ESTIMATES FOR $\pi(x)$ FOR LARGE VALUES OF x AND RAMANUJAN'S PRIME COUNTING INEQUALITY

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Abstract

In this paper we use refined approximations for Chebyshev's ϑ-function to establish new explicit estimates for the prime counting function $\pi(x)$, which improve several bounds of similar shape for all sufficiently large values of x. As an application, we find an upper bound for the number H_0 which is defined to be the smallest positive integer so that a certain inequality due to Ramanujan holds for every $x \geq H_0$.

1. Introduction

Let $\pi(x)$ denote the number of primes not exceeding x. Since there are infinitely many primes, we have $\pi(x) \to \infty$ as $x \to \infty$. Gau stated a conjecture concerning an asymptotic behavior of $\pi(x)$, namely

$$\pi(x) \sim \mathrm{li}(x) \qquad (x \to \infty), \tag{1.1}$$

where the *logarithmic integral* $\mathrm{li}(x)$ is defined as

$$\mathrm{li}(x) = \int_0^x \frac{\mathrm{d}t}{\log t} = \lim_{\varepsilon \to 0} \left\{ \int_0^{1-\varepsilon} \frac{\mathrm{d}t}{\log t} + \int_{1+\varepsilon}^x \frac{\mathrm{d}t}{\log t} \right\} = \int_2^x \frac{\mathrm{d}t}{\log t} + 1.04516\ldots, \tag{1.2}$$

where $\log x$ is the natural logarithm of x. Hadamard and de la Vallée-Poussin independently proved the asymptotic formula (1.1) which is known as the *Prime Number Theorem*. In a later paper, where the existence of a zero-free region for the Riemann zeta-function $\zeta(s)$ to the left of the line $\mathrm{Re}(s) = 1$ was proved, de la Vallée-Poussin also estimated the error term in the Prime Number Theorem by showing

$$\pi(x) = \mathrm{li}(x) + O(x \exp(-\delta_0 \sqrt{\log x})), \tag{1.3}$$

where δ_0 is a positive absolute constant. Panaitopol [13, p. 55] gave another asymptotic formula for the prime counting function $\pi(x)$ by showing that for each positive integer m, we have

$$\pi(x) = \frac{x}{\log x - k_0 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \ldots - \frac{k_m}{\log^m x}} + O\left(\frac{x}{\log^{m+2} x}\right), \tag{1.4}$$

where $k_0 = 1$ and the positive integers $k_1, \ldots, k_m$ are defined by the recurrence formula

$$k_m + 1!k_{m-1} + 2!k_{m-2} + \ldots + (m-1)!k_1 = m \cdot m!.$$

For instance, we have $k_1 = 1$, $k_2 = 3$, and $k_3 = 13$. Setting $m = n + 2$, we see that the asymptotic formula (1.4) implies that

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} - \ldots - \frac{k_n}{\log^n x}} \tag{1.5}$$

for every nonnegative integer n and all sufficently large values of x. The first result in this direction is due to Rosser and Schoenfeld [14, Corollary 1]. They showed that the inequality

$$\pi(x) > \frac{x}{\log x} \tag{1.6}$$

holds for every $x \geq 17$. Dusart [7, p. 55] obtained $\pi(x) > x/(\log x - 1)$ for every $x \geq 5393$. The current best effective upper bound which corresponds to the first terms of (1.5) is given in [2, Corollary 3.5] and states that

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x}} \tag{1.7}$$

for every $x \geq 468\,049$. In the following theorem, we make progress towards finding the smallest positive integer N_0 so that the inequality (1.5) holds for $n = 2$ and every $x \geq N_0$.

Theorem 1. *The inequality*

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x}} \tag{1.8}$$

holds for every x such that $65\,405\,887 \leq x \leq 2.7358 \cdot 10^{40}$ and every $x \geq 4.8447 \cdot 10^{19377}$.

