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ON EISENSTEIN PRIMES

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Abstract
In this paper, we show that there are infinitely many primes of the form $\ell^2-\ell m+m^2$ such that $2\ell-m$ is prime, and that the number of such primes satisfies the expected asymptotic formula. We show this by adapting the proof of a result of Fouvry and Iwaniec who showed that there are infinitely many primes of the form ℓ^2+m^2 with ℓ prime.

1. Introduction and Statement of Results

By following a proof similar to that of a theorem of Fermat regarding representations of primes as sums of two squares, it is possible to show that all primes congruent to 1 (mod 3) are representable as $\ell^2-\ell m+m^2 = N(\pi), \pi = \ell+m\omega$ Here, $N = N_{\mathbb{Q}(\omega)/\mathbb{Q}}$ refers to the norm in the quadratic field $\mathbb{Q}(\omega)$, where $\omega = (-1+\sqrt{-3})/2$. In this paper, we show that there are infnitely many such primes s.t. $\Re\pi$ is half a prime.

In particular, we show the following result, which indicates that the number of such primes satisfies the sort of asymptotic formula one would expect from congruence considerations and the prime number theorem:

Theorem 1. *We have*

$$\sum\sum_{\ell^2-\ell m+m^2\leq x} \Lambda(2\ell-m)\Lambda(\ell^2-\ell m+m^2) \sim \sigma x$$

for some $\sigma > 0$.

We shall prove Theorem 1 by following along the lines of the proof of Theorem 20.3 in [2], by using $\mathbb{Q}(\omega)$ rather than $\mathbb{Q}(i)$ when working with the bilinear forms that arise in Section 20.4 of [2]. A related result was proven by Fouvry and Iwaniec in [3] where it is shown that there are infinitely many primes of the form ℓ^2+m^2 such that ℓ is prime.

2. Preliminaries

Let $\gamma_\ell = \log \ell$ when ℓ is a prime greater than 2 and 0 otherwise. Then, let

$$a_n = \sum_{\ell^2-\ell m+m^2=n} \gamma_{2\ell-m} = \sum_{r^2+3s^2=4n} \gamma_r.$$

Let

$$A(x) = \sum_{n\le x} a_n$$

and let

$$A_d(x) = \sum_{\substack{n\le x\\ n\equiv 0 \pmod{d}}} a_n.$$

Let $\rho(d) = |\{v \in \mathbb{Z}/(d) : v^2 + 3 \equiv 0 \pmod{d}\}|$. We expect that $A_d(x)$ is well approximated by

$$M_d(x) = \frac{\rho(4d)}{4d} \sum_{r\le\sqrt{4x}} \frac{1}{2}\gamma_r \sqrt{\frac{4x-r^2}{3}}$$

so we let the remainder terms $r_d(x)$ be such that

$$A_d(x) = M_d(x) + r_d(x).$$

For d even, this is clearly equal to 0, while for d odd, since $\rho(d)$ is multipicative, this is equal to

$$\frac{\rho(d)}{4d} \sum_{r\le\sqrt{4x}} \gamma_r \sqrt{\frac{4x-r^2}{3}}.$$

We then have the following:

Proposition 1. *Suppose that for some* $\sqrt{x} < D \le x(\log x)^{-20}$

$$R(x;D) = \sup_{y\le x} \sum_{d\le D} |r_d(y)| \ll A(x)\log^{-2} x \tag{1}$$

and let

$$T(x;D) = \sum_{\ell\le D} \Bigg| \sum_{\substack{\ell m\le x\\ xD^{-1}<m\le x^2D^{-2}}} a_{\ell m}\mu(m)\Bigg|. \tag{2}$$

Then, we have that

$$\sum_{n\le x} a_n\Lambda(n) = HA(x)\left\{1 + O((\log x)^{-1})\right\} + O(T(x,D)\log x) \tag{3}$$

where

$$H = \prod_p (1-g(p))\left(1-\frac{1}{p}\right)^{-1} \tag{4}$$

for $g(d) = \rho(4d)/(4d)$.

