ON A CONJECTURE OF DE KONINCK

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Abstract
For a positive integer \( n \), let \( \sigma(n) \), \( \omega(n) \) and \( \gamma(n) \) denote the sum of divisors, the number of distinct prime factors and the product of the distinct prime divisors of \( n \), respectively. In this paper, we focus on positive integers \( n \) such that \( \sigma(n) = \gamma(n)^2 \) and \( \omega(n) = 5 \).

1. Introduction
For a positive integer \( n \), let \( \sigma(n) \), \( \omega(n) \) and \( \gamma(n) \) denote the sum of divisors, the number of the distinct prime factors and the product of distinct prime divisors of \( n \), respectively. Let
\[
\mathcal{K} = \{ n : \sigma(n) = \gamma(n)^2 \}.
\]
De Koninck conjectured \( \mathcal{K} = \{ 1, 1782 \} \). It is included in Richard Guy’s compendium ([5], Section B11).

In 2012, Broughan, De Koninck, Kátai and Luca [1] showed that the only solution with at most four distinct prime factors is \( n = 1782 \). Moreover, they showed that if \( n > 1 \) is in \( \mathcal{K} \), then \( n \) is not fourth power free.

\textbf{Theorem A} ([1, Lemma 1].) If \( n > 1 \) is in \( \mathcal{K} \), then
\[
n = 2^e p_1 \prod_{i=2}^{s} p_i^{\alpha_i},
\]
where \( e \geq 1 \), \( \alpha_i \) is even for all \( i = 3, \ldots, s \). Furthermore, either \( \alpha_2 \) is even in which case \( p_1 \equiv 3 \pmod{8} \), or \( \alpha_2 \equiv 1 \pmod{4} \) and \( p_1 \equiv p_2 \equiv 1 \pmod{4} \).

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By Theorem A, we know that if \( n > 1 \) and \( n \in K \), then the prime factorization of \( n \) has the form
\[
n = 2^a p q \prod_{i=1}^{s} p_i^{\alpha_i},
\]
where \( \alpha \geq 1 \), \( \alpha_i(1 \leq i \leq s) \) are even and \( p \equiv q \equiv 1 \pmod{4} \), or
\[
n = 2^a p \prod_{i=1}^{s} p_i^{\alpha_i},
\]
where \( \alpha \geq 1 \), \( \alpha_i(2 \leq i \leq s) \) are even and either \( p \equiv p_1 \equiv \alpha_1 \equiv 1 \pmod{4} \), \( \alpha_1 \geq 5 \) or \( p \equiv 3 \pmod{8} \) and \( \alpha_1 \) is even.

In 2014, Broughan, Delbourgo and Zhou [2] determined some necessary conditions that an integer \( n > 1 \) must meet in order to belong to \( K \) (see also [3]).

**Theorem B** ([2, Theorem 1].) If \( n \in K \), then \( n \) is divisible by the fourth power of an odd prime.

In 2015, Chen and Tong [4] proved some new theorems on De Koninck’s conjecture.

**Theorem C** ([4, Theorem 1.2].) If \( n > 1 \), \( n \neq 1782 = 2 \cdot 3^4 \cdot 11 \) and \( n \in K \) with the form (3), then \( n \) is divisible by the fourth powers of at least two odd primes.

**Theorem D** ([4, Theorem 1.4].) If \( n > 1 \), \( n \neq 1782 = 2 \cdot 3^4 \cdot 11 \) and \( n \in K \) with the form (3), then at least two exponents of the odd primes in the prime factorization of \( n \) are equal to 2.

In this paper, we obtain the following results:

**Theorem 1.** There is no \( n \in K \) such that \( n = 2^a p q p_1^4 p_2^4 \), where \( \alpha \geq 1 \) and \( p \equiv q \equiv 1 \pmod{4} \).

**Corollary 1.** If \( \omega(n) = 5 \) and \( n \in K \), then the prime factorization of \( n \) has either the form
\[
n = 2^a p q p_1^2 p_2^2,
\]
where \( \alpha \geq 1 \), \( \alpha_2 \geq 4 \) are even and \( p \equiv q \equiv 1 \pmod{4} \), or the form
\[
n = 2^a p p_1^{\alpha_1} p_2^{\alpha_2},
\]
where \( \alpha \geq 1 \), \( \alpha_1, \alpha_2 \geq 4 \) are even, \( (\alpha_1, \alpha_2) \neq (4, 4) \) and \( p \equiv q \equiv 1 \pmod{4} \).

