the derivative on $[0,1]$. Hence our starting point was a graph similar to figure 1, with the height of $\operatorname{up}'(x)$ equal an unknown k. These conditions translate in the functional differential equation $y'(x) = k(y(2x+1) - y(2x-1))$. Only for the value $k = 2$ there is a non null solution.

In fact, the unique solution $\operatorname{up}(x)$ is infinitely differentiable of compact support equal to $[-1,1]$, and for all $x \in \mathbb{R}$ satisfies the equation

$$\operatorname{up}'(x) = 2\big(\operatorname{up}(2x+1) - \operatorname{up}(2x-1)\big), \qquad (1)$$

with $\operatorname{up}(0) = 1$.

The Fabius function $F\colon \mathbb{R} \to \mathbb{R}$, defined by Fabius [4], is the unique solution to the functional differential equation $F'(x) = 2F(2x)$ with $F(x) = 0$ for $x < 0$ and $F(1) = 1$. This function coincides with the function $\theta(x)$ in [2].

There is a connection between these two functions (see [2, Theorem 4]). Any of them can be easily defined in terms of the other, namely

$$F(x) = \sum_{n=0}^{\infty} (-1)^{w(n)} \operatorname{up}(x - 2n - 1), \qquad (2)$$

where $w(n)$ is the sum of the digits of n expressed in base 2.

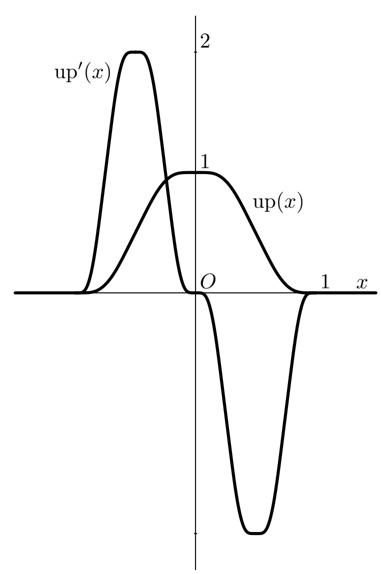

Figure 1: Rvachëv function

We may also define $\operatorname{up}(x)$ in terms of $F(x)$,

$$\operatorname{up}(t) = \begin{cases} F(t+1) & \text{for } -1 \le t \le 0, \\ F(1-t) & \text{for } 0 \le t \le 1, \\ 0 & \text{for } |t| > 1. \end{cases} \qquad (3)$$

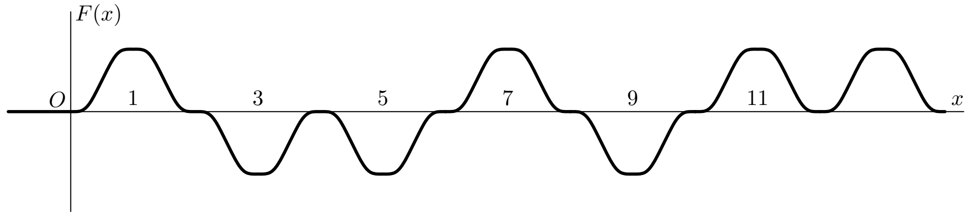

Figure 2: Fabius function

This follows from (2) noticing that the support of the summands in (2) are essentially disjoint.

The Fabius function (2) appears when one attempts to compute the derivatives $\operatorname{up}^{(n)}(x)$. We have $F'(x) = 2F(2x)$ for all $x \in \mathbb{R}$. Therefore ([2, eq. (22)])

$$F^{(k)}(x) = 2^{\binom{k+1}{2}} F(2^k x), \tag{4}$$

and ([2, eq. (23)])

$$\operatorname{up}^{(n)}(x) = 2^{\binom{n+1}{2}} F(2^n(x+1)), \qquad -1 \le x \le 1. \tag{5}$$

The Fourier transform of $\operatorname{up}(x)$ is an entire function, namely (see [16, eq.(0.2)] or [2, eq. (4)])

$$\widehat{\operatorname{up}}(z) = \prod_{n=0}^{\infty} \frac{\sin(\pi z/2^n)}{\pi z/2^n}, \qquad \text{with} \quad \widehat{\operatorname{up}}(0) = 1. \tag{6}$$

The inversion formula yields the following expression

$$\operatorname{up}(x) = \int_{\mathbb{R}} \widehat{\operatorname{up}}(t) e^{2\pi i t x}\, dt. \tag{7}$$

1.2. Summary of the Paper

The function $\operatorname{up}(x)$ takes rational values at dyadic points [13].[2] The object of this paper is the study of the arithmetical properties of these rational values. Some of the results are also contained in [2]. In particular, we solve here a question (see Question in page 9) posed in MathOverflow by Vladimir Reshetnikov [11].

Proposition 1 states a recurrence for factors c_n in the coefficients of the power series of the Fourier transform $\widehat{\operatorname{up}}(z)$, see the definition of c_n in (8). In addition,

[2] This Russian paper was not accessed directly. Rather, it came to notice through the works of V. A. Rvachëv [16]. In [2] it was independently proven that the values of $\operatorname{up}(k/2^n)$ are rational numbers.

it states that c_n can be expressed in terms of a sequence of natural numbers F_n defined in (13). There is another sequence of rational numbers d_n related to Fabius function (23) and Rvachëv function (22). Proposition 3 gives a formula for d_n in terms of c_n. Proposition 4 shows that d_n can be written (25) in terms of natural numbers G_n.