Proof. This is Theorem 18.6 in [2] for our particular sequence. $\square$

3. The Remainder Term

In this section, we verify that (1) holds. From this point on, $e(\alpha) = e^{2\pi i\alpha}$. First, we study the distribution of the roots of the congruence $v^2 + 3 \equiv 0 \pmod{d}$ by studying Weyl sums related to these quadratic roots. In order to do so, we will establish a well-spacing of the points $v/d \pmod{1}$, similar to that established in [1, 2, 3]. It is easy to show that for odd d, the roots to $v^2 + 3 \equiv 0 \pmod{d}$ are in a bijection with representations

$$d = r^2 + rs + s^2 = \frac{(r-s)^2 + 3(r+s)^2}{4}$$

subject to $(r, s) = 1, -r - s < r - s \leq r + s$ where $v(r - s) \equiv (r + s) \pmod{d}$.

To show this, note that it is sufficient to verify it only when $d = p$ for primes $p \equiv 0, 1 \pmod{3}$. The case $p = 3$ is easily dealt with, so we assume $p \equiv 1 \pmod{3}$.

In this case, note that since there exists $v \neq 0$ s.t. $v^2 + 3 \equiv 0 \pmod{p}$, we have that $p|(v + \sqrt{-3})(v - \sqrt{-3})$ in $\mathbb{Z}[\omega]$ so it follows that since $p \nmid v + \sqrt{-3}, p \nmid v - \sqrt{-3}$, p is not a prime in $\mathbb{Z}[\omega]$. Then, it follows that there exists $\mathfrak{a} \in \mathbb{Z}[\omega]$ with norm p. Multiplying $\mathfrak{a}$ with units in $\mathbb{Z}[\omega]$ to control the sign of the corresponding value of v and the relative size of the real and imaginary parts yields the desired result.

It then follows that

$$\frac{v}{d} \equiv -\frac{4\overline{(r-s)}}{r+s} + \frac{r-s}{d(r+s)} \pmod{1}$$

where $\overline{r-s}$ is such that $(r - s)\overline{(r - s)} \equiv 1 \pmod{r + s}$.

Note that we then have that

$$\frac{|r-s|}{d(r+s)} \leq \frac{1}{(r+s)^2}.$$

Now, restrict d to the range $4D < d \leq 9D$. It then follows that $2D^{1/2} < r + s < 3D^{1/2}$, so for any two points $v_1/d_1, v_2/d_2$, $\max\left\{\frac{r_1+s_1}{r_2+s_2}, \frac{r_2+s_2}{r_1+s_1}\right\} \leq \frac{3}{2}$

$$\left\|\frac{v_1}{d_1} - \frac{v_2}{d_2}\right\| \geq \frac{4}{(r_1+s_1)(r_2+s_2)} - \frac{1}{(r_1+s_1)^2} - \frac{1}{(r_2+s_2)^2} \gg \frac{1}{D}.$$

Then by the large sieve inequality of Davenport and Halberstam, we have the following result.

Lemma 1. *For all $\alpha_1, \alpha_2, \cdots \in \mathbb{C}$, we have that*

$$\sum_{\substack{D<d\leq 2D \\ d\equiv 1 \pmod{2}}} \sum_{v^2+3\equiv 0 \pmod{d}} \left| \sum_{n\leq N} \alpha_n e\left(\frac{vn}{d}\right) \right|^2 \ll (D+N)\left(\sum_n \alpha_n^2\right).$$

Applying Cauchy's inequality yields

Proposition 2. *For all $\alpha_1, \alpha_2, \dots \in \mathbb{C}$, we have that*

$$\sum_{\substack{D<d\leq 2D\\ d\equiv 1 \ (\mathrm{mod}\ 2)}} \sum_{v^2+3\equiv 0 \ (\mathrm{mod}\ d)} \left| \sum_{n\leq N} \alpha_n e\left(\frac{vn}{d}\right) \right| \ll D^{1/2}(D+N)^{1/2} \left(\sum_n \alpha_n^2\right)^{1/2}. \quad (5)$$

Now, let

$$\rho_h(d) = \sum_{v^2+3\equiv 0 \ (\mathrm{mod}\ d)} e\left(\frac{vh}{d}\right).$$

Then, by the triangle inequality, the following holds.