2. **Proof of Theorem 1**

We need the following lemma.
Lemma 1. (See [4, Lemma 2.2].) Let $\gamma_1, \gamma_2, \gamma$ be three primes. If $\gamma_2 \mid \gamma_1^{-1} + \gamma_1^{-2} + \cdots + \gamma_1 + 1$, then either $\gamma_2 = \gamma$ or $\gamma \mid \gamma_2 - 1$.

Suppose that $n \in \mathbb{K}$ and $n = 2^\alpha p_1 p_2^4$, where $\alpha \geq 1$ and $p \equiv q \equiv 1 \pmod{4}$, then

$$
(2^{\alpha+1} - 1) \frac{p + 1}{2} \frac{q + 1}{2} (p_1^4 + p_1^3 + p_1^2 + p_1 + 1)(p_2^4 + p_2^3 + p_2^2 + p_2 + 1) = p^2 q^2 p_1^4 p_2^2.
$$

Thus we have the following observations:

(i) $3 \nmid n$ and $p, q \equiv 1 \pmod{3}$. Indeed, by $p \equiv q \equiv 1 \pmod{4}$, we know that if $3 \mid n$, then by the symmetry of $p_1$ and $p_2$, we may assume that $p_1 = 3$. Since $\sigma(3^4) = 11^2$, we have $p_2 = 11$. Noting that $\sigma(11^4) = 5 \times 3221$, by (4) we may assume that $p = 5$ and $q = 3221$, thus $\frac{p^2 q^2}{2} = 3^2 \times 179$, hence (4) cannot hold. Therefore, $3 \nmid n$ and $p, q \equiv 1 \pmod{3}$.

(ii) $5 \nmid n$. Indeed, if $5 \mid n$, then we may assume that $p_1 = 1$ and observe that $\sigma(5^4) = 11 \times 71$. Noting that $11, 71 \equiv 1 \pmod{4}$, we have (4) cannot hold. Hence $5 \nmid n$.

(iii) $\alpha \geq 4$. Indeed, by (i) and (ii) we have $\alpha \neq 1, 3$. If $\alpha = 2$, then we may assume that $p_1 = 7$. Observe that $\sigma(p_1^2) = 2801$. Noting that if $p = 2801$, then $p + 1 = 2 \times 3 \times 467$, which is impossible; if $p_2 = 2801$ then $\sigma(p_2^2) = 5 \times 1956611 \times 6294091$, which is also impossible. Hence $\alpha \geq 4$.

Let

$$Q = (p_1^4 + p_1^3 + p_1^2 + p_1 + 1)(p_2^4 + p_2^3 + p_2^2 + p_2 + 1).$$

By (4), we have

$$pq = \frac{2^{\alpha+1} - 1}{2^2} \frac{Q}{p_1^4 p_2^2} \frac{p + 1}{p} \frac{q + 1}{q} > \frac{(2^{\alpha+1} - 1)Q}{2^2 p_1^4 p_2^2}.$$ 

If $(2^{\alpha+1} - 1)Q \geq p^2 q^2$, then

$$(2^{\alpha+1} - 1)Q > \frac{(2^{\alpha+1} - 1)^2 Q^2}{2^4 p_1^4 p_2^2},$$

thus $(2^{\alpha+1} - 1)Q < 2^4 p_1^4 p_2^4$, which is impossible. Hence $(2^{\alpha+1} - 1)Q < p^2 q^2$.

Since $5 \nmid n$, by Lemma 1 we have gcd$(Q, p_1^2 p_2^2) = 1$, thus $Q \mid p^2 q^2$. Since $p_1^4 + p_1^3 + p_1^2 + p_1 + 1 \neq p_2^4 + p_2^3 + p_2^2 + p_2 + 1$, we have $Q = pq, p^2 q$ or $q^2 p$. By the the symmetry of $p$ and $q$, it is sufficient to consider $Q = pq$ or $p^2 q$. We now consider the following three cases.

