ON A CONJECTURE OF DE KONINICK

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Abstract

For a positive integer n, let $\sigma(n)$, $\omega(n)$ and $\gamma(n)$ denote the sum of divisors, the number of distinct prime factors and the product of the distinct prime divisors of n, respectively. In this paper, we focus on positive integers n such that $\sigma(n) = \gamma(n)^2$ and $\omega(n) = 5$.

1. Introduction

For a positive integer n, let $\sigma(n)$, $\omega(n)$ and $\gamma(n)$ denote the sum of divisors, the number of the distinct prime factors and the product of distinct prime divisors of n, respectively. Let

$$\mathcal{K} = \{n : \sigma(n) = \gamma(n)^2\}.$$

De Koninck conjectured $\mathcal{K} = \{1, 1782\}$. It is included in Richard Guy's compendium ([5], Section B11).

In 2012, Broughan, De Koninck, Kátai and Luca [1] showed that the only solution with at most four distinct prime factors is $n = 1782$. Moreover, they showed that if $n > 1$ is in $\mathcal{K}$, then n is not fourth power free.

Theorem A *([1, Lemma 1].) If $n > 1$ is in $\mathcal{K}$, then*

$$n = 2^e p_1 \prod_{i=2}^{s} p_i^{\alpha_i}, \tag{1}$$

where $e \geq 1$, α_i is even for all $i = 3, \ldots, s$. Furthermore, either α_2 is even in which case $p_1 \equiv 3 \pmod 8$, or $\alpha_2 \equiv 1 \pmod 4$ and $p_1 \equiv p_2 \equiv 1 \pmod 4$.

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By Theorem A, we know that if $n > 1$ and $n \in \mathcal{K}$, then the prime factorization of n has the form

$$n = 2^{\alpha} pq \prod_{i=1}^{s} p_i^{\alpha_i}, \tag{2}$$

where $\alpha \geq 1$, $\alpha_i (1 \leq i \leq s)$ are even and $p \equiv q \equiv 1 \pmod 4$, or

$$n = 2^{\alpha} p \prod_{i=1}^{s} p_i^{\alpha_i}, \tag{3}$$

where $\alpha \geq 1$, $\alpha_i (2 \leq i \leq s)$ are even and either $p \equiv p_1 \equiv \alpha_1 \equiv 1 \pmod 4$, $\alpha_1 \geq 5$ or $p \equiv 3 \pmod 8$ and α_1 is even.

In 2014, Broughan, Delbourgo and Zhou [2] determined some necessary conditions that an integer $n > 1$ must meet in order to belong to $\mathcal{K}$ (see also [3]).

Theorem B *([2, Theorem 1].) If $n \in \mathcal{K}$, then n is divisible by the fourth power of an odd prime.*

In 2015, Chen and Tong [4] proved some new theorems on De Koninck's conjecture.

Theorem C *([4, Theorem 1.2].) If $n > 1$, $n \neq 1782 = 2 \cdot 3^4 \cdot 11$ and $n \in \mathcal{K}$ with the form (3), then n is divisible by the fourth powers of at least two odd primes.*

Theorem D*([4, Theorem 1.4].) If $n > 1$, $n \neq 1782 = 2 \cdot 3^4 \cdot 11$ and $n \in \mathcal{K}$ with the form (3), then at least two exponents of the odd primes in the prime factorization of n are equal to 2.*

In this paper, we obtain the following results:

Theorem 1. *There is no $n \in \mathcal{K}$ such that $n = 2^{\alpha} pq p_1^4 p_2^4$, where $\alpha \geq 1$ and $p \equiv q \equiv 1 \pmod 4$.*

Corollary 1. *If $\omega(n) = 5$ and $n \in \mathcal{K}$, then the prime factorization of n has either the form*

$$n = 2^{\alpha} pq p_1^2 p_2^{\alpha_2},$$

where $\alpha \geq 1$, $\alpha_2 \geq 4$ are even and $p \equiv q \equiv 1 \pmod 4$, or the form

$$n = 2^{\alpha} pq p_1^{\alpha_1} p_2^{\alpha_2},$$

where $\alpha \geq 1$, $\alpha_1, \alpha_2 \geq 4$ are even, $(\alpha_1, \alpha_2) \neq (4, 4)$ and $p \equiv q \equiv 1 \pmod 4$.