Proposition 3. *For all $\alpha_1, \alpha_2, \dots \in \mathbb{C}$, we have that*

$$\sum_{d\leq D} \left| \sum_{h\leq N} \alpha_h \rho_h(d) \right| \ll D^{1/2}(D+N)^{1/2} \left(\sum_n \alpha_n^2\right)^{1/2}. \quad (6)$$

Now, we prove that (1) holds by proving the following:

Proposition 4. *For all $D \leq x$*

$$\sum_{d\leq D} |r_d(x)| \ll D^{1/4} x^{3/4} (\log x)^4. \quad (7)$$

Proof. Note that

$$A_d(x) = \sum_{\substack{\frac{r^2+3s^2}{4}\leq x\\ \frac{r^2+3s^2}{4}\equiv 0 \ (\mathrm{mod}\ d)}} \gamma_r.$$

It is more convenient for now to consider only the contribution of the terms with $(r,d)=1$. To that end, note that it is possible to replace $A_d(x)$ with

$$A_d^*(x) = \sum_{\substack{\frac{r^2+3s^2}{4}\leq x\\ \frac{r^2+3s^2}{4}\equiv 0 \ (\mathrm{mod}\ d)\\ (r,d)=1}} \gamma_r.$$

We can do this since we have that

$$\sum_{d\leq D} |A_d(x) - A_d^*(x)| \leq \sum_{d\leq D} \sum_{r|d} |\gamma_r| \sum_{\substack{r^2+3s^2\leq 4x\\ r^2+3s^2\equiv 0 \ (\mathrm{mod}\ 4d)}} 1.$$

When r is an odd prime and $r|d$, $r^2+3s^2 \equiv 0 \pmod{d}$, we have that $s = rt$ for some t. Therefore, it follows that

$$\sum_{d\le D} |A_d(x) - A_d^*(x)| \le \sum_{r\le 2\sqrt{x}} \gamma_r \sum_{t\le\sqrt{4x/3}/r} \tau(r^2(1+3t^2)) \ll x^{1/2+\varepsilon}.$$

Now, rather than approximating $A_d^*(x)$, we shall approximate

$$A_d^*(f) = \sum_{\substack{r^2+3s^2\equiv 0 \pmod{4d}\\(r,d)=1}} \gamma_r f\left(\frac{r^2+3s^2}{4}\right)$$

for some smooth f supported on $[1,x]$ satisfying

$$f(u) = 1, \text{ for } y \le u \le x-y$$
$$f^{(j)}(u) \ll y^{-j}$$

where $y = \min\{x^{3/4}D^{1/4}, \frac{1}{2}x\}$. Note that consdering $A_d^*(f)$ instead of $A_d^*(x)$ is sufficient for proving the desired result since

$$\sum_{d\le D}|A_d^*(f)-A_d^*(x)| \le \sum_{\ell^2-\ell m+m^2\in I} \tau(\ell^2-\ell m+m^2) \le \sum_{n\in I} r(n)\tau(n) \ll \sum_{n\in I}\tau(n)^2 \ll y(\log x)^3$$

where $I = \mathbb{Z}\cap([1,y]\cup[x-y,x])$, and $r(n) = |\{\mathfrak{n}\in\mathbb{Z}[\omega] | N\mathfrak{n} = n\}|$. Note that since γ_r is supported on odd primes, we have that

$$A_d^*(f) = \sum_{v^2+3\equiv 0 \pmod{4d}} \sum_{(r,d)=1} \gamma_r \sum_{s\equiv vr \pmod{4d}} f\left(\frac{r^2+3s^2}{4}\right).$$