Case 1:

$$p_1^4 + p_1^3 + p_1^2 + p_1 + 1 = p \quad (5)$$

$$p_2^4 + p_2^3 + p_2^2 + p_2 + 1 = q \quad (6)$$

$$\frac{(2^{\alpha+1} - 1)\left(p + 1\right)\left(q + 1\right)}{2} = pq p_1^4 p_2^2. \quad (7)$$
In this case, we have the following facts:

(a) $p_1 \mid \frac{p+1}{2}$ and $p_2 \mid \frac{q+1}{2}$. Assume $p_1 \mid \frac{p+1}{2}$, then $p_1 \mid p + 1$, however, by (5) we have $p_1 \mid p - 1$, thus $p_1 \mid \gcd(p + 1, p - 1) = 2$, which is impossible. Similarly, we have $p_2 \mid \frac{q+1}{2}$.

(b) If $p_2 \mid p_1 - 1$, then $p_2 \mid \frac{q+1}{2}$. Since $p_2 \mid p_1 - 1$, by (5) we have $p_2 \mid p - 5$.

Assume $p_2 \mid \frac{q+1}{2}$, then $p_2 \mid \gcd(p + 1, p - 5) = 2$ or 6, which is impossible.

According to the number of prime divisors of $\frac{p+1}{2}$, we consider the following three subcases:

**Case 1.1: $\frac{q+1}{2}$ has only one prime divisor.** By (a) we have $\frac{q+1}{2} = p$ or $p_1$.

If $\frac{q+1}{2} = p$, then

$$
(2^{a+1} - 1)\frac{p+1}{2} = qp_1^2p_2^2.
$$

(8)

Since $q = 2p - 1 > p$, we have $q \nmid \frac{p+1}{2}$. Since $p_2 \mid q - 1 = 2p - 2$, we have $p_2 \mid p - 1$, thus $p_2 \nmid p + 1$, hence $p_2 \nmid \frac{q+1}{2}$. Combined with (a), we know that (8) cannot hold.

If $\frac{q+1}{2} = p_1$, then

$$
(2^{a+1} - 1)\frac{p+1}{2} = pq_1p_2^2.
$$

(9)

Since $p_2 \mid q - 1 = 2p_1 - 2$, we have $p_2 \mid p_1 - 1$, and by (b) we have $p_2 \nmid \frac{q+1}{2}$. Combined with (a) we have $\frac{q+1}{2} = q$, that is, $p + 1 = 4p_1 - 2$, which contradicts with (5). Hence (9) cannot hold.

**Case 1.2: $\frac{q+1}{2}$ has two prime divisors.** By (a) we have $\frac{q+1}{2} = p_1^2$ or $pp_1$.

If $\frac{q+1}{2} = p_1^2$, then

$$
(2^{a+1} - 1)\frac{p+1}{2} = pq_1p_2^2.
$$

(10)

Noting that $p_2 \mid q - 1 = 2(p_1 + 1)(p_1 - 1)$, if $p_2 \nmid p_1 - 1$, then by (b) we have $p_2 \nmid \frac{q+1}{2}$.

If $p_2 \mid p_1 + 1$, then by (5) we have $p_2 \mid p - 1$, thus $p_2 \nmid \frac{q+1}{2}$. By (10) we have $\frac{q+1}{2} = q$, that is, $p + 1 = 4p_1 - 2$, which contradicts (5). Hence (10) cannot hold.

If $\frac{q+1}{2} = pp_1$, then

$$
(2^{a+1} - 1)\frac{p+1}{2} = qp_1p_2^2.
$$

(11)

Noting that $q - 1 = 2pp_1 - 2 = 2p_1(p + 1) - 2p_1 - 2$, we have

$$
\gcd(p + 1, q - 1) = \gcd(p + 1, 2p_1 + 2)
= \gcd(p_1^2 + p_1^2 + p_1 + 2, 2p_1 + 2)
= \gcd(p_1^2(p_1 + 1) + p_1(p_1 + 1) + 2, 2p_1 + 2)
= 2 \text{ or } 4.
$$

If $p_2 \mid \frac{p+1}{2}$, then $p_2 \mid p + 1$, however, $p_2 \mid q - 1$, thus $p_2 \mid \gcd(p + 1, q - 1)$, which is impossible. Hence $p_2 \nmid \frac{p+1}{2}$. By (11) and (a), we have $\frac{q+1}{2} = q$, that is, $p + 1 = 2q = 4pp_1 - 2 > p + 1$, a contradiction. Hence (11) cannot hold.
Case 1.3: $\frac{2^{n+1}}{2}$ has three prime divisors. By (a) we have $\frac{2^{n+1}}{2} = pp_1^2$. Then

$$(2^{n+1} - 1)\frac{p + 1}{2} = qp_2^2.$$  \hspace{1cm} (12)