2. Proof of Theorem 1

We need the following lemma.

Lemma 1. *(See [4, Lemma 2.2].) Let $\gamma_1, \gamma_2, \gamma$ be three primes. If $\gamma_2 \mid \gamma_1^{\gamma-1} + \gamma_1^{\gamma-2} + \cdots + \gamma_1 + 1$, then either $\gamma_2 = \gamma$ or $\gamma \mid \gamma_2 - 1$.*

Suppose that $n \in \mathcal{K}$ and $n = 2^\alpha pqp_1^4p_2^4$, where $\alpha \geq 1$ and $p \equiv q \equiv 1 \pmod 4$, then

$$(2^{\alpha+1}-1)\frac{p+1}{2}\frac{q+1}{2}(p_1^4+p_1^3+p_1^2+p_1+1)(p_2^4+p_2^3+p_2^2+p_2+1) = p^2q^2p_1^2p_2^2. \quad (4)$$

Thus we have the following observations:

(i) $3 \nmid n$ and $p, q \equiv 1 \pmod 3$. Indeed, by $p \equiv q \equiv 1 \pmod 4$, we know that if $3 \mid n$, then by the symmetry of p_1 and p_2, we may assume that $p_1 = 3$. Since $\sigma(3^4) = 11^2$, we have $p_2 = 11$. Noting that $\sigma(11^4) = 5 \times 3221$, by (4) we may assume that $p = 5$ and $q = 3221$, thus $\frac{q+1}{2} = 3^2 \times 179$, hence (4) cannot hold. Therefore, $3 \nmid n$ and $p, q \equiv 1 \pmod 3$.

(ii) $5 \nmid n$. Indeed, if $5 \mid n$, then we may assume that $p_1 = 5$ and observe that $\sigma(5^4) = 11 \times 71$. Noting that $11, 71 \equiv 3 \pmod 4$, we have (4) cannot hold. Hence $5 \nmid n$.

(iii) $\alpha \geq 4$. Indeed, by (i) and (ii) we have $\alpha \neq 1, 3$. If $\alpha = 2$, then we may assume that $p_1 = 7$. Observe that $\sigma(p_1^4) = 2801$. Noting that if $p = 2801$, then $p + 1 = 2 \times 3 \times 467$, which is impossible; if $p_2 = 2801$ then $\sigma(p_2^4) = 5 \times 1956611 \times 6294091$, which is also impossible. Hence $\alpha \geq 4$.

Let

$$Q = (p_1^4+p_1^3+p_1^2+p_1+1)(p_2^4+p_2^3+p_2^2+p_2+1).$$

By (4), we have

$$pq = \frac{2^{\alpha+1}-1}{2^2}\frac{Q}{p_1^2p_2^2}\frac{p+1}{p}\frac{q+1}{q} > \frac{(2^{\alpha+1}-1)Q}{2^2p_1^2p_2^2}.$$

If $(2^{\alpha+1}-1)Q \geq p^2q^2$, then

$$(2^{\alpha+1}-1)Q > \frac{(2^{\alpha+1}-1)^2Q^2}{2^4p_1^4p_2^4},$$

thus $(2^{\alpha+1}-1)Q < 2^4p_1^4p_2^4$, which is impossible. Hence $(2^{\alpha+1}-1)Q < p^2q^2$.

Since $5 \nmid n$, by Lemma 1 we have $\gcd(Q, p_1^2p_2^2) = 1$, thus $Q \mid p^2q^2$. Since $p_1^4+p_1^3+p_1^2+p_1+1 \neq p_2^4+p_2^3+p_2^2+p_2+1$, we have $Q = pq, p^2q$ or q^2p. By the the symmetry of p and q, it is sufficient to consider $Q = pq$ or p^2q. We now consider the following three cases.

Case 1:

$$p_1^4+p_1^3+p_1^2+p_1+1 = p \quad (5)$$

$$p_2^4+p_2^3+p_2^2+p_2+1 = q \quad (6)$$

$$(2^{\alpha+1}-1)\frac{p+1}{2}\frac{q+1}{2} = pqp_1^2p_2^2. \quad (7)$$

In this case, we have the following facts:

(a) $p_1 \nmid \frac{p+1}{2}$ and $p_2 \nmid \frac{q+1}{2}$. Assume $p_1 \mid \frac{p+1}{2}$, then $p_1 \mid p+1$, however, by (5) we have $p_1 \mid p-1$, thus $p_1 \mid \gcd(p+1, p-1) = 2$, which is impossible. Similarly, we have $p_2 \nmid \frac{q+1}{2}$.