Now, let

$$A_d(f) = \sum_{v^2+3\equiv 0 \pmod{4d}} \sum_{r} \gamma_r \sum_{s\equiv vr \pmod{4d}} f\left(\frac{r^2+3s^2}{4}\right).$$

We can replace $A_d^*(f)$ with $A_d(f)$ with an error of $O(x^{1/2+\varepsilon})$ by a similar argument to that with which we replaced $A_d(x)$ with $A_d^*(x)$, which is small enough. We then have that by the Poisson summation formula

$$A_d(f) = \frac{1}{4d}\sum_r \gamma_r \sum_{k\in\mathbb{Z}} \rho_{kr}(4d) F_r\left(\frac{k}{4d}\right)$$

where

$$F_r(v) = \int_{\mathbb{R}} f\left(\frac{r^2+3t^2}{4}\right) e(-vt)dt = 2\int_0^\infty f\left(\frac{r^2+3t^2}{4}\right)\cos(2\pi vt)dt.$$

Note that the contribution to the sum due to the frequency $k = 0$ is equal to $M_d(x) + O(y)$, so it is sufficient to bound the contribution from the sum over frequencies $k \neq 0$. To that end, note that by the change of variable $t = w\sqrt{x}/k$,

$$F_r\left(\frac{k}{4d}\right) = \frac{2\sqrt{x}}{k}\int_0^\infty f\left(\frac{r^2 + \frac{3xw^2}{k^2}}{4}\right)\cos\left(\frac{2\pi w\sqrt{x}}{4d}\right)dw. \tag{8}$$

Integrating by parts twice yields that this equals

$$\frac{16\sqrt{x}d^2}{\pi^2 k^3}\int_0^\infty \left(f' + \frac{2w^2x}{k^2}f''\right)\left(\frac{r^2 + \frac{3xw^2}{k^2}}{4}\right)\cos\left(\frac{\pi w\sqrt{x}}{2d}\right)dw. \tag{9}$$

Now, let

$$R(f,D) = \sum_{D<d\leq 2D}\left|\frac{1}{4d}\sum_r \gamma_r \sum_{k\in\mathbb{Z}\setminus\{0\}}\rho_{kr}(4d)F_r\left(\frac{k}{4d}\right)\right|.$$

We then have that

$$R(f,D) \ll \frac{1}{D}\sum_{D<d\leq 2D}\left|\sum_{kr\neq 0}\sum \gamma_r F_r\left(\frac{k}{4d}\right)\right|.$$

To estimate this, we split this into sums with $|k|$ restricted to dyadic intervals. In particular, we write

$$R_n(f,D) = \frac{1}{D}\sum_{D<d\leq 2D}\left|\sum_{2^n\leq|k|<2^{n+1}}\sum_r \gamma_r\rho_{kr}(4d)F_r\left(\frac{k}{4d}\right)\right|.$$

Note that $R(f,D) = \sum_{n\geq 0} R_n(f,D)$. Then, we have that by (8) and Proposition 3, $R_n(f,D)$ is

$$\frac{1}{D}\sum_{D<d\leq 2D}\left|\sum_{2^n\leq|k|<2^{n+1}}\sum_r \gamma_r\rho_{kr}(4d)\frac{2\sqrt{x}}{k}\int_0^\infty f\left(\frac{r^2 + \frac{3xw^2}{k^2}}{4}\right)\cos\left(\frac{\pi w\sqrt{x}}{2d}\right)dw\right|$$

$$\ll \frac{\sqrt{x}}{D}\int_0^{2^{n+1}}\sum_{D<d\leq 2D}\left|\sum_{2^n\leq|k|<2^{n+1}}\sum_r \gamma_r\rho_{kr}(4d)f\left(\frac{r^2 + \frac{3xw^2}{k^2}}{4}\right)\right|dw$$

$$\ll \frac{x^{1/2}(\log x)^3}{D^{1/2}}(D + 2^n\sqrt{x})^{1/2}(2^n\sqrt{x})^{1/2}.$$