Reshetnikov's question asks if certain numbers R_n defined in (28) are integers. Proposition 5 states that R_n can be written in terms of d_n. Theorem 1 states that each R_{2n+1} is a multiple of F_n. Proposition 6 gives an explicit expression of R_{2n} depending on F_n. Theorem 2 states that each R_n is a natural number, solving Question 1.

The user Pierrot Bolnez asked a question in Mathematica StackExchange [3] about how to compute $F(x)$. Reshetnikov answered, posting there a very efficient code he attributes to Rvachëv. Proposition 7 is the result of our efforts to reverse engineer this code. We do not have access to Rvachëv's exposition of his method.

Let x be a dyadic rational number $x = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}$ for integers $n_1 > n_2 > \cdots > n_k$, and $y = x - 2^{n_1}$. Proposition 7 expresses the difference $F(x) - F(y)$ as a known rational number. In a finite number of steps we arrive to $F(0) = 0$. This reasoning proves that $F(x)$ is rational and allows us to compute its value easily.

If we want to compute $F(x)$ for an arbitrary real number with a given degree of precision, we may use the fact that $|F(x) - F(y)| \leq 2|x - y|$ to reduce the computation of $F(x)$ to computing $F(y)$ for a dyadic rational number y which depends on the precision required. Proposition 7 gives the best known way to compute $F(x)$.

Theorem 3 states that an explicit integer multiple of $\mathrm{up}(a/2^n)$ is a natural number. Conjecture 1 states an unsolved generalization of Theorem 3. Proposition 10 states that all F_n are odd. Theorem 6 gives a formula for the exact power of 2 that divides $\mathrm{up}(1 - 2^{-n})$. Proposition 11 states recurrences for c_n and d_n using the well-known Bernoulli numbers.

1.3. Notations

We use $w(n)$ to denote the sum of the digits of n expressed in binary notation. We denote by $\nu_2(r)$ the dyadic valuation of the rational number r, that is to say, the exponent of 2 in the factorization of r. The double factorial $n!!$ denotes the product of integers from 1 to n that have the same parity as n. The sequence of numbers B_k denotes the Bernoulli numbers: $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_{2k+1} = 0$

2. The Values of the Fabius Function and Two Sequences of Rational Numbers

2.1. The Sequence of Rational Numbers c_n

The Fourier transform $\widehat{\mathrm{up}}(z)$ is an entire function. In [2, eq. (4) and (6)] it is shown that

$$\widehat{\mathrm{up}}(z) = \prod_{n=0}^{\infty} \frac{\sin(\pi z/2^n)}{\pi z/2^n} = \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{(2n)!} (2\pi z)^{2n}. \tag{8}$$

These coefficients c_n are very important for the study of the Fabius function. They are given by the integrals [2, eq. (34)]

$$c_n = \int_{\mathbb{R}} t^{2n}\, \mathrm{up}(t)\, dt, \tag{9}$$

and satisfy the recurrence [2, eq. (7)]

$$(2n+1)2^{2n} c_n = \sum_{k=0}^{n} \binom{2n+1}{2k} c_k. \tag{10}$$

The next Proposition is contained in [2] but is a little cryptic, so it is expanded here.

Proposition 1. *The numbers c_n satisfy the recurrence*

$$c_0 = 1, \quad (2n+1)(2^{2n}-1)c_n = \sum_{k=0}^{n-1} \binom{2n+1}{2k} c_k. \tag{11}$$

For any natural number n we have

$$c_n = \frac{F_n}{(2n+1)!!} \prod_{\nu=1}^{n} (2^{2\nu}-1)^{-1}, \tag{12}$$

where F_n are natural numbers.

Proof. The last term in the sum of (10) is equal to $(2n+1)c_n$. We obtain (11) from (10) by adding $-(2n+1)c_n$ to both sides. We have $c_0 = \widehat{\mathrm{up}}(0)$ and this is equal to 1. Then the recurrence can be used to compute the numbers c_n:

$$1,\ \frac{1}{9},\ \frac{19}{675},\ \frac{583}{59\,535},\ \frac{132\,809}{32\,531\,625},\ \frac{46\,840\,699}{24\,405\,225\,075},\ \frac{4\,068\,990\,560\,161}{4\,133\,856\,862\,760\,625},\ \dots$$

The recurrence shows that $c_n \geq 0$ for all n.

Equation (12) follows from the following consideration. Define

$$F_n := c_n(2n+1)!! \prod_{\nu=1}^{n} (2^{2\nu}-1). \tag{13}$$

It remains to show that the numbers F_n are natural. We understand that an empty product is equal to 1, so that $F_0 = 1$. For $n \geq 1$, substituting (12) in the recurrence (11) yields, for $n \geq 1$,

$$\frac{F_n}{(2n-1)!!}\prod_{\nu=1}^{n-1}(2^{2\nu}-1)^{-1} = \sum_{k=0}^{n-1}\binom{2n+1}{2k}\frac{F_k}{(2k+1)!!}\prod_{\nu=1}^{k}(2^{2\nu}-1)^{-1},$$

so that

$$F_n = \sum_{k=0}^{n-1} F_k\binom{2n+1}{2k}\frac{(2n-1)!!}{(2k+1)!!}\prod_{\nu=k+1}^{n-1}(2^{2\nu}-1), \qquad n \geq 1. \tag{14}$$

This expression for F_n shows by induction that all the F_n are natural numbers. The first F_n's are

$$1,\ 1,\ 19,\ 2\,915,\ 2\,788\,989,\ 14\,754\,820\,185,\ 402\,830\,065\,455\,939,\ \dots$$

□

2.2. Relation to Values of the Fabius Function

Notice that by (3) $F(2^{-n}) = \operatorname{up}(1-2^{-n})$. In [2, Theorem 6] it is shown that

$$n\int_0^1 x^{n-1}\operatorname{up}(x)\,dx = n!\,2^{\binom{n}{2}}\operatorname{up}(1-2^{-n}). \tag{15}$$