(b) If $p_2 \mid p_1 - 1$, then $p_2 \nmid \frac{p+1}{2}$. Since $p_2 \mid p_1 - 1$, by (5) we have $p_2 \mid p-5$. Assume $p_2 \mid \frac{p+1}{2}$, then $p_2 \mid \gcd(p+1, p-5) = 2$ or 6, which is impossible.

According to the number of prime divisors of $\frac{q+1}{2}$, we consider the following three subcases:

`Case 1.1`: $\frac{q+1}{2}$ has only one prime divisor. By (a) we have $\frac{q+1}{2} = p$ or p_1.

If $\frac{q+1}{2} = p$, then

$$(2^{\alpha+1} - 1)\frac{p+1}{2} = qp_1^2 p_2^2. \tag{8}$$

Since $q = 2p - 1 > p$, we have $q \nmid \frac{p+1}{2}$. Since $p_2 \mid q - 1 = 2p - 2$, we have $p_2 \mid p - 1$, thus $p_2 \nmid p + 1$, hence $p_2 \nmid \frac{p+1}{2}$. Combined with (a), we know that (8) cannot hold.

If $\frac{q+1}{2} = p_1$, then

$$(2^{\alpha+1} - 1)\frac{p+1}{2} = pqp_1p_2^2. \tag{9}$$

Since $p_2 \mid q - 1 = 2p_1 - 2$, we have $p_2 \mid p_1 - 1$, and by (b) we have $p_2 \nmid \frac{p+1}{2}$. Combined with (a) we have $\frac{p+1}{2} = q$, that is, $p + 1 = 4p_1 - 2$, which contradicts with (5). Hence (9) cannot hold.

`Case 1.2`: $\frac{q+1}{2}$ has two prime divisors. By (a) we have $\frac{q+1}{2} = p_1^2$ or pp_1.

If $\frac{q+1}{2} = p_1^2$, then

$$(2^{\alpha+1} - 1)\frac{p+1}{2} = pqp_2^2. \tag{10}$$

Noting that $p_2 \mid q-1 = 2(p_1+1)(p_1-1)$, if $p_2 \mid p_1 - 1$, then by (b) we have $p_2 \nmid \frac{p+1}{2}$. If $p_2 \mid p_1 + 1$, then by (5) we have $p_2 \mid p-1$, thus $p_2 \nmid \frac{p+1}{2}$. By (10) we have $\frac{p+1}{2} = q$, that is, $p + 1 = 4p_1^2 - 2$, which contradicts (5). Hence (10) cannot hold.

If $\frac{q+1}{2} = pp_1$, then

$$(2^{\alpha+1} - 1)\frac{p+1}{2} = qp_1p_2^2. \tag{11}$$

Noting that $q - 1 = 2pp_1 - 2 = 2p_1(p+1) - 2p_1 - 2$, we have

$$\begin{aligned}
\gcd(p+1, q-1) &= \gcd(p+1, 2p_1+2)\\
&= \gcd(p_1^4 + p_1^3 + p_1^2 + p_1 + 2, 2p_1 + 2)\\
&= \gcd(p_1^3(p_1+1) + p_1(p_1+1) + 2, 2p_1 + 2)\\
&= 2 \text{ or } 4.
\end{aligned}$$

If $p_2 \mid \frac{p+1}{2}$, then $p_2 \mid p + 1$, however, $p_2 \mid q - 1$, thus $p_2 \mid \gcd(p + 1, q - 1)$, which is impossible. Hence $p_2 \nmid \frac{p+1}{2}$. By (11) and (a), we have $\frac{p+1}{2} = q$, that is, $p + 1 = 2q = 4pp_1 - 2 > p + 1$, a contradiction. Hence (11) cannot hold.