Noting that $q - 1 = 2pp_1^2 - 2 = 2p^2(p + 1) - 2p_1^2 - 2$, we have

$$\text{gcd}(p + 1, q - 1) = \text{gcd}(p + 1, 2p_1^2 + 2) = \text{gcd}(p_1^4 + p_1^3 + p_1^2 + p_1 + 2, 2p_1^2 + 2) = \text{gcd}(p_1^2(p_1^2 + 1) + p_1(p_1^2 + 1) + 2, 2p_1^2 + 2) = 2 \text{ or } 4.$$

If $p_2 \mid \frac{p+1}{2}$, then $p_2 \mid p + 1$, however, $p_2 \mid q - 1$, thus $p_2 \mid \text{gcd}(p + 1, q - 1)$, which is impossible. Hence $p_2 \nmid \frac{p+1}{2}$. By (12) we have $\frac{p+1}{2} = q$ and $2^{n+1} - 1 = p_2^2$, which is impossible. Hence (12) cannot hold.

Case 2:

$$p_i^4 + p_i^3 + p_i^2 + p_i + 1 = p^2$$ \hspace{1cm} (13)

$$p_i^4 + p_i^2 + p_i + 1 = q$$ \hspace{1cm} (14)

$$(2^{n+1} - 1)\frac{p + 1}{2} = qp_2^2.$$ \hspace{1cm} (15)

In this case, we have $p_2 \nmid \frac{p+1}{2}$. Depending on the number of prime divisors of $\frac{2^{n+1}}{2}$, we consider the following two subcases:

Case 2.1: $\frac{2^{n+1}}{2}$ has only one prime divisor. Then $\frac{2^{n+1}}{2} = p_1$ and we have

$$(2^{n+1} - 1)\frac{p + 1}{2} = qp_1p_2^2.$$ \hspace{1cm} (16)

By $p \equiv 1 \pmod{4}$, we have

$$\text{gcd}(p + 1, p^2 - 5) = \text{gcd}(p + 1, (p + 1)(p - 1) - 4) = \text{gcd}(p + 1, 4) = 2.$$

If $p_2 \mid \frac{p+1}{2}$, then $p_2 \mid p + 1$, however, $p_2 \mid q - 1 = 2p_1 - 2$, thus $p_2 \mid p_1 - 1$. By (13) we have $p_1 - 1 \mid p^2 - 5$. Thus $p_2 \mid \text{gcd}(p + 1, p^2 - 5)$, which is impossible. Hence $p_2 \nmid \frac{p+1}{2}$.

By (16) we have $\frac{p+1}{2} = q$ or $qp_1$. If $\frac{p+1}{2} = q$, then $p = 2q - 1 = 4p_1 - 3$, which contradicts (13). If $\frac{p+1}{2} = qp_1$, then $p = 4p_1^2 - 2p_1 - 1$, which is also impossible. Hence (16) cannot hold.

Case 2.2: $\frac{2^{n+1}}{2}$ has exactly two prime divisors. Then $\frac{2^{n+1}}{2} = p_1^2$ and we have

$$(2^{n+1} - 1)\frac{p + 1}{2} = qp_2^2.$$ \hspace{1cm} (17)
Noting that $2^{α+1} - 1 \equiv 3 \pmod{4}$, we have $2^{α+1} - 1 = p_2$ or $qp_2$.

If $2^{α+1} - 1 = p_2$, then $\frac{p_2 + 1}{2} = q$, and thus

$$p_1^4 + p_1^3 + p_1^2 + p_1 = p^2 - 1 = (4p_1^2p_2 - 2p_2 - 1)^2 - 1,$$

so we have $p_1 \mid p_2 + 1$. Moreover, $p_1 \mid q + 1$. Hence, $p_1 \mid \gcd(p_2 + 1, p_2^4 + p_2^3 + p_2^2 + p_2 + 2) = 2$, which is impossible.

If $2^{α+1} - 1 = qp_2$, then $\frac{p_2 + 1}{2} = p_2$. Thus

$$p_1^4 + p_1^3 + p_1^2 + p_1 = p^2 - 1 = 4p_2^2 - 4p_2,$$

and we have $p_1 \mid p_2 - 1$. Moreover, $p_1 \mid q + 1$. Thus, $p_1 \mid \gcd(p_2 - 1, p_2^4 + p_2^3 + p_2^2 + p_2 + 2)$. Noting that

$$\gcd(p_2 - 1, p_2^4 + p_2^3 + p_2^2 + p_2 + 2) = \gcd(p_2 - 1, 6),$$

so we have $p_1 = 2, 3$ or 6, which is impossible.