Since $\operatorname{up}(x)$ is an even function, it follows that

$$\frac{c_n}{2} = \int_0^1 x^{2n}\operatorname{up}(x)\,dx = (2n)!\,2^{\binom{2n+1}{2}}\operatorname{up}(1-2^{-2n-1}). \tag{16}$$

Therefore, as noticed in [2] we have

$$F(2^{-2n-1}) = \operatorname{up}(1-2^{-2n-1}) = \frac{2^{-\binom{2n+1}{2}}}{2(2n)!}\frac{F_n}{(2n+1)(2n-1)\cdots 1}\prod_{k=1}^{n}(2^{2k}-1)^{-1}. \tag{17}$$

The values of $F(2^{-n}) = \operatorname{up}(1-2^{-n})$ depend of another interesting sequence of rational numbers.

2.3. The Sequence of Rational Numbers d_n

The sequence of numbers d_n is introduced in [2, eq.(38) and (39)] as the coefficients of the power series

$$f(x) = \sum_{n=0}^{\infty}\frac{d_n}{n!}x^n \tag{18}$$

satisfying

$$f(2x) = \frac{e^x - 1}{x} f(x). \tag{19}$$

So these numbers are given also by the recurrence [2, eq. (40)]

$$d_0 = 1, \quad (n+1)(2^n - 1)d_n = \sum_{k=0}^{n-1} \binom{n+1}{k} d_k. \tag{20}$$

The d_n are connected with $\operatorname{up}(t)$ in [2, eq. (36)]. It is shown there that the function

$$f(x) := 1 + x\int_0^1 \operatorname{up}(t)e^{xt}\,dt = e^{\frac{x}{2}}\widehat{\operatorname{up}}\Big(\frac{ix}{4\pi}\Big) \tag{21}$$

satisfies (19). We give the simple proof that was omitted in [2].

Proposition 2. *If f is defined as in* (21)*, then we have* (19)*.*

Proof. From (6) it follows that

$$\widehat{\operatorname{up}}(z) = \frac{\sin \pi z}{\pi z}\widehat{\operatorname{up}}(z/2).$$

Therefore

$$f(2x) = e^x\widehat{\operatorname{up}}\Big(\frac{ix}{2\pi}\Big) = e^x\frac{\sin(ix/2)}{ix/2}\widehat{\operatorname{up}}\Big(\frac{ix}{4\pi}\Big) = e^{\frac{x}{2}}\frac{e^{-x/2} - e^{x/2}}{2i(ix/2)}f(x) = \frac{e^x - 1}{x}f(x).$$

□

By (21) we have

$$f(x) = 1 + x\sum_{n=0}^\infty \frac{x^n}{n!}\int_0^1 t^n \operatorname{up}(t)\,dt.$$

So, equating the coefficients of equal powers with the expansion (18) we obtain [2, eq. (42)]

$$d_n = n\int_0^1 t^{n-1}\operatorname{up}(t)\,dt. \tag{22}$$

Together with [2, eq. (33)] this yields

$$d_n = n!\,2^{\binom{n}{2}}\operatorname{up}(1 - 2^{-n}) = n!\,2^{\binom{n}{2}}F(2^{-n}). \tag{23}$$

The values of the d_n can be computed easily applying the recurrence (20). The first terms are

$$1, \frac{1}{2}, \frac{5}{18}, \frac{1}{6}, \frac{143}{1\,350}, \frac{19}{270}, \frac{1\,153}{23\,814}, \frac{583}{17\,010}, \frac{1\,616\,353}{65\,063\,250}, \frac{132\,809}{7\,229\,250}, \frac{134\,926\,369}{9\,762\,090\,030}, \dots$$

Proposition 3. *The values of d_n can be computed in terms of c_n as:*

$$d_n = \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}c_k. \tag{24}$$

Proof. For $n = 0$ we have $c_0 = d_0 = 1$ and (24) is trivial in this case. For $n \geq 1$ we have the following computation, starting from (22)

$$\begin{aligned} d_n &= n\int_0^1 t^{n-1}\,\mathrm{up}(t)\,dt = \int_0^1 \mathrm{up}(t)\,d(t^n) = t^n\,\mathrm{up}(t)\Big|_{t=0}^{t=1} - \int_0^1 t^n\,\mathrm{up}'(t)\,dt \\ &= -\int_0^1 2t^n\big(\mathrm{up}(2t+1) - \mathrm{up}(2t-1)\big)\,dt = 2\int_0^1 t^n\,\mathrm{up}(2t-1)\,dt \\ &= \int_{-1}^1 \mathrm{up}(x)\Big(\frac{x+1}{2}\Big)^n\,dx = \frac{1}{2^n}\sum_{k=0}^n \int_{-1}^1 \binom{n}{k}\,\mathrm{up}(x)x^k\,dx \\ &= \frac{1}{2^n}\sum_{0\leq h\leq n/2}\binom{n}{2h}\int_{-1}^1 \mathrm{up}(x)x^{2h}\,dx = \frac{1}{2^n}\sum_{0\leq h\leq n/2}\binom{n}{2h}c_h. \end{aligned}$$

□

As in the case of the c_n, there is an interesting sequence of natural numbers hiding in d_n.