Similarly, we also have that by (9) and Proposition 3 $R_n(f,D)$ is

$$\ll \frac{D\sqrt{x}}{2^{3n}}\int_0^{2^{n+1}}\sum_{D<d\leq 2D}\left|\sum_{2^n\leq|k|<2^{n+1}}\sum_r \gamma_r\rho_{kr}(4d)\left(f' + \frac{2w^2x}{k^2}f''\right)\left(\frac{r^2 + \frac{3xw^2}{k^2}}{4}\right)\right|dw$$

$$\ll \frac{x^{3/2}(\log x)^3 D^{3/2}}{y^2 2^{2n}}(D + 2^n\sqrt{x})^{1/2}(2^n\sqrt{x})^{1/2}.$$

The desired result then follows since we have that

$$\sum_{n>\log(D\sqrt{x}/y)/\log 2} R_n(f, D) \ll \sum_{n>\log(D\sqrt{x}/y)/\log 2} \frac{x^2(\log x)^3 D^{3/2}}{y^2 2^n} \ll D^{1/4}x^{3/4}(\log x)^3$$

and

$$\sum_{0\leq n\leq\log(D\sqrt{x}/y)/\log 2} R_n(f, D) \ll \frac{x^{5/4}(\log x)^4}{y^{1/2}} D^{1/2}x^{1/4}(D\sqrt{x}/y)^{1/2} \ll D^{1/4}x^{3/4}(\log x)^4.$$

$\square$

4. The Bilinear Form

Now, we shall bound the billinear form in (2) by estimating the sum

$$B_1(M, N) = \sum_{N\leq n\leq N'} \left| \sum_{M<m\leq M'} a_{mn}\mu(m) \right| \tag{10}$$

for some arbitrary $M < M' \leq 2M, N < N' \leq 2N$. In particular, we show the following result.

Proposition 5. *For δ a sufficiently small positve number, we have that*

$$B(M, N) \ll MN(\log MN)^{-A} \tag{11}$$

for all $A > 0$, where $M = N^{\delta}$.

This implies that $T(x; D) \ll x(\log x)^{-A}$ for the same reason (20.20) implies Proposition 20.8 in [2], and therefore, from it follows the main theorem.

Proof. First, note that it is sufficient to estimate

$$B_1(M, N) = \sum_{N<n\leq N'} \left| \sum_{\substack{M<m\leq M' \\ (m,n)=1}} a_{mn}\mu(m) \right|,$$

since if $(m, n) = d$, if $d < M^{1/2}$, we can just transfer the factor of d to n, and otherwise use the trivial bound. Writing $\gamma(\mathfrak{a})$ to denote $\gamma_{2\,\mathrm{Re}\,\mathfrak{a}}$, note that we have that

$$a_n = \sum_{N\mathfrak{a}=n} \gamma(\mathfrak{a})$$

so by unique factorization in $\mathbb{Q}(\omega)$, we have that for relatively prime m, n,

$$a_{mn} = \frac{1}{6} \sum_{N\mathfrak{m}=m} \sum_{N\mathfrak{n}=n} \gamma(\mathfrak{m}\mathfrak{n})$$

where the factor of $1/6$ accounts for the six units $\pm 1, \pm\omega, \pm\omega^2$ in $\mathbb{Z}[\omega]$. It follows that

$$B_1(M,N) = \frac{1}{6} \sum_{N<N(\mathfrak{n})\leq N'} \Bigg| \sum_{\substack{M<N(\mathfrak{m})\leq M' \\ (\mathfrak{m},\mathfrak{n})=1}} \gamma(\mathfrak{m}\mathfrak{n})\mu(\mathfrak{m}) \Bigg|.$$