`Case 1.3`: $\frac{q+1}{2}$ has three prime divisors. By (a) we have $\frac{q+1}{2} = pp_1^2$. Then

$$(2^{\alpha+1}-1)\frac{p+1}{2} = qp_2^2. \tag{12}$$

Noting that $q-1 = 2pp_1^2 - 2 = 2p_1^2(p+1) - 2p_1^2 - 2$, we have

$$\begin{aligned} \gcd(p+1, q-1) &= \gcd(p+1, 2p_1^2+2) \\ &= \gcd(p_1^4+p_1^3+p_1^2+p_1+2, 2p_1^2+2) \\ &= \gcd(p_1^2(p_1^2+1)+p_1(p_1^2+1)+2, 2p_1^2+2) \\ &= 2 \text{ or } 4. \end{aligned}$$

If $p_2 \mid \frac{p+1}{2}$, then $p_2 \mid p+1$, however, $p_2 \mid q-1$, thus $p_2 \mid \gcd(p+1, q-1)$, which is impossible. Hence $p_2 \nmid \frac{p+1}{2}$. By (12) we have $\frac{p+1}{2} = q$ and $2^{\alpha+1}-1 = p_2^2$, which is impossible. Hence (12) cannot hold.

`Case 2`:

$$\begin{aligned} p_1^4+p_1^3+p_1^2+p_1+1 &= p^2 \qquad (13)\\ p_2^4+p_2^3+p_2^2+p_2+1 &= q \qquad (14)\\ (2^{\alpha+1}-1)\frac{p+1}{2}\frac{q+1}{2} &= qp_1^2p_2^2. \qquad (15) \end{aligned}$$

In this case, we have $p_2 \nmid \frac{q+1}{2}$. Depending on the number of prime divisors of $\frac{q+1}{2}$, we consider the following two subcases:

`Case 2.1`: $\frac{q+1}{2}$ has only one prime divisor. Then $\frac{q+1}{2} = p_1$ and we have

$$(2^{\alpha+1}-1)\frac{p+1}{2} = qp_1p_2^2. \tag{16}$$

By $p \equiv 1 \pmod 4$, we have

$$\gcd(p+1, p^2-5) = \gcd(p+1, (p+1)(p-1)-4) = \gcd(p+1, 4) = 2.$$

If $p_2 \mid \frac{p+1}{2}$, then $p_2 \mid p+1$, however, $p_2 \mid q-1 = 2p_1-2$, thus $p_2 \mid p_1-1$. By (13) we have $p_1 - 1 \mid p^2 - 5$. Thus $p_2 \mid \gcd(p+1, p^2-5)$, which is impossible. Hence $p_2 \nmid \frac{p+1}{2}$.

By (16) we have $\frac{p+1}{2} = q$ or qp_1. If $\frac{p+1}{2} = q$, then $p = 2q-1 = 4p_1-3$, which contradicts (13). If $\frac{p+1}{2} = qp_1$, then $p = 4p_1^2 - 2p_1 - 1$, which is also impossible. Hence (16) cannot hold.

`Case 2.2`: $\frac{q+1}{2}$ has exactly two prime divisors. Then $\frac{q+1}{2} = p_1^2$ and we have

$$(2^{\alpha+1}-1)\frac{p+1}{2} = qp_2^2. \tag{17}$$

Noting that $2^{\alpha+1}-1\equiv 3 \pmod 4$, we have $2^{\alpha+1}-1=p_2$ or qp_2.

If $2^{\alpha+1}-1=p_2$, then $\frac{p+1}{2}=qp_2$, and thus

$$p_1^4+p_1^3+p_1^2+p_1=p^2-1=(4p_1^2p_2-2p_2-1)^2-1,$$

so we have $p_1 \mid p_2+1$. Moreover, $p_1 \mid q+1$. Hence, $p_1 \mid \gcd(p_2+1, p_2^4+p_2^3+p_2^2+p_2+2)=2$, which is impossible.

If $2^{\alpha+1}-1=qp_2$, then $\frac{p+1}{2}=p_2$. Thus

$$p_1^4+p_1^3+p_1^2+p_1=p^2-1=4p_2^2-4p_2,$$

and we have $p_1 \mid p_2-1$. Moreover, $p_1 \mid q+1$. Thus, $p_1 \mid \gcd(p_2-1, p_2^4+p_2^3+p_2^2+p_2+2)$. Noting that

$$\gcd(p_2-1, p_2^4+p_2^3+p_2^2+p_2+2)=\gcd(p_2-1,6),$$

so we have $p_1=2,3$ or 6, which is impossible.