**Case 3:**

$$p_1^4 + p_1^3 + p_1^2 + p_1 + 1 = p \quad (18)$$

$$p_2^4 + p_2^3 + p_2^2 + p_2 + 1 = pq \quad (19)$$

$$\frac{(2^{α+1} - 1)p + 1}{2} q + 1 = \gcd(p^2, p_2^2). \quad (20)$$

In this case, we have $p_1 \mid \frac{2^{α+1}}{2}$. According to the number of prime divisors of $\frac{2^{α+1}}{2}$, we consider the following three subcases:

**Case 3.1:** $\frac{2^{α+1}}{2}$ has only one prime divisor. Then $\frac{2^{α+1}}{2} = p_1$ or $p_2$.

If $\frac{2^{α+1}}{2} = p_1$, then

$$\frac{(2^{α+1} - 1)p + 1}{2} = qp_1^2. \quad (21)$$

If $p_2 \mid \frac{2^{α+1}}{2}$, then $p_2 \mid p + 1$; however, $p_2 \mid pq - 1$, thus $p_2 \mid \gcd(p + 1, pq - 1)$. Noting that

$$\gcd(p + 1, pq - 1) = \gcd(p + 1, q(p + 1) - (q + 1)) = \gcd(p + 1, q + 1) = \gcd(p + 1, 2p_1) = \gcd(p - 1 + 2, 2p_1) = 2,$$

we have $p_2 = 2$, which is impossible. Hence $p_2 \nmid \frac{2^{α+1}}{2}$. By (21) and the fact that $p_1 \nmid \frac{2^{α+1}}{2}$, we have $\frac{2^{α+1}}{2} = q$, that is, $p + 1 = 2q = 4p_1 - 2$, which contradicts (18).

If $\frac{2^{α+1}}{2} = p_2$, then

$$\frac{(2^{α+1} - 1)p + 1}{2} = q^2p_2. \quad (22)$$
Since \( p_1 \mid \frac{p+1}{2} \) and \( 2^\alpha + 1 - 1 \equiv 3 \pmod{4} \), we know that \( p_2 \mid \frac{p+1}{2} \). Thus \( \frac{p+1}{2} = q \).

Hence
\[
p_2^2 + p_2^3 + p_2^2 + p_2 + 1 = pq = 8p_2^2 - 10p_2 + 3,
\]
which is impossible.

**Case 3.2:** \( \frac{p+1}{2} \) has exactly two prime divisors. Then \( \frac{p+1}{2} = p_1^2 \) or \( p_2^2 \) or \( p_1p_2 \).

If \( 2p_2^2 = p_1^2 \), then
\[
(2^\alpha + 1)\frac{p+1}{2} = qp_2^2. \tag{23}
\]
If \( p_2 \mid \frac{p+1}{2} \), then \( p_2 \mid p + 1 \), however, \( p_2 \mid pq - 1 \), thus \( p_2 \mid \gcd(p + 1, pq - 1) \). Noting that
\[
\gcd(p + 1, pq - 1) = \gcd(p + 1, q(p + 1) - (q + 1)) = \gcd(p + 1, q + 1) = \gcd(p + 1, 2p_2^2) = \gcd(p - 1 + 2, 2p_2^2) = 2,
\]
and we have \( p_2 = 2 \), which is impossible. Hence \( p_2 \not\mid \frac{p+1}{2} \). By (23) we have \( \frac{p+1}{2} = q \), thus \( 2^\alpha + 1 = p_2^2 \), which is impossible.

If \( 2p_2^2 = p_2^2 \), then
\[
(2^\alpha + 1)\frac{p+1}{2} = qp_2^2. \tag{24}
\]
Since \( p_1 \mid \frac{p+1}{2} \) we have \( \frac{p+1}{2} = q \), thus \( 2^\alpha + 1 = p_1^2 \), which is impossible.

If \( 2p_2^2 = p_1p_2 \), then
\[
(2^\alpha + 1)\frac{p+1}{2} = qp_1p_2. \tag{25}
\]
If \( \frac{p+1}{2} = p_2 \), then
\[
p_2^4 + p_2^3 + p_2^2 + p_2 + 1 = pq = (2p_2^2 - 1)(2p_1p_2 - 1) = 4p_1p_2^2 - 2p_2^2 - 2p_1p_2 + 1,
\]
thus \( p_2 \mid 2p_1 + 3 \), that is, \( \frac{p+1}{2} \mid 2p_1 + 3 \), which is impossible.

