LARGE SETS OF t-DESIGNS AND A RAMSEY-TYPE PROBLEM

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Abstract

For a given r-uniform hypergraph H and for given integers $p > q > 0$, $R^p_q(H;r)$ is the smallest positive integer m such that in every p-coloring of the edges of the complete r-uniform hypergraph K^r_m there is a copy of H with edges colored by at most q colors. Here we extend some previous results when $H = K^r_{r+1}$.

Budden, Stender and Zhang recently [1] considered the following hypergraph version of a graph Ramsey-type number initiated in [3, 4]. For a given r-uniform hypergraph H and for given integers $p > q > 0$, $R^p_q(H;r)$ is the smallest positive integer m such that in every p-coloring of the edges of the complete r-uniform hypergraph K^r_m there is a copy of H with edges colored by at most q colors. For $q = 1$ this definition gives the traditional p-color Ramsey number of H.

It was proved in [1] that $7 \leq R^3_2(K^3_4;3) \leq 8$. The upper bound was derived from the result that the K^3_4 versus $K^3_4 - e$ Ramsey number ($K^3_4 - e$ is the hypergraph with three triples on four vertices) is 8 [8]. In fact, one can derive more from a Turán number. Turán's famous problem [9] is to determine $T(n)$, the smallest positive integer ℓ with the following property: there exist ℓ triples in an n-element set S such that every 4-subset of S contains at least one of the ℓ triples. For the state of the art on $T(n)$, see [7].

Proposition 1. *We have $R^3_2(K^3_4;3) = 7$.*

Proof. The lower bound is the 3-coloring of K^3_6 given in [1]. On the other hand, in every 3-coloring of $K = K^3_7$ there is a color class, say red, with at most 11 triples. Because $T(7) = 12$ is known from [6], there must be a $K^3_4 \subset K$ containing no red triples. □

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In the rest of this note we consider the case $H = K^r_{r+1}, p = r+1, q = r$. Set $f(r) = R^{r+1}_r(K^r_{r+1}; r)$, i.e. $f(r)$ is the smallest integer m such that in every $(r+1)$-coloring of the edges of K^r_m there is a copy of K^r_{r+1} with its edges colored by at most r colors. It is known that $f(2) = 5$ ([4]), $f(3) = 5$ ([1]). Since among the $r+2$ edges containing a fixed set of $r-1$ vertices in an $(r+1)$-colored K^r_{2r+1} there are two edges with the same color, one gets the following upper bound.

Proposition 2. *([1]) For all $r \geq 1$, $f(r) \leq 2r+1$.*

Here we give a formula for $f(r)$ that connects it to the existence problem of large sets of t-designs and provides $f(r)$ for infinitely many r.

For $1 \leq t < k < v$, a $t-(v,k)$ design is a set of k-subsets of a v-set that covers each t-subset of the v-set exactly once. A large set of $t-(v,k)$ designs is a partition of the set of all k-subsets of a v-set into $t-(v,k)$ designs. The *size* N of a large set is the number of $t-(v,k)$ designs in the partition,

$$N = \frac{\binom{v}{k}\binom{k}{t}}{\binom{v}{t}}.$$

Here we defined only a special case of designs needed for our purposes; for more details see Chapter 4.4 of [5].

Let $g(r)$ be the smallest v in the form $v = r+t+1$ for which there is no large set of $t-(v,t+1)$ designs. Note that these large sets (if they exist) must have size $r+1$, since

$$(r+1) = \frac{\binom{r+t+1}{t+1}\binom{t+1}{t}}{\binom{r+t+1}{t}}.$$

Theorem 1. *For all $r \geq 1$, $f(r) = g(r)$.*

Proof. Suppose that $v = g(r) = r+t+1$ and we have an $(r+1)$-coloring c on the edges of K^r_v. Let K be the vertex set of K^r_v. We claim that some $K^r_{r+1} \subset K^r_v$ is colored with at most r colors. Indeed, otherwise every $(r+1)$-element subset S of K would receive all of the $r+1$ colors on their r-element subsets. The coloring c naturally defines a coloring c^* on the complements of the edges of K^r_v by defining $c^*(K \setminus e) = c(e)$ for all $e \in E(K^r_v)$. Since $|K| - |e| = v - r = t+1$, c^* colors the $(t+1)$-element subsets of K. Moreover, for any t-element subset $T \subset K$ and for any color, say red, there is a unique red edge $e \subset K \setminus T$ in coloring c, thus the complement of e is the unique red $(t+1)$-element set in the coloring c^* that covers T. Thus each color class of c^* is a $t-(v,t+1)$ design and the color classes define a large set of designs (of size $r+1$), contradicting to the definition of $g(r)$. □

Corollary 1. *We have $f(4) = 7$.*

Proof. There is a large set of $1-(6,2)$ designs: the factorization of K_6. However, as known from Cayley [2], there is no large set of $2-(7,3)$ designs. □

Corollary 2. *For all $r \geq 1$, $f(r) = r+2$ if r is odd and $f(r) = r+3$ if $r \equiv 2 \pmod 6$.*

Proof. If r is odd, there is no $1-(r+2,2)$ design, let alone a large set of them. If $r \equiv 2 \pmod 6$ then there are large sets of $1-(r+2,2)$ designs: the factorizations of K_{r+2}. However, no $2-(r+3,3)$ design exists, since it would be a Steiner triple system on $r+3$ points implying $r+3 \equiv 1$ or $3 \pmod 6$. However, in our case $r+3 \equiv 5 \pmod 6$. □

Corollary 3. *We have $f(6) > 9$.*

Proof. Large sets of $1-(8,2), 2-(9,3)$ designs exist. See [5], Large sets of t-designs, Chapter 4.4. □

Remark. To determine $f(6)$ one has to decide whether there are large sets (of size 7) for the $3-(10,4), 4-(11,5), 5-(12,6)$ designs. If the last one exists then the upper bound $f(6) \leq 13$ in Proposition 2 would be sharp.

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References

[1] M. Budden, M. Stender and Y. Zhang, Weakened Ramsey numbers and their hypergraph analogues, *Integers* **17** (2017), 1-8.

[2] A. Cayley, On triadic arrangements of seven and fifteen things, *London, Edinburgh and Dublin Philos. Mag. Sci.* **37** (1850), 50-53.

[3] K. Chung, M. Chung and C. Liu, A generalization of Ramsey theory for graphs - with stars and complete graphs as forbidden subgraphs, *Congressus Numerantium* **19** (1977), 155-161.

[4] K. Chung and C. Liu, A generalization of Ramsey theory for graphs, *Discrete Mathematics* **21** (1978), 117-127.

[5] C. J. Colbourn, J. H. Dinitz editors, *Handbook of Combinatorial Designs*, CRC Press, 2006.

[6] Gy. Katona, T. Nemetz and M. Simonovits, On a problem of Turán in the theory of graphs, (in Hungarian, English summary), *Matematikai Lapok* **15** (1964), 228-238.

[7] P. Keevash, Hypergraph Turán problems, *Surveys in Combinatorics, London Math. Soc. Lecture Note Ser. 392*, Cambridge University Press, Cambridge, 2011.

[8] S. Radziszowski, Small Ramsey numbers, revision 15, *Electronic J. of Combinatorics* **DS 1.15** (2017), 1-104.

[9] P. Turán, Research problem, *Közl. MTA Mat. Kutató Int.* **6** (1961), 177–181.