Proposition 4. *There is a sequence of natural numbers (G_n) such that*

$$d_n = \frac{G_n}{(n+1)!}\prod_{k=1}^n (2^k-1)^{-1}. \tag{25}$$

Proof. For $n \geq 0$ define G_n by the equation (25). The recurrence (20) then implies

$$G_n = \sum_{k=0}^{n-1} G_k\binom{n+1}{k}\frac{n!}{(k+1)!}\prod_{j=k+1}^{n-1}(2^j-1), \qquad n \geq 1.$$

By induction we obtain that all the G_n are natural numbers. □

The first numbers G_n starting from $n = 0$ are

$$1,\ 1,\ 5,\ 84,\ 4\,004,\ 494\,760,\ 150\,120\,600,\ 107\,969\,547\,840,\ 179\,605\,731\,622\,464,\dots$$

By (23) we obtain the explicit value

$$F(2^{-n}) = \mathrm{up}(1-2^{-n}) = \frac{2^{-\binom{n}{2}}G_n}{n!\,(n+1)!}\prod_{k=1}^n (2^k-1)^{-1}, \qquad n \geq 0. \tag{26}$$

Equations (17) and (26) yields

$$\frac{G_{2n+1}}{2n+1} = 2^n (n+1)! F_n \prod_{k=0}^{n} (2^{2k+1} - 1), \qquad n \geq 0. \tag{27}$$

So that $(2n+1)F_n \mid G_{2n+1}$.

3. Reshetnikov's Question

Recently Vladimir Reshetnikov posed a question (question 261649 in MathOverflow) about the Fabius function. The results in [2] make it easy to answer. Note that [2] has been translated to English and posted in arXiv:1702.05442. We refer to this paper for some of the results.

The question of Reshetnikov is the following:

Question 1 (V. Reshetnikov)**.** For any natural number n define the number

$$R_n = 2^{\binom{n-1}{2}} (2n)!\, F(2^{-n}) \prod_{m=1}^{\lfloor n/2 \rfloor} (2^{2m} - 1). \tag{28}$$

It is true that all R_n are natural numbers?

The first R_n (starting from $n = 1$) certainly are:

1, 5, 15, 1 001, 5 985, 2 853 675, 26 261 235, 72 808 620 885, 915 304 354 965, ...

Applying some of the results in [2] we will show that in fact all R_n are natural numbers.

In Section 5 we consider the problem of the computation of the exact values of the function $F(x)$ at dyadic points. This will allow us to state another conjecture related to this question of Reshetnikov.

4. Answer to Reshetnikov's Question

Reshetnikov's numbers can be computed in terms of the d_n. This will be essential in the solution of Reshetnikov's question.

Proposition 5. *For all $n \geq 1$ we have*

$$R_n = 2 d_n (2n-1)!! \prod_{k=1}^{\lfloor n/2 \rfloor} (2^{2k} - 1). \tag{29}$$

Proof. In the definition (28) of the R_n substitute the value of $F(2^{-n})$ obtained from (23). $\square$

First we consider the case with odd n.

Theorem 1. *For any $n \geq 0$ the number R_{2n+1} is an integer multiple of F_n. Hence R_{2n+1} is an integer due to Proposition 1.*

Proof. Since up is an even function, we have by (22)

$$d_{2n+1} = (2n+1)\int_0^1 t^{2n}\,\mathrm{up}(t)\,dt = \frac{2n+1}{2}c_n.$$

Therefore by (29)

$$R_{2n+1} = 2d_{2n+1}(4n+1)!!\prod_{k=1}^{n}(2^{2k}-1) = (2n+1)c_n(4n+1)!!\prod_{k=1}^{n}(2^{2k}-1).$$

By (12)

$$R_{2n+1} = \frac{F_n}{(2n-1)!!}\prod_{\nu=1}^{n}(2^{2\nu}-1)^{-1}(4n+1)!!\prod_{k=1}^{n}(2^{2k}-1).$$

Simplifying

$$R_{2n+1} = F_n\frac{(4n+1)!!}{(2n-1)!!}. \tag{30}$$

So we conclude that R_{2n+1} is a natural number and $F_n \mid R_{2n+1}$. $\square$

Proposition 6. *For $n \geq 1$ we have*

$$R_{2n} = \sum_{k=0}^{n}\frac{2F_k}{2^{2n}}\binom{2n}{2k}\frac{(4n-1)!!}{(2k+1)!!}\prod_{\ell=k+1}^{n}(2^{2\ell}-1), \tag{31}$$

so that the denominator of R_{2n} can be only a power of 2.

Proof. By (29) and (24) we have

$$R_{2n} = 2d_{2n}(4n-1)!!\prod_{\ell=1}^{n}(2^{2\ell}-1) = \frac{2}{2^{2n}}\sum_{k=0}^{n}\binom{2n}{2k}c_k(4n-1)!!\prod_{\ell=1}^{n}(2^{2\ell}-1),$$

so that by (12)

$$R_{2n} = \sum_{k=0}^{n}\frac{2}{2^{2n}}\binom{2n}{2k}(4n-1)!!\frac{F_k}{(2k+1)!!}\prod_{r=1}^{k}(2^{2r}-1)^{-1}\prod_{\ell=1}^{n}(2^{2\ell}-1).$$

Simplifying we obtain (31). Since the factor $\frac{(4n-1)!!}{(2k+1)!!}$ is an integer in (31) the assertion about the denominator follows. $\square$

Theorem 2. *For any n the number R_n is a natural number.*

Proof. By Theorem 1 it remains only to prove that R_{2n} is a natural number. For a rational number r let $\nu_2(r)$ be the exponent of 2 in the prime decomposition of r. By Proposition 6, R_{2n} is a natural number if and only if $\nu_2(R_{2n}) \geq 0$. By (29) we have $\nu_2(R_n) = \nu_2(2d_n)$. We will show by induction that $\nu_2(2d_n) \geq 0$ and this will prove the Theorem.