Integration by parts in (1.3) implies that the asymptotic expansion

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \ldots + \frac{(m-1)!x}{\log^m x} + O\left(\frac{x}{\log^{m+1} x}\right) \tag{1.9}$$

holds for every positive integer m, which, in turn, implies that for each positive integer n, there exists a smallest positive integer $g_1(n)$ so that

$$\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \frac{24x}{\log^5 x} + \ldots + \frac{(n-1)!x}{\log^n x}$$

for every $x \geq g_1(n)$. Again, the inequality (1.6) yields the first lower bound for $\pi(x)$, which corresponds to the first term of (1.9). Dusart [7, p. 55] found that $g_1(2) = 599$. In [10, Corollary 5.2], he improved his own result by showing that $g_1(3) = 88\,783$. In the following theorem, we go one step further by finding an upper bound for the smallest positive integer $g_1(4)$.

Theorem 2. *The inequality*

$$\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} \tag{1.10}$$

holds for every x such that $10\,384\,261 \leq x \leq 2.7358 \cdot 10^{40}$ and every $x \geq 1.0584 \cdot 10^{2918}$.

In the second part of the paper, we undertake a study of an inequality established by Ramanujan. In one of his notebooks (see Berndt [4]), Ramanujan used (1.9) with $n = 5$ to find that

$$\pi(x)^2 - \frac{ex}{\log x}\pi\left(\frac{x}{e}\right) = -\frac{x^2}{\log^6 x} + O\left(\frac{x}{\log^7 x}\right),$$

and concluded that the inequality

$$\pi(x)^2 < \frac{ex}{\log x}\pi\left(\frac{x}{e}\right) \tag{1.11}$$

holds for all sufficiently large values of x. The inequality (1.11) is called *Ramanujan's prime counting inequality.* A legitimate question is to find the smallest integer H_0 so that the inequality (1.11) holds for every real $x \geq H_0$. Under the assumption that the Riemann hypothesis is true (RH), Hassani [11, Theorem 1.2] has given the upper bound

$$RH \ \Rightarrow \ H_0 \leq 138\,766\,146\,692\,471\,228.$$

Dudek and Platt [6, Lemma 3.2] refined Hassani's result by showing that

$$RH \ \Rightarrow \ H_0 \leq 1.15 \cdot 10^{16}. \tag{1.12}$$

Wheeler, Keiper and Galway (see Berndt [4, p. 113]) attempted to determine the value of H_0, but they failed. Nevertheless, Galway found that the largest prime up to 10^{11} for which the inequality (1.11) fails is $x = 38\,358\,837\,677$. Hence

$$H_0 > 38\,358\,837\,677.$$

Dudek and Platt [6, Theorem 1.3] showed by computation that $x = 38\,358\,837\,682$ is the largest integer (not necessarily prime) counterexample below 10^{11} and that there are no more failures at integer values before $1.15 \cdot 10^{16}$. Hence the inequality

(1.11) holds unconditionally for every $x \in I_0$, where $I_0 = [38\,358\,837\,683, 1.15 \cdot 10^{16}]$. Combined with (1.12),

$$RH \;\Rightarrow\; H_0 = 38\,358\,837\,683.$$

Based on a result of Büthe [5, Theorem 2], we extend the interval I_0 in which the inequality (1.11) holds unconditionally by proving the following theorem.

Theorem 3. *Ramanujan's prime counting inequality* (1.11) *holds unconditionally for every x such that* $38\,358\,837\,683 \leq x \leq 10^{19}$.

In addition, Dudek and Platt [6, Theorem 1.2] claimed to give an upper bound for H_0 which does not depend on the assumption that the Riemann hypothesis is true, namely

$$H_0 \leq e^{9658}. \tag{1.13}$$

After the present author raised some doubts about the correctness of the proof of (1.13), one of the authors confirmed (email communication) that the proof of (1.13) given in [6] is not correct. This motivated us to write this paper, where we prove the following even stronger result. In our proof, explicit estimates for the prime counting function, which hold for all sufficiently large values of x, play an important role.

Theorem 4. *Ramanujan's prime counting inequality* (1.11) *holds unconditionally for every $x \geq e^{9032}$; i.e.,*

$$H_0 \leq e^{9032}.$$

2. Preliminaries

In order to prove Theorems 1 and 2, we first consider Chebyshev's ϑ-function, which is defined by

$$\vartheta(x) = \sum_{p \leq x} \log p,$$

where p runs over primes not exceeding x. The prime counting function and Chebyshev's ϑ-function are connected by identities

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t}\, \mathrm{d}t \tag{2.1}$$

and

$$\vartheta(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t}\, \mathrm{d}t, \tag{2.2}$$

which hold for every $x \geq 2$ (see, for instance, Apostol [1, Theorem 4.3]). Using (2.2), it is easy to see that the Prime Number Theorem is equivalent to

$$\vartheta(x) \sim x \qquad (x \to \infty). \tag{2.3}$$

De la Vallée-Poussin estimated the error term in (2.3) by proving $\vartheta(x) = x + O(x \exp(-\delta_1 \sqrt{\log x}))$, where δ_1 is a positive absolute constant. In this direction, we give the following result.