The coprimality condition can easily be dropped by a similar argument by which it was added, so it follows that it is sufficient to show that

$$B_2(M,N) = \sum_{N<N(\mathfrak{n})\leq N'} \Bigg| \sum_{M<N(\mathfrak{m})\leq M'} \gamma(\mathfrak{m}\mathfrak{n})\mu(\mathfrak{m}) \Bigg| \ll MN(\log MN)^{-A}.$$

By Cauchy, we have that it is sufficient to show that

$$B_3(M,N) = \sum_{N<N(\mathfrak{n})\leq N'} \Bigg| \sum_{M<N(\mathfrak{m})\leq M'} \gamma(\mathfrak{m}\mathfrak{n})\mu(\mathfrak{m}) \Bigg|^2 \ll M^2N(\log MN)^{-A}.$$

Expanding and reversing the order of summation, we have that

$$B_3(M,N) = \sum_{M<N(\mathfrak{m}_1),N(\mathfrak{m}_2)\leq M'} \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)S(\mathfrak{m}_1,\mathfrak{m}_2),$$

where

$$S(\mathfrak{m}_1,\mathfrak{m}_2) = \sum_{N<N(\mathfrak{n})\leq N'} \gamma(\mathfrak{n}\mathfrak{m}_1)\gamma(\mathfrak{n}\mathfrak{m}_2).$$

Now, let ℓ_1, ℓ_2 be such that

$$\mathfrak{n}\mathfrak{m}_1 + \overline{\mathfrak{n}\mathfrak{m}}_1 = \ell_1$$

$$\mathfrak{n}\mathfrak{m}_2 + \overline{\mathfrak{n}\mathfrak{m}}_2 = \ell_2,$$

and let $\Delta(\mathfrak{m}_1,\mathfrak{m}_2) = \Delta = i(\mathfrak{m}_1\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1\mathfrak{m}_2)$. Note that $\ell_1, \ell_2 \leq 4\sqrt{MN}$. When $\Delta = 0$, note that the contribution $B_0(M,N)$ satisfies

$$B_0(M,N) \ll N(\log N)^2 \sum\sum_{\operatorname{Im}\overline{\mathfrak{m}}_1\mathfrak{m}_2=0} 1$$

which is clearly $\ll M^2N(\log MN)^{-A}$. Otherwise, we have that

$$\overline{\mathfrak{n}} = \frac{i(\ell_1\mathfrak{m}_2 - \ell_2\mathfrak{m}_1)}{\Delta}$$

so it follows that

$$\ell_1\mathfrak{m}_2 \equiv \ell_2\mathfrak{m}_1 \pmod{\Delta}$$

and that

$$\Delta^2 N < N(\ell_1\mathfrak{m}_2 - \ell_2\mathfrak{m}_1) \leq \Delta^2 N'.$$

Therefore

$$S(\mathfrak{m}_1, \mathfrak{m}_2) = \sum_{\substack{\ell_1\mathfrak{m}_2 \equiv \ell_2\mathfrak{m}_1 \pmod{\Delta} \\ \Delta^2 N < N(\ell_1\mathfrak{m}_2 - \ell_2\mathfrak{m}_1) \leq \Delta^2 N'}} \gamma_{\ell_1}\gamma_{\ell_2}.$$

Now, we state Proposition 20.9 in [2].

Proposition 6. *We have*

$$\sum_{q\leq Q} \max_{\substack{a\in\mathbb{Z},(a,q)=1 \\ \mathfrak{a}\in\mathbb{C} \\ y\in\mathbb{R}}} \left| \sum\sum_{\substack{\ell_1,\ell_2\leq x \\ |\ell_1-\mathfrak{a}\ell_2|\leq y \\ \ell_1\equiv a\ell_2 \pmod{q}}} \gamma_{\ell_1}\gamma_{\ell_2} - \varphi(q)^{-1} \sum\sum_{\substack{\ell_1,\ell_2\leq x \\ |\ell_1-\mathfrak{a}\ell_2|\leq y}} \gamma_{\ell_1}\gamma_{\ell_2} \right| \ll x^2(\log x)^{-A}$$

where $Q = x(\log x)^{-B}$ *for some* $B > 0$ *that depends on* A.