Since the numbers F_n are natural, (30) implies that $\nu_2(2d_{2n+1}) = \nu_2(R_{2n+1}) \geq 0$. We have $\nu_2(2d_0) = \nu_2(2) = 1 \geq 0$. For any other even argument $2d_{2n}$ we use induction. By (20) we have for $n \geq 1$

$$(2n+1)(2^{2n}-1)2d_{2n} = \sum_{k=0}^{2n-1} \binom{2n+1}{k} 2d_k. \tag{32}$$

Hence

$$\nu_2(2d_{2n}) = \nu_2\bigl((2n+1)(2^{2n}-1)2d_{2n}\bigr) = \nu_2\Bigl(\sum_{k=0}^{2n-1} \binom{2n+1}{k} 2d_k\Bigr), \qquad n \geq 1.$$

By induction hypothesis $\nu_2(2d_k) \geq 0$ for $0 \leq k \leq 2n-1$. It follows that $\nu_2(2d_{2n}) \geq 0$ for all $n \geq 1$. □

5. Computation of $\mathrm{up}(q/2^n)$

At dyadic points $t = m/2^n$ the function takes rational values. This was proved in [2, Th. 7], where a procedure to compute these values was given. In [2] it was desired to express $\mathrm{up}(q/2^n)$, $q \in \mathbb{Z}$, in terms of $\mathrm{up}(1-2^{-k-1})$. This was achieved by substituting for c_k an expression depending on $\mathrm{up}(1-2^{-k-1})$. This is correct, but for computations it is preferable not to make this last substitution. In this way the formula in [2, last line in the proof of Thm. 7] can be written as

$$\mathrm{up}(q2^{-n}) = \frac{2^{-\binom{n+1}{2}}}{n!} \sum_{h=0}^{q+2^n-1} (-1)^{w(h)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \bigl(2(q-h) + 2^{n+1} - 1\bigr)^{n-2k} c_k. \tag{33}$$

This is valid for $-2^{-n} \leq q \leq 2^n$. The sum in h has few terms for q near -2^n. Therefore since $\mathrm{up}(1-x) = \mathrm{up}(-1+x)$ we obtain, with $q = a - 2^n$,

$$F(\tfrac{a}{2^n}) = \mathrm{up}(1 - \tfrac{a}{2^n}) = \frac{2^{-\binom{n+1}{2}}}{n!} \sum_{h=0}^{a-1} (-1)^{w(h)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2a - 2h - 1)^{n-2k} c_k. \tag{34}$$

The numbers c_k can be computed with the recurrence (11). This equation, together with $\mathrm{up}(t) + \mathrm{up}(1-t) = 1$ for $0 \leq t \leq 1$, allows one to easily compute the values at dyadic points. Another way has been proposed by Haugland recently in [6]. Notice also that our formula (33) can be written as

$$\operatorname{up}(1-\tfrac{a}{2^n}) = \frac{2^{-\binom{n+1}{2}}}{n!}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}c_k\sum_{h=0}^{a-1}(-1)^{w(h)}(2a-2h-1)^{n-2k}. \tag{35}$$

Equation (35) shows that the values at dyadic points of $\operatorname{up}(x)$, and therefore also those of $F(x)$, are rational numbers. In principle it allows the explicit computation of $\operatorname{up}(a/2^n)$ and therefore of $F(a/2^n)$ for all integers a.

Nevertheless there is a better way to compute the function. This method brought to attention by V. Reshetnikov, is attributed to V. A. Rvachëv. Reshetnikov posted an algorithm in Mathematica to compute $F(x)$ in an answer to a question of Pierrot Bolnez [3]. We have reverse engineered the program of Reshetnikov. The result is our Proposition 7. Since there is no explanation in English of this algorithm it is useful to include it here.

We need a Lemma.

Lemma 1. *Let $x>0$ be a real number and let k be the unique integer such that $2^k\le x<2^{k+1}$. Then for any non negative integer n we have*

$$\int_{2^k}^x (x-t)^n F^{(n+1)}(t)\,dt = -\int_0^y (y-t)^n F^{(n+1)}(t)\,dt, \qquad \text{with}\quad y = x-2^k. \tag{36}$$

Proof. Putting $t=2^k(1+u)$ in the first integral we obtain

$$\int_{2^k}^x (x-t)^n F^{(n+1)}(t)\,dt = 2^k\int_0^{2^{-k}y}(y-2^ku)^nF^{(n+1)}(2^k(1+u))\,du.$$

Letting $a=2^{-k}y$, we have $0\le a<1$ and we obtain

$$\int_{2^k}^x (x-t)^n F^{(n+1)}(t)\,dt = 2^{k(n+1)}\int_0^a (a-u)^nF^{(n+1)}(2^k(1+u))\,du.$$

Differentiating $F^{(k)}(t)=2^{\binom{k+1}{2}}F(2^kt)$ repeatedly we obtain

$$F^{(k+n+1)}(t) = 2^{\binom{k+1}{2}}2^{k(n+1)}F^{(n+1)}(2^kt). \tag{37}$$

It follows that

$$2^{k(n+1)}\int_0^a (a-u)^nF^{(n+1)}(2^k(1+u))\,du = 2^{-\binom{k+1}{2}}\int_0^a(a-u)^nF^{(k+n+1)}(u+1)\,du.$$

When $0<u<a$ we have $1<u+1<2$. For $0<x<2$ we have by (2) $F(x)=\operatorname{up}(x-1)$. Therefore $F'(x)=2\operatorname{up}(2x-1)-2\operatorname{up}(2x-3)$. This implies for