`Case 3:`

$$\begin{aligned} p_1^4+p_1^3+p_1^2+p_1+1 &= p \qquad (18)\\ p_2^4+p_2^3+p_2^2+p_2+1 &= pq \qquad (19)\\ (2^{\alpha+1}-1)\frac{p+1}{2}\frac{q+1}{2} &= qp_1^2p_2^2. \qquad (20)\end{aligned}$$

In this case, we have $p_1 \nmid \frac{p+1}{2}$. According to the number of prime divisors of $\frac{q+1}{2}$, we consider the following three subcases:

`Case 3.1`: $\frac{q+1}{2}$ has only one prime divisor. Then $\frac{q+1}{2}=p_1$ or p_2.

If $\frac{q+1}{2}=p_1$, then

$$(2^{\alpha+1}-1)\frac{p+1}{2}=qp_1p_2^2. \qquad (21)$$

If $p_2 \mid \frac{p+1}{2}$, then $p_2 \mid p+1$; however, $p_2 \mid pq-1$, thus $p_2 \mid \gcd(p+1, pq-1)$. Noting that

$$\begin{aligned} \gcd(p+1,pq-1) &= \gcd(p+1, q(p+1)-(q+1))\\ &= \gcd(p+1,q+1)\\ &= \gcd(p+1,2p_1)\\ &= \gcd(p-1+2,2p_1)\\ &= 2,\end{aligned}$$

we have $p_2=2$, which is impossible. Hence $p_2 \nmid \frac{p+1}{2}$. By (21) and the fact that $p_1 \nmid \frac{p+1}{2}$, we have $\frac{p+1}{2}=q$, that is, $p+1=2q=4p_1-2$, which contradicts (18).

If $\frac{q+1}{2}=p_2$, then

$$(2^{\alpha+1}-1)\frac{p+1}{2}=qp_1^2p_2. \qquad (22)$$

Since $p_1 \nmid \frac{p+1}{2}$ and $2^{\alpha+1}-1 \equiv 3 \pmod 4$, we know that $p_2 \nmid \frac{p+1}{2}$. Thus $\frac{p+1}{2} = q$. Hence

$$p_2^4 + p_2^3 + p_2^2 + p_2 + 1 = pq = 8p_2^2 - 10p_2 + 3,$$

which is impossible.

`Case 3.2`: $\frac{q+1}{2}$ has exactly two prime divisors. Then $\frac{q+1}{2} = p_1^2$ or p_2^2 or p_1p_2.

If $\frac{q+1}{2} = p_1^2$, then

$$(2^{\alpha+1}-1)\frac{p+1}{2} = qp_2^2. \tag{23}$$

If $p_2 \mid \frac{p+1}{2}$, then $p_2 \mid p+1$, however, $p_2 \mid pq-1$, thus $p_2 \mid \gcd(p+1, pq-1)$. Noting that

$$\begin{aligned}
\gcd(p+1, pq-1) &= \gcd(p+1, q(p+1)-(q+1)) \\
&= \gcd(p+1, q+1) \\
&= \gcd(p+1, 2p_1^2) \\
&= \gcd(p-1+2, 2p_1^2) \\
&= 2,
\end{aligned}$$

and we have $p_2 = 2$, which is impossible. Hence $p_2 \nmid \frac{p+1}{2}$. By (23) we have $\frac{p+1}{2} = q$, thus $2^{\alpha+1} - 1 = p_2^2$, which is impossible.

If $\frac{q+1}{2} = p_2^2$, then

$$(2^{\alpha+1}-1)\frac{p+1}{2} = qp_1^2. \tag{24}$$

Since $p_1 \nmid \frac{p+1}{2}$ we have $\frac{p+1}{2} = q$, thus $2^{\alpha+1} - 1 = p_1^2$, which is impossible.