Proposition 1. *Let $R = 5.573412$. Then*

$$|\vartheta(x) - x| < \frac{\sqrt{8}}{\sqrt{\pi\sqrt{R}}}\, x(\log x)^{1/4} e^{-\sqrt{(\log x)/R}} \tag{2.4}$$

for every $x \geq 3$.

Proof. By Mossinghoff and Trudgian [12, Theorem 1], there are no zeros of the Riemann zeta-function $\zeta(s)$ for $|\mathrm{Im}(s)| \geq 2$ and

$$\mathrm{Re}(s) \geq 1 - \frac{1}{R \log|\mathrm{Im}(s)|}.$$

Applying this to [9, Theorem 1.1], we get the required inequality for every $x \geq e^{390}$. Further, Trudgian [16, Theorem 1] showed that the inequality

$$|\vartheta(x) - x| < \frac{\sqrt{8}}{\sqrt{17\pi\sqrt{6.455}}}\, x(\log x)^{1/4} e^{-\sqrt{(\log x)/6.455}}$$

holds for every $x \geq 149$. By comparing the right hand side of the last inequality with the right hand side of (2.4), we get the desired inequality for every x with $149 \leq x \leq e^{390}$. For the remaining case, where x belongs to the interval $[3, 149)$, we check the desired inequality with a computer. $\square$

Now we use Proposition 1 to obtain the following result which we use in the proof of Theorem 1.

Corollary 1. *For every $x > 1$, we have*

$$|\vartheta(x) - x| < \frac{580115x}{\log^5 x}.$$

Proof. By Proposition 1, we get the required inequality for every $x \geq e^{5801.149}$. In [3, Proposition 2.5], it is shown that $|\vartheta(x) - x| < 100x/\log^4 x$ for every $x \geq 70\,111$, which implies the correctness of the desired inequality for every x satisfying $70\,111 \leq x \leq e^{5801.15}$. We finish by direct computation. $\square$

3. Proof of Theorem 1

Let k be a positive integer, and let η_k and $x_1(k)$ be positive real numbers so that

$$|\vartheta(x) - x| < \frac{\eta_k x}{\log^k x} \tag{3.1}$$

for every $x \geq x_1(k)$. By (2.1) and (3.1), we have

$$J_{k,-\eta_k,x_1(k)}(x) \leq \pi(x) \leq J_{k,\eta_k,x_1(k)}(x) \tag{3.2}$$

for every $x \geq x_1(k)$, where

$$\begin{aligned} J_{k,\eta_k,x_1(k)}(x) &= \pi(x_1(k)) - \frac{\vartheta(x_1(k))}{\log x_1(k)} + \frac{x}{\log x} + \frac{\eta_k x}{\log^{k+1} x} && (1) \\ &\quad + \int_{x_1(k)}^{x} \left(\frac{1}{\log^2 t} + \frac{\eta_k}{\log^{k+2} t} \, \mathrm{d}t \right). && (3.3) \end{aligned}$$

The function $J_{k,\eta_k,x_1(k)}$ given in (3.3) was already introduced by Rosser and Schoenfeld [14, p. 81] (for the case $k = 1$), and by Dusart [8] in general, and plays an important role in the following proof of Theorem 1.