$0 < u < 1$ that $F'(u) = 2\,\mathrm{up}(2u-1)$ and $F'(u+1) = -2\,\mathrm{up}(2u-1) = -F'(u)$. Hence

$$2^{-\binom{k+1}{2}} \int_0^a (a-u)^n F^{(k+n+1)}(u+1)\,du = -2^{-\binom{k+1}{2}} \int_0^a (a-u)^n F^{(k+n+1)}(u)\,du.$$

We have by (37)

$$\begin{aligned} -2^{-\binom{k+1}{2}} \int_0^a (a-u)^n F^{(k+n+1)}(u)\,du &= -\int_0^a (a-u)^n 2^{k(n+1)} F^{(n+1)}(2^k u)\,du \\ &= -\int_0^y (y-v)^n F^{(n+1)}(v)\,dv. \end{aligned}$$

□

Proposition 7. *Let $x > 0$ be a real number. Let n be the unique integer such that $2^{-n} \le x < 2^{-n+1}$ and put $y = x - 2^{-n}$. Then we have*

$$F(x) = -F(y) + \sum_{0\le k\le n} 2^{\binom{k+1}{2} - \binom{n-k}{2}} \frac{d_{n-k}}{(n-k)!} \frac{y^k}{k!}, \tag{38}$$

where the sum is 0 when $n < 0$.

Proof. We use Taylor's Theorem with integral remainder,

$$F(x) = F(y + 2^{-n}) = \sum_{0\le k\le N} \frac{F^{(k)}(2^{-n})}{k!} y^k + \int_{2^{-n}}^x \frac{(x-t)^N}{N!} F^{(N+1)}(t)\,dt.$$

When $n < 0$, the number 2^{-n} is an even natural number. For $k \ge 0$ we have the derivative $F^{(k)}(2^{-n}) = 0$. Hence for any $N \ge 0$,

$$F(x) = \int_{2^{-n}}^x \frac{(x-t)^N}{N!} F^{(N+1)}(t)\,dt.$$

By Lemma (1),

$$\int_{2^{-n}}^x \frac{(x-t)^N}{N!} F^{(N+1)}(t)\,dt = -\int_0^y \frac{(x-t)^N}{N!} F^{(N+1)}(t)\,dt.$$

Again, by Taylor's Theorem, applied at the point 0 where all derivatives are also $= 0$, this is equal to $-F(y)$ as we wanted to prove.

For $n \ge 0$ we apply again Taylor's Theorem with $N = n$. In this case

$$F(x) = F(y + 2^{-n}) = \sum_{0\le k\le n} \frac{F^{(k)}(2^{-n})}{k!} y^k + \int_{2^{-n}}^x \frac{(x-t)^n}{n!} F^{(n+1)}(t)\,dt.$$

Lemma (1) gives us

$$\int_{2^{-n}}^{x} \frac{(x-t)^n}{n!} F^{(n+1)}(t)\,dt = -\int_0^y \frac{(y-t)^n}{n!} F^{(n+1)}(t)\,dt = -F(y).$$

In the last equality we used Taylor's Theorem at the point $t = 0$ where all derivatives of F are equal to 0.

To finish the proof notice that by (4) and (23) we have for $0 \le k \le n$,

$$F^{(k)}(2^{-n}) = 2^{\binom{k+1}{2}} F(2^k 2^{-n}) = 2^{\binom{k+1}{2}} \frac{d_{n-k}}{(n-k)!2^{\binom{n-k}{2}}}.$$

□

Remark 1. Notice that Proposition 7 reduces the computation of $F(x)$ to that of $F(y)$. If x is dyadic, a repeated use ends in $F(0)$, the number of steps being the number of times that 1 appears in the dyadic representation of x. If x is not dyadic, a finite number of steps reduces the computation to that of $F(y)$ for some y small enough so that $F(y)$ is less than the absolute error requested. This procedure is implemented by Reshetnikov in his answer to Pierrot Bolnez's question in Mathematica StackExchange.

6. The Least Common Denominator of up$(a/2^n)$

Reshetnikov's conjecture tries to determine the denominator of the dyadic number $F(2^{-n})$. It is more natural to consider the common denominator D_n of all numbers $F(a/2^n)$ where a is an odd number. This is a finite number because we only have to consider the case of $a = 2k+1$ for $k = 0, 1, \dots, 2^{n-1}-1$.

Therefore we make the following definition.

Definition 6.1. Let D_n be the least common multiple of the denominators of the fractions up$(\frac{2k+1}{2^n})$ with $2^{-n} < 2k+1 \le 2^n$, $k \in \mathbb{Z}$.

The first D_n starting from $D_0 = 1$ are

$$1,\ 2,\ 72,\ 288,\ 2\,073\,600,\ 33\,177\,600,\ 2\,809\,213\,747\,200,\ 179\,789\,679\,820\,800,$$
$$704\,200\,217\,922\,109\,440\,000,\ 180\,275\,255\,788\,060\,016\,640\,000,$$
$$6\,231\,974\,256\,792\,696\,936\,191\,754\,240\,000,$$
$$6\,381\,541\,638\,955\,721\,662\,660\,356\,341\,760\,000,$$
$$292\,214\,732\,887\,898\,713\,986\,916\,575\,925\,267\,070\,976\,000\,000,$$
$$1\,196\,911\,545\,908\,833\,132\,490\,410\,294\,989\,893\,922\,717\,696\,000\,000,$$
$$963\,821\,659\,256\,803\,158\,077\,786\,940\,841\,300\,728\,342\,971\,034\,894\,336\,000\,000,\dots$$

Theorem 3. *For any natural number $n \geq 1$ and any integer $-2^n < a < 2^n$ the number*

$$\operatorname{up}(\tfrac{a}{2^n})\, n!\, 2^{\binom{n+1}{2}}(2\lfloor n/2\rfloor+1)!! \prod_{k=1}^{\lfloor n/2\rfloor}(2^{2k}-1) \tag{39}$$

is natural.