Now we can split up $S(\mathfrak{m}_1, \mathfrak{m}_2)$ into classes restricted to

$$\ell_1 \equiv a\ell_2 \pmod{\Delta}$$

for $a \in (\mathbb{Z}/(\Delta))^*$ such that $a\mathfrak{m}_2 \equiv \mathfrak{m}_1 \pmod{\Delta}$ and apply Proposition 6. It then follows that

$$B_3(M,N) \ll B_4(M,N) + O(NM^2(\log MN)^{-A})$$

where

$$B_4(M,N) = \sum\sum_{M<N(\mathfrak{m}_1),N(\mathfrak{m}_2)\leq M'} \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\frac{\eta(\Delta)}{\varphi(\Delta)} \sum\sum_{\substack{\ell_1,\ell_2\leq x \\ \Delta^2 N<N(\ell_1\mathfrak{m}_2-\ell_2\mathfrak{m}_1)\leq\Delta^2 N'}} \gamma_{\ell_1}\gamma_{\ell_2}$$

where $\eta(\Delta)$ is the total number of $a \in (\mathbb{Z}/(\Delta))^*$ such that $a\mathfrak{m}_2 \equiv \mathfrak{m}_1 \pmod{\Delta}$. By the prime number theorem, we have that the inner sum satisfies

$$\sum\sum_{\substack{\ell_1,\ell_2\leq x \\ \Delta^2 N<N(\ell_1\mathfrak{m}_2-\ell_2\mathfrak{m}_1)\leq\Delta^2 N'}} \gamma_{\ell_1}\gamma_{\ell_2} = X + O(MN(\log MN)^{-A})$$

where

$$X = \int\int_{\Delta\sqrt{N}<|\ell_1\mathfrak{m}_2-\ell_2\mathfrak{m}_1|\leq\Delta\sqrt{N'}} d\ell_1 d\ell_2 = |\Delta| \int\int_{N<|u+\omega v|\leq N'} du dv = \frac{1}{2}\pi\sqrt{3}|\Delta|(N'-N).$$

It therefore now remains to estimate

$$S_1 = \sum_{M<N(\mathfrak{m}_1),N(\mathfrak{m}_2)\le M'}\sum \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\frac{\eta(\Delta)|\Delta|}{\varphi(\Delta)}.$$

Splitting this up for all $(\mathfrak{m}_1,\mathfrak{m}_2)=\mathfrak{d}$, we then have that

$$\begin{aligned} S_1 &= \sum_{\mathfrak{d}}\mu^2(d)\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M'\\(\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2,\mathfrak{d})=1}}\sum \mu(\mathfrak{m}_1\mathfrak{d})\mu(\mathfrak{m}_2\mathfrak{d})\frac{\eta(\Delta N(\mathfrak{d}))|\Delta|N(\mathfrak{d})}{\varphi(\Delta N(\mathfrak{d}))}\\ &= \sum_{\mathfrak{d}}\mu^2(d)\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M'\\(\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2)=1}}\sum \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\frac{\eta(\Delta N(\mathfrak{d}))|\Delta|N(\mathfrak{d})}{\varphi(\Delta N(\mathfrak{d}))}. \end{aligned}$$