Proof of Theorem 1. First, let $k = 5$, $x_1 = 10^{13}$, and

$$f(x) = \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} - \frac{13}{\log^3 x} + \frac{580044}{\log^4 x}}.$$

Further, we set $g(x) = J_{5,-580115,x_1}(x) - f(x)$. Then,

$$g'(x) = \frac{s(\log x)}{(\log^5 x - \log^4 x - \log^3 x - 3\log^2 x - 13\log x + 580044)^2 \log^7 x},$$

where the polynomial $s(y)$ is given by

$$\begin{aligned} s(y) = {} & 580\,576y^{10} - 6\,381\,045y^9 - 4\,060\,210y^8 - 15\,661\,259y^7 - 336\,607\,082\,789y^6 \\ & + 4\,037\,979\,215\,095y^5 - 2\,691\,881\,529\,325y^4 - 1\,345\,840\,694\,825y^3 \\ & - 1\,345\,478\,703\,065y^2 - 195\,224\,040\,181\,960\,440y + 975\,901\,480\,963\,513\,200. \end{aligned}$$

Since $s(y) > 0$ for every $y \geq \log x_1 \geq 28$, we get

$$J'_{5,-580115,x_1}(x) \geq f'(x) \tag{3.4}$$

for every $x \geq x_1$. By Dusart [10, Table 2], we have $\vartheta(x_1) < 9\,999\,996\,988\,294$. Applying this and the fact that $\pi(x_1) = 346\,065\,536\,839$ to (3.3), we get $J_{5,-580115,x_1}(x_1) - f(x_1) > 3.786 \cdot 10^8$. We combine this with (3.4) to see that $J_{5,-580115,x_1}(x) > f(x)$ for every $x \geq x_1$. Now we use (3.2) and Corollary 1 to get $\pi(x) \geq f(x)$ for every $x \geq x_1$, which implies the validity of (1.8) for every $x \geq 4.8447 \cdot 10^{19377}$.

A comparison of the lower bound in (1.8) with the lower bound obtained in [3, Theorem 1.3] implies that the desired inequality (1.8) holds for every x with $19\,033\,744\,403 \leq x \leq 2.7358 \cdot 10^{40}$.

Let p_n denote the nth prime. To complete the proof, we check that $\pi(p_n) > h(p_{n+1})$ for every integer n satisfying $\pi(65\,405\,887) \leq n \leq \pi(19\,033\,744\,403) + 1$, where $h(x)$ denotes the right-hand side of (1.8). $\square$

Remark. The method employed in the proof of Theorem 1 can also be used to find further lower bounds for $\pi(x)$ given by truncating the asymptotic expansion (1.4) at later terms (with $\log^n x$ in the denominator, where $n \geq 3$). However, these bounds will only hold when x is exceptionally large.

Under the assumption that the Riemann hypothesis is true, Schoenfeld [15, Corollary 1] showed that

$$\pi(x) > \mathrm{li}(x) - \frac{\sqrt{x}\log x}{8\pi} \tag{3.5}$$

for every $x \geq 2657$. Using this inequality, we obtain the following result.

Proposition 2. *Under the assumption that the Riemann hypothesis is true, the inequality* (1.8) *holds for every* $x \geq 65\,405\,887$.

Proof. We denote the right-hand side of (1.8) by $g(x)$ and set $h(x) = -\log^8 x + 208\pi\sqrt{x}\log^2 x + 96\pi\sqrt{x}\log x + 144\pi\sqrt{x}$. Then $h(x) > 0$ for every $x \geq 233\,671\,227\,509$. Further, let $f(x) = \mathrm{li}(x) - \sqrt{x}\log x/(8\pi) - g(x)$. We have $f(10^{12}) > 0$ and $f'(x) \geq h(x)/(16\pi\sqrt{x}(\log^3 x - \log^2 x - \log x - 3)^2 \log x) > 0$ for every $x \geq 233\,671\,227\,509$. Hence

$$\mathrm{li}(x) - \frac{\sqrt{x}}{8\pi}\log x > g(x) \tag{3.6}$$

for every $x \geq 10^{12}$. Since we assume that the Riemann hypothesis is true, we can apply (3.5) to (3.6) and get the assertion for every $x \geq 10^{12}$. Finally, it suffices to apply Theorem 1. $\square$

4. Proof of Theorem 2

Throughout this section, let n be a positive integer, $R = 5.573412$, $c_0 = 3\sqrt{2}/\sqrt{\pi\sqrt{R}}$, and $d_0 = 2/\log 2 - \mathrm{li}(2)$. Proposition 1 implies that the inequality

$$|\vartheta(x) - x| < \frac{a_n(x)x}{\log^n x} \tag{4.1}$$

holds for every $x \geq 3$, where the function $a_n : [2, \infty) \to (0, \infty)$ is defined by

$$a_n(x) = \frac{\sqrt{8}}{\sqrt{\pi\sqrt{R}}}(\log x)^{n+1/4} e^{-\sqrt{(\log x)/R}}.$$

We can show by a straightforward calculation that the function $a_n(x)$ has a global maximum at $x_0 = e^{(4n+1)^2 R/4}$. For the proof of Theorem 2, we need the following inequality involving $a_n(x)$.