Equivalently,

$$D_n \text{ divides the number } n!\, 2^{\binom{n+1}{2}}(2\lfloor n/2\rfloor+1)!! \prod_{k=1}^{\lfloor n/2\rfloor}(2^{2k}-1).$$

Proof. By (33) we know that

$$\operatorname{up}(\tfrac{a}{2^n})\, n!\, 2^{\binom{n+1}{2}}$$

is a linear combination with integer coefficients of the rational numbers c_k with $0 \leq k \leq n/2$. We have $c_0 = 1$ and, for $n \geq 1$, by (12) the numbers

$$F_k = c_k(2k+1)!! \prod_{r=1}^{k}(2^{2r}-1)$$

are natural. Therefore, if $m = \lfloor n/2\rfloor$ then all the numbers

$$c_k(2m+1)!! \prod_{r=1}^{m}(2^{2r}-1), \qquad 0 \leq k \leq n/2,$$

are natural. It follows that the number in (39) is an integer. Since it is also positive it is a natural number. □

Remark 2. Theorem 3 applies to any a and not only for $2^n - 1$ as in Reshetnikov's conjecture. The power of 2 is better in Reshetnikov's conjecture than in Theorem 3, but for other primes Theorem 3 is better than Reshetnikov's conjecture. We believe that the power of 2 can be improved in Theorem 3.

Proposition 8. *For any natural number n, the number D_n is a multiple of the denominator of $d_n/(2^{\binom{n}{2}}\, n!)$.*

Proof. This follows directly from equation (23) and the definition of D_n, since D_n is a multiple of the denominator of $\operatorname{up}(1-2^{-n})$. □

The quotients $D_n/\operatorname{den}(d_n/(2^{\binom{n}{2}}\, n!))$ for $1 \leq n \leq 17$ are

$$1,\ 1,\ 1,\ 1,\ 1,\ 5,\ 1,\ 1,\ 1,\ 5,\ 1,\ 1,\ 1,\ 55,\ 1,\ 13,\ 11,\ \ldots$$

The assertions in the following conjecture appear to be true.

Conjecture 1. (a) The quotient $A_n = 2^{-\binom{n}{2}} D_n$ satisfies $A_{2n} = A_{2n+1}$ for $n \geq 1$.

(b) Let $K_n = A_{2n-1}$ for $n \geq 1$, so that by (a) we have $K_1 = A_1$, and for $n \geq 2$

$$K_n = 2^{-\binom{2n-1}{2}} D_{2n-1} = 2^{-\binom{2n-2}{2}} D_{2n-2}.$$

Then $2 \times (2n-1)!$ divides K_n.

(c) The quotients $H_n = \frac{K_n}{2(2n-1)!}$ are odd integers.

The first few H_n starting from $H_1 = 1$ are

$$1,\ 3,\ 135,\ 8\,505,\ 3\,614\,625,\ 2\,218\,656\,825,\ 317\,988\,989\,443\,125,$$
$$148\,846\,103\,258\,477\,625,\ 8\,607\,025\,920\,921\,468\,665\,625,\ \dots$$

If these conjectures are true we obtain for $n \geq 1$

$$D_{2n-1} = 2^{1+\binom{2n-1}{2}}(2n-1)!\,H_n, \quad D_{2n} = 2^{1+\binom{2n}{2}}(2n+1)!\,H_{n+1}.$$

7. On the Nature of the Numbers d_n

In this section we show that the numerator and denominator of $2d_n$ are odd numbers.

We have noticed in the proof of Theorem 2 that $\nu_2(2d_n) = \nu_2(R_n)$. So that by equation (30) it follows that $\nu_2(2d_{2n+1}) = \nu_2(F_n)$. We will show now that the numbers F_n are odd. This will prove our result for $2d_n$ when n is an odd number. For this proof we will use Lucas' Theorem in [5, Thm. 1].

Theorem 4 (Lucas)**.** *Let p be a prime and n and k two positive integers that we write as*

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_a p^a, \quad k = k_0 + k_1 p + k_2 p^2 + \cdots + k_a p^a, 0 \leq n_j, k_j < p$$

in base p. Then we have the congruence

$$\binom{n}{k} \equiv \binom{n_a}{k_a} \cdots \binom{n_2}{k_2}\binom{n_1}{k_1}\binom{n_0}{k_0} \pmod{p}.$$

From Lucas's Theorem we get the following application:

Proposition 9. (a) *For $n \geq 1$ the number of k with $0 \leq k \leq n$ such that the binomial coefficient $\binom{2n+1}{2k}$ is odd is exactly $2^{w(n)}$.*

(b) *For $n \geq 1$ the number k with $0 \leq k \leq 2n+1$ such that the binomial coefficients $\binom{2n+1}{k}$ are odd is $2^{w(n)+1}$.*

Proof. Let the binary expansion of n be $\varepsilon_r\varepsilon_{r-1}\dots\varepsilon_0$, that is,

$$n = \varepsilon_r 2^r + \varepsilon_{r-1}2^{r-1} + \cdots \varepsilon_0, \quad \text{with } \varepsilon_j \in \{0,1\}.$$