Since we have that

$$\eta(\Delta N(\mathfrak{d})) = \sum_{\substack{a\in(\mathbb{Z}/(\Delta N(\mathfrak{d})))^*\\ a\equiv\mathfrak{m}_2\mathfrak{m}_1^{-1} \pmod{\overline{\mathfrak{d}}\Delta}}} 1 = N(\mathfrak{d})\prod_{p|N(\mathfrak{d}),p\nmid\Delta}\left(1-\frac{1}{p}\right),$$

it follows that

$$S_1 = \sum_{\mathfrak{d}}\mu^2(d)N(\mathfrak{d})\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M'\\(\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2)=1}}\sum \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\frac{|\Delta|}{\varphi(\Delta)}.$$

By multiplicativity, we have that

$$\frac{|\Delta|}{\varphi(\Delta)} = \sum_{d|\Delta}\mu^2(d)\varphi(d)^{-1}.$$

Using this and reversing the order of summation, we have that

$$\begin{aligned} S_1 &= \sum_{\mathfrak{d}}\mu^2(d)N(\mathfrak{d})\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M'\\(\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2)=1}}\sum \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\sum_{d|\Delta}\mu^2(d)\varphi(d)^{-1}\\ &= \sum_{\mathfrak{d}}\mu^2(d)N(\mathfrak{d})\sum_{d\le 2M}\varphi(d)^{-1}\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M'\\(\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2)=1\\ \mathfrak{m}_1\overline{\mathfrak{m}}_2\equiv\overline{\mathfrak{m}}_1\mathfrak{m}_2 \pmod d}}\sum \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\\ &= \sum_{\mathfrak{d}}\mu^2(d)N(\mathfrak{d})\sum_{d\le 2M}\frac{1}{d\varphi(d)}\sum_{\chi}\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M'\\(\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2)=1}}\sum \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\psi(\mathfrak{m}_1)\overline{\psi}(\mathfrak{m}_2), \end{aligned}$$

where χ runs over the characters of $\mathbb{Z}[\omega]/(d)$ and $\psi(\mathfrak{m})=\chi(\mathfrak{m})\overline{\chi}(\overline{\mathfrak{m}})$, where the last statement follows by orthogonality. To estimate this, we use the following version of the Siegel-Walfisz Theorem that follows from the main result in [4].

Proposition 7. *For any character ψ on ideals of $\mathbb{Z}[i]$, we have*

$$\sum_{N(\mathfrak{m})\le x} \mu(\mathfrak{m})\psi(\mathfrak{m}) \ll_A x(\log x)^{-A}$$

for all $A > 0$.

Now, let

$$S^*_{\mathfrak{d},d,\psi}(M) = \sum\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M' \\ (\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2,\mathfrak{d})=1}} \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\psi(\mathfrak{m}_1)\overline{\psi}(\mathfrak{m}_2).$$

Then, it is easy to see that $S^*_{\mathfrak{d},d,\psi}(M) = S_{\mathfrak{d},d,\psi}(M) + O(M^{1+\varepsilon})$ where

$$S_{\mathfrak{d},d,\psi}(M) = \sum\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M' \\ (\mathfrak{m}_1,\mathfrak{m}_2)=1}} \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\psi(\mathfrak{m}_1)\overline{\psi}(\mathfrak{m}_2).$$

We then have that

$$\sum_{\mathfrak{d}_1\in\mathbb{Z}[\omega]\setminus\{0\}} \mu^2(\mathfrak{d}_1)S_{\mathfrak{d},d,\psi}(M/N(\mathfrak{d}_1))$$

$$= \left(\sum_{M<N(m_1\mathfrak{d})\le M'} \mu(\mathfrak{m}_1)\psi(\mathfrak{m}_1)\right)\left(\sum_{M<N(m_2\mathfrak{d})\le M'} \mu(\mathfrak{m}_2)\overline{\psi}(\mathfrak{m}_2)\right),$$

so by a variant of Möbius inversion, we have that

$$S_{\mathfrak{d},d,\psi}(M) \ll (M/N(\mathfrak{d}))^2(\log M/N(\mathfrak{d}))^{-A}.$$

The desired result follows.

□

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