If $\frac{q+1}{2} = p_1p_2$, then

$$(2^{\alpha+1}-1)\frac{p+1}{2} = qp_1p_2. \tag{25}$$

If $\frac{p+1}{2} = p_2$, then

$$p_2^4 + p_2^3 + p_2^2 + p_2 + 1 = pq = (2p_2-1)(2p_1p_2-1) = 4p_1p_2^2 - 2p_2 - 2p_1p_2 + 1,$$

thus $p_2 \mid 2p_1 + 3$, that is, $\frac{p+1}{2} \mid 2p_1 + 3$, which is impossible.
If $\frac{p+1}{2} = qp_2$, then

$$p_2^4 + p_2^3 + p_2^2 + p_2 + 1 = pq = 8p_1^2p_2^3 - 8p_1p_2^2 - 2p_1p_2 + 2p_2 + 1,$$

we have $p_2 \mid 2p_1 - 1$. Thus $p_2 \mid \gcd(p+1, 2p_1 - 1)$.

Noting that

$$\begin{aligned}
& \gcd(2p_1-1, p+1) \\
= \ & \gcd(2p_1-1, 2p_1^4+2p_1^3+2p_1^2+2p_1+4) \\
= \ & \gcd(2p_1-1, p_1^3(2p_1-1)+3p_1^3+2p_1^2+2p_1+4) \\
= \ & \gcd(2p_1-1, 6p_1^3+4p_1^2+4p_1+8) \\
= \ & \gcd(2p_1-1, 3p_1^2(2p_1-1)+7p_1^2+4p_1+8) \\
= \ & \gcd(2p_1-1, 14p_1^2+8p_1+16) \\
= \ & \gcd(2p_1-1, 7p_1(2p_1-1)+15p_1+16) \\
= \ & \gcd(2p_1-1, 47),
\end{aligned}$$

we have $p_2 = 47$, and thus $\sigma(p_2^4) = 11 \cdot 31 \cdot 14621$, which contradicts (19).

If $\frac{p+1}{2} = q$, then

$$p_1^4+p_1^3+p_1^2+p_1+2 = p+1 = 2q = 4p_1p_2-3.$$

Thus $p_1 \mid 5$, and $p_1 = 5$, which contradicts (ii).

`Case 3.3`: $\frac{q+1}{2}$ has three prime divisors. Then $\frac{q+1}{2} = p_1^2p_2$ or $p_1p_2^2$.

If $\frac{q+1}{2} = p_1^2p_2$, then

$$(2^{\alpha+1}-1)\frac{p+1}{2} = qp_2. \tag{26}$$

Noting that $2^{\alpha+1}-1 \equiv 3 \pmod 4$ and $q \equiv 1 \pmod 4$, we have $q \neq 2^{\alpha+1}-1$. By (26) we have $\frac{p+1}{2} = q$, that is,

$$p_1^4+p_1^3+p_1^2+p_1+2 = p+1 = 2q = 4p_1^2p_2-2.$$

Then $p_1 \mid 4$, which is impossible.

If $\frac{q+1}{2} = p_1p_2^2$, then

$$(2^{\alpha+1}-1)\frac{p+1}{2} = qp_1. \tag{27}$$

Since $p_1 \nmid \frac{p+1}{2}$ we have $\frac{p+1}{2} = q$, that is,

$$p_1^4+p_1^3+p_1^2+p_1+2 = p+1 = 2q = 4p_1p_2^2-2,$$

then $p_1 \mid 4$, which is impossible.

This completes the proof of Theorem 1.

3. Proof of Corollary 1

By Theorems A, C, D, we know that if $\omega(n) = 5$ and $n \in \mathcal{K}$, then

$$n = 2^\alpha pq \prod_{i=1}^{2} p_i^{\alpha_i},$$

where $\alpha \geq 1$, α_1 and α_2 are even and $p \equiv q \equiv 1 \pmod 4$.

By Theorem B and Theorem 1, we know that

$$n = 2^{\alpha} pq p_1^2 p_2^{\alpha_2},$$

where $\alpha \geq 1$, $\alpha_2 \geq 4$ are even and $p \equiv q \equiv 1 \pmod 4$, or

$$n = 2^{\alpha} pq p_1^{\alpha_1} p_2^{\alpha_2},$$

where $\alpha \geq 1$, $\alpha_1, \alpha_2 \geq 4$ are even, $(\alpha_1, \alpha_2) \neq (4, 4)$ and $p \equiv q \equiv 1 \pmod 4$.

This completes the proof of Corollary 1.

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