Proposition 3. *For every $x \geq 851$, we have*

$$\int_3^x \frac{a_n(t)}{\log^{n+2} t}\,\mathrm{d}t \leq \frac{\sqrt{2}}{\sqrt{\pi\sqrt{R}}} \cdot \frac{x}{(\log x)^{3/4} e^{\sqrt{\log x/R}}}.$$

Proof. Let $x \geq 851$. From the definition of $a_n(x)$, we have

$$\int_3^x \frac{a_n(t)}{\log^{n+2} t}\,\mathrm{d}t = \frac{\sqrt{8}}{\sqrt{\pi\sqrt{R}}} \int_3^x (\log t)^{-7/4} e^{-\sqrt{\log t/R}}\,\mathrm{d}t.$$

The substitution $t = e^{Ry}$ gives

$$\int_3^x \frac{a_n(t)}{\log^{n+2} t}\,\mathrm{d}t = \frac{\sqrt{8}}{R\sqrt{\pi}} \int_{\log 3/R}^{\log x/R} \frac{e^{Ry}}{y^{7/4} e^{\sqrt{y}}}\,\mathrm{d}y. \tag{4.2}$$

For convenience, we write $b = \log 3/R$ and $c = \log x/R$, and define $f : [3, c] \to (0, \infty), y \mapsto e^{Ry}/(y^{7/4}e^{\sqrt{y}})$. It is easy to see that the function $f(y)$ is convex on the interval $[b, c]$. So we get

$$\int_b^c f(y)\,\mathrm{d}y \leq \frac{c-b}{2}(f(b) + f(c)). \tag{4.3}$$

The function $g : [3, \infty) \to (0, \infty), y \mapsto y/(y^{11/4}e^{\sqrt{y/R}})$ is strictly increasing for every $x \geq 22.75$ and fulfills $g(851) \geq g(3)$. Hence $g(y) \geq g(3)$ for every $y \geq 851$, which is equivalent to $bf(c) \geq cf(b)$. Applying this inequality to (4.3), we get

$$\int_b^c f(y)\,\mathrm{d}y \leq \frac{cf(c)}{2}.$$

Combining this with (4.2) and the definition of the function $f(y)$, we finish the proof. $\square$

Now we use the identity (2.1) and Proposition 3 to obtain the following estimates for $\pi(x)$.

Proposition 4. *For every $x \geq 2$, we have*

$$\pi(x) > \mathrm{li}(x) - \frac{c_0 x}{(\log x)^{3/4} e^{\sqrt{\log x/R}}} \tag{4.4}$$

and

$$\pi(x) < \mathrm{li}(x) + \frac{c_0 x}{(\log x)^{3/4} e^{\sqrt{\log x/R}}} + d_0. \tag{4.5}$$

Proof. First, let $x \geq 851$. Since $\vartheta(t)/(t\log^2 t) > 0$ for every $t \geq 2$, we use the identity (2.1) to get

$$\pi(x) > \frac{\vartheta(x)}{\log x} + \int_3^x \frac{\vartheta(t)}{t\log^2 t}\,\mathrm{d}t.$$

By (4.1), we have

$$\pi(x) > \frac{x}{\log x} - \frac{a_n(x)x}{\log^{n+1} x} + \int_3^x \frac{\mathrm{d}t}{\log^2 t} - \int_3^x \frac{a_n(t)}{\log^{n+2} t}\,\mathrm{d}t.$$

Now we can apply Proposition 3 and the identity

$$\int_3^x \frac{\mathrm{d}t}{\log^2 t} = \mathrm{li}(x) - \frac{x}{\log x} - \mathrm{li}(3) + \frac{3}{\log 3}$$

to obtain the inequality

$$\pi(x) > \mathrm{li}(x) - \frac{a_n(x)x}{\log^{n+1} x} - \mathrm{li}(3) + \frac{3}{\log 3} - \frac{\sqrt{2}}{\sqrt{\pi\sqrt{R}}} \cdot \frac{x}{(\log x)^{3/4} e^{\sqrt{\log x/R}}}. \tag{4.6}$$

Since $3/\log 3 - \mathrm{li}(3) > 0$, the inequality (4.6) implies (4.4) for every $x \geq 851$. For smaller values of x, we check the inequality (4.4) with a computer.