If \( \frac{p+1}{2} = qp_2 \), then
\[
p_2^4 + p_2^3 + p_2^2 + p_2 + 1 = pq = 8p_1^2p_2^3 - 8p_1p_2^2 - 2p_1p_2 + 2p_2 + 1,
\]
we have \( p_2 \mid 2p_1 - 1 \). Thus \( p_2 \mid \gcd(p + 1, 2p_1 - 1) \).
Noting that
\[
gcd(2p_1 - 1, p + 1)
= \gcd(2p_1 - 1, 2p_1^4 + 2p_1^3 + 2p_1^2 + 2p_1 + 4)
= \gcd(2p_1 - 1, p_1^3(2p_1 - 1) + 3p_1^2 + 2p_1 + 4)
= \gcd(2p_1 - 1, 6p_1^3 + 4p_1^2 + 4p_1 + 8)
= \gcd(2p_1 - 1, 3p_1^2(2p_1 - 1) + 7p_1^2 + 4p_1 + 8)
= \gcd(2p_1 - 1, 14p_1^2 + 8p_1 + 16)
= \gcd(2p_1 - 1, 7p_1(2p_1 - 1) + 15p_1 + 16)
= \gcd(2p_1 - 1, 147),
\]
we have \( p_2 = 47 \), and thus \( \sigma(p_2^2) = 11 \cdot 31 \cdot 14621 \), which contradicts (19).

If \( \frac{2^i + 1}{2} = q \), then
\[
p_1^4 + p_1^3 + p_1^2 + p_1 + 2 = p + 1 = 2q = 4p_1p_2 - 3.
\]
Thus \( p_1 | 5 \), and \( p_1 = 5 \), which contradicts (ii).

**Case 3.3:** \( \frac{2^i + 1}{2} \) has three prime divisors. Then \( \frac{2^i + 1}{2} = p_1^2p_2 \) or \( p_1p_2^2 \).

If \( \frac{2^i + 1}{2} = p_1^2p_2 \), then
\[
\left(2^{a+1} - 1\right)\frac{p + 1}{2} = qp_2. \tag{26}
\]
Noting that \( 2^{a+1} - 1 \equiv 3 \pmod{4} \) and \( q \equiv 1 \pmod{4} \), we have \( q \neq 2^{a+1} - 1 \). By (26) we have \( \frac{2^i + 1}{2} = q \), that is,
\[
p_1^4 + p_1^3 + p_1^2 + p_1 + 2 = p + 1 = 2q = 4p_1^2p_2 - 2.
\]
Then \( p_1 | 4 \), which is impossible.

If \( \frac{2^i + 1}{2} = p_1p_2^2 \), then
\[
\left(2^{a+1} - 1\right)\frac{p + 1}{2} = qp_1. \tag{27}
\]
Since \( p_1 | \frac{2^i + 1}{2} \), we have \( \frac{2^i + 1}{2} = q \), that is,
\[
p_1^4 + p_1^3 + p_1^2 + p_1 + 2 = p + 1 = 2q = 4p_1p_2^2 - 2,
\]
then \( p_1 | 4 \), which is impossible.

This completes the proof of Theorem 1.

### 3. Proof of Corollary 1

By Theorems A, C, D, we know that if \( \omega(n) = 5 \) and \( n \in K \), then
\[
n = 2^ap_0 \cdot \prod_{i=1}^{2} p_i^{a_i},
\]
where $\alpha \geq 1$, $\alpha_1$ and $\alpha_2$ are even and $p \equiv q \equiv 1 \pmod{4}$.

By Theorem B and Theorem 1, we know that

$$n = 2^{\alpha} p q p_1^{\alpha_1} p_2^{\alpha_2},$$

where $\alpha \geq 1$, $\alpha_1 \geq 4$ are even and $p \equiv q \equiv 1 \pmod{4}$, or

$$n = 2^{\alpha} p q p_1^{\alpha_1} p_2^{\alpha_2},$$

where $\alpha \geq 1$, $\alpha_1, \alpha_2 \geq 4$ are even, $(\alpha_1, \alpha_2) \neq (4, 4)$ and $p \equiv q \equiv 1 \pmod{4}$.

This completes the proof of Corollary 1.

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**References**