The expansion of $2n+1$ is $\varepsilon_r\varepsilon_{r-1}\dots\varepsilon_0 1$. Let $\binom{2n+1}{2k}$ be odd for $k \le n$, then $k \le 2^{r+1}$. Hence its binary expansion is of the form $k = \delta_r \cdots \delta_0$. Therefore Lucas's Theorem implies

$$1 \equiv \binom{2n+1}{2k} \equiv \binom{\varepsilon_r}{\delta_r}\cdots\binom{\varepsilon_0}{\delta_0}\binom{1}{0} \pmod 2.$$

Here each factor $\binom{0}{1} = 0$ and each factor $\binom{1}{\delta} = 1$ for any value of δ. So $\binom{2n+1}{2k} \equiv 1$ is equivalent to the assertion that $\delta_j = 0$ when $\varepsilon_j = 0$. Therefore the number of $k \le n$ such that $\binom{2n+1}{2k}$ is odd is 2^m, where m is the number of digits equal to 1 in the expansion of n. But this is to say $m = w(n)$.

The second assertion can be proved in the same way. □

Proposition 10. *All the numbers F_n are odd.*

Proof. We have $F_0 = F_1 = 1$. We proceed by induction assuming that we have proved that all F_k are odd numbers for $0 \le k \le n-1$. Equation (14) gives F_n as a sum of terms. Each term is a product of an odd number times a binomial coefficient $\binom{2n+1}{2k}$, with k running on $0 \le k \le n-1$. The missing summand with $k = n$ (since for for $k = n$ we have $\binom{2n+1}{2n} = 2n+1$) is an odd number. So, by Proposition 9, the summands adding to F_n are even except for $2^{w(n)} - 1$ of them. Since $w(n) \ge 1$ this implies that F_n is odd. □

Theorem 5. *For all $n \ge 1$ we have $\nu_2(2d_n) = 0$, that is, the rational number $2d_n$ is the quotient of two odd numbers.*

Proof. We have seen that F_n is an odd number and that this implies $\nu_2(2d_{2n+1}) = 0$. So we only need to show this for $2d_{2n}$ and $n \ge 1$ (notice that $2d_0 = 2$, so that $\nu_2(2d_0) = 1$).

We have shown (Theorem 2) that R_n are integers. By Proposition 5 there is an odd number

$$N_n = (2n-1)!! \prod_{k=1}^{\lfloor n/2 \rfloor} (2^{2k} - 1),$$

such that $2d_n N_n = R_n$ is an integer. It is easy to see that for $k \le n$ we have $N_k \mid N_n$. Therefore for all $1 \le k \le n$ we have that $d_k N_n$ is an integer.

We proceed by induction, assuming we have proved that $\nu(2d_k) = 0$ for $1 \le k < 2n$ and we have to show that $\nu_2(2d_{2n}) = 0$. We multiply (32) for $n \ge 1$ by N_{2n} and obtain

$$\nu_2(2d_{2n}) = \nu_2\big((2n+1)(2^{2n}-1)2d_{2n}N_{2n}\big) = \nu_2\Big(\sum_{k=0}^{2n-1}\binom{2n+1}{k}2d_k N_{2n}\Big).$$

By the induction hypothesis $2d_k N_{2n}$ is an odd natural number for $1 \le k \le 2n-1$ and it is even for $k=0$. Let L be number of indices k with $1 \le k \le 2n-1$ and such that the binomial coefficient $\binom{2n+1}{k}$ is odd. We only need to show that this L is odd, and this follows easily from Propostition 9. □

Theorem 6. *The number R_n is odd for $n \ge 1$. Or equivalently*

$$\nu_2\big(\operatorname{up}(1-2^{-n})\big) = -\binom{n}{2} - 1 - \nu_2(n!). \tag{40}$$

Proof. We have seen that $2d_n$ is a quotient of two odd numbers and R_n is a natural number. Therefore equation (29) implies that R_n is an odd number.

Equation (23) yields

$$0 = \nu_2(2d_n) = 1 + \nu_2(n!) + \binom{n}{2} + \nu_2\big(\operatorname{up}(1-2^{-n})\big).$$

□

8. Relation With Bernoulli Numbers

The numbers c_n and d_n are related to the Bernoulli numbers. Perhaps this may be useful to solve Conjecture 1.

Proposition 11. *The numbers d_n and c_n are given by the recurrences*

$$c_0 = 1, \quad c_n = \frac{1}{2^{2n}-1}\sum_{k=1}^{n} 2^{2n-2k}(2^{2k}-2)\binom{2n}{2k} B_{2k} c_{n-k}, \tag{41}$$

and

$$d_0 = 1, \quad d_n = \frac{n2^{n-2}}{2^n-1} d_{n-1} - \frac{1}{2^n-1}\sum_{k=1}^{\lfloor n/2\rfloor}\binom{n}{2k} 2^{n-2k} B_{2k} d_{n-2k}, \tag{42}$$

where B_k are the Bernoulli numbers $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, ...

Proof. By (8) and (19) the functions

$$\widehat{\operatorname{up}}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{(2n)!}(2\pi z)^{2n}, \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n$$

satisfy the equations

$$\frac{\pi z}{\sin(\pi z)}\widehat{\operatorname{up}}(z) = \widehat{\operatorname{up}}(\tfrac{z}{2}), \qquad \frac{x}{e^x - 1} f(2x) = f(x).$$

Using the well known power series

$$\frac{\pi z}{\sin(\pi z)} = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{B_{2n}(2^{2n}-2)}{(2n)!} (\pi z)^{2n}, \qquad \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

and comparing coefficients of the same power of the variable we obtain our recurrences. □

Note: After this paper was written I have noticed the paper by Yoneda [19] that, unaware of previous works, defines again Fabius function, and also the paper by Alkauskas [1] that contains interesting results.

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