In order to prove that the inequality (4.5) holds for every $x \geq 2$, we use (2.1) and integration by parts in (1.3) to get

$$\pi(y) - \mathrm{li}(y) = \frac{\vartheta(y) - y}{\log y} + d_0 + \int_2^y \frac{\vartheta(t) - t}{t\log^2 t}\,\mathrm{d}t \tag{4.7}$$

for every $y \geq 2$. First we consider the case $x \geq 851$. By Büthe [5, Theorem 2], we have $\vartheta(t) < t$ for every t satisfying $1 \leq t \leq 10^{19}$. Combining this with (4.7) and (4.1), we obtain the inequality

$$\pi(x) - \mathrm{li}(x) < \frac{a_n(x)x}{\log^{n+1} x} + d_0 + \int_3^x \frac{a_n(t)}{\log^{n+2} t}\,\mathrm{d}t.$$

Using Proposition 3, we get

$$\pi(x) - \mathrm{li}(x) < \frac{a_n(x)x}{\log^{n+1} x} + d_0 + \frac{\sqrt{2}}{\sqrt{\pi\sqrt{R}}} \cdot \frac{x}{(\log x)^{3/4} e^{\sqrt{\log x/R}}}.$$

Substituting the definition of $a_n(x)$, we get the inequality (4.5) for every $x \geq 851$. Again, we check the desired inequality for smaller values of x with a computer. □

The function $x/\log^{n+2} x$ is strictly increasing on the interval (e^{n+2}, ∞) and tends to infinity as $x \to \infty$. Therefore, there exists a positive integer $A_0(n)$ so that

$$\frac{x}{\log^{n+2} x} \geq \frac{1}{(n+1)!} \sum_{k=1}^{n+1} \frac{2(k-1)!}{\log^k 2} \tag{4.8}$$

for every $x \geq A_0(n)$, and we get the following proposition.

Proposition 5. *For every $x \geq \max\{27, A_0(n)\}$, we have*

$$\pi(x) > \sum_{k=1}^{n+1} \frac{(k-1)!x}{\log^k x} - \frac{c_0 x}{(\log x)^{3/4} e^{\sqrt{\log x / R}}}, \tag{4.9}$$

and for every $x \geq 4$, we have

$$\pi(x) < \sum_{k=1}^{n} \frac{(k-1)!x}{\log^k x} + \frac{n!\sqrt{x}}{\log^{n+1} 2} + \frac{n! 2^{n+1} x}{\log^{n+1} x} + \frac{c_0 x}{(\log x)^{3/4} e^{\sqrt{\log x / R}}}. \tag{4.10}$$

Proof. We start with the proof of (4.9). Let $x \geq \max\{27, A_0(n)\}$. We use integration by parts in (1.2) to get

$$\mathrm{li}(x) \geq \sum_{k=1}^{n+1} \frac{(k-1)!x}{\log^k x} + (n+1)! \int_2^3 \frac{\mathrm{d}t}{\log^{n+2} t} + (n+1)! \int_3^x \frac{\mathrm{d}t}{\log^{n+2} t} - \sum_{k=1}^{n+1} \frac{2(k-1)!}{\log^k 2}.$$

Since $1/\log^m t$ is strictly decreasing on the interval $[2, x]$ for every positive integer m, we obtain

$$\mathrm{li}(x) \geq \sum_{k=1}^{n+1} \frac{(k-1)!x}{\log^k x} + \frac{(n+1)!}{\log^{n+2} 3} + \frac{(x-3) \cdot (n+1)!}{\log^{n+2} x} - \sum_{k=1}^{n+1} \frac{2(k-1)!}{\log^k 2}. \tag{4.11}$$

We have $1/\log^{n+2} 3 \geq 3/\log^{n+2} t$ for every $t \geq 27$. Applying this to (4.11), we get

$$\mathrm{li}(x) \geq \sum_{k=1}^{n+1} \frac{(k-1)!x}{\log^k x} + \frac{(n+1)!x}{\log^{n+2} x} - \sum_{k=1}^{n+1} \frac{2(k-1)!}{\log^k 2}. \tag{4.12}$$

Since $x \geq A_0(n)$, we can apply the inequality (4.8) to (4.12) to obtain the inequality

$$\mathrm{li}(x) \geq \sum_{k=1}^{n+1} \frac{(k-1)!x}{\log^k x}.$$

Now we use (4.4) to complete the proof of (4.9).

Next we verify the correctness of (4.10). Let $x \geq 4$. Again we use integration by parts in (1.2) to get

$$\mathrm{li}(x) = \mathrm{li}(2) + \sum_{k=1}^{n} \frac{(k-1)!x}{\log^k x} + n! \int_2^x \frac{\mathrm{d}t}{\log^{n+1} t} - \sum_{k=1}^{n} \frac{2(k-1)!}{\log^k 2}. \tag{4.13}$$

In the first part of the proof, we notice that $1/\log^{n+1} t$ is strictly decreasing on the interval $[2, x]$. So,

$$\int_2^x \frac{\mathrm{d}t}{\log^{n+1} t} = \int_2^{\sqrt{x}} \frac{\mathrm{d}t}{\log^{n+1} t} + \int_{\sqrt{x}}^x \frac{\mathrm{d}t}{\log^{n+1} t} \leq \frac{\sqrt{x}}{\log^{n+1} 2} + \frac{2^{n+1} x}{\log^{n+1} x}.$$

Combining this with (4.13), (4.5), and the definition of d_0, we obtain the inequality (4.10). ∎

Now we give the proof of Theorem 2 in which Proposition 5 plays an important role.

Proof of Theorem 2. Let $n = 4$. It is easy to see that we can choose $A_0(4) = 132\,718\,993$. Further, we set $A_1(4) = e^{6719}$. Then we have

$$\frac{c_0 x}{(\log x)^{3/4} e^{\sqrt{\log x/R}}} \leq \frac{n! x}{(\log x)^{n+1}} \tag{4.14}$$

for every $x \geq A_1(4)$. Applying (4.14) to (4.9), we see that the inequality (1.10) holds for every $x \geq e^{6719}$.

Next we verify that the inequality (1.10) is fulfilled for every x with $10\,384\,261 \leq x \leq 2.7358 \cdot 10^{40}$. We denote the right-hand side of the inequality (1.10) by $U(x)$. For $y > 0$, let $R(y) = U(y) \log y / y$ and $S(y) = (y^4 - y^3 - y^2 - 3y)/y^3$. We have $S(t) > 0$ for every $t > 2.14$ and $y^5 R(y) S(y) = y^6 - T(y)$, where $T(y) = 11y^2 + 12y + 18$. Then, by Theorem 1,

$$\pi(x) > \frac{x}{S(\log x)} > \frac{x}{S(\log x)} \left(1 - \frac{T(\log x)}{\log^6 x}\right) = U(x), \tag{4.15}$$

which completes the proof for every x satisfying $65\,405\,887 \leq x \leq 2.7358 \cdot 10^{40}$. Finally, we use a computer to check that $\pi(p_n) > U(p_{n+1})$ for every integer n such that $\pi(10\,384\,261) \leq n \leq \pi(65\,405\,887)$, where p_n denotes the nth prime number. □

A consequence of Proposition 2 and (4.15) is the following result concerning Theorem 2.

Proposition 6. *Under the assumption that the Riemann hypothesis is true, the inequality* (1.10) *holds for every* $x \geq 10\,384\,261$.

Proof. We assume that the Riemann hypothesis is true. By (4.15) and Proposition 2, we get the inequality (1.10) for every $x \geq 65\,405\,887$. Then it suffices to apply Theorem 2. □

5. Proof of Theorem 3

In the following proof of Theorem 3, we use a recent result of Büthe [5, Theorem 2] and the explicit estimate (1.7) for the prime counting function $\pi(x)$.

Proof of Theorem 3. First, we show that the inequality (1.11) holds for every x with $1.62 \cdot 10^{12} \leq x \leq 10^{19}$. By Büthe [5, Theorem 2], we have

$$\pi(t) < \mathrm{li}(t) \tag{5.1}$$

for every t such that $2 \leq t \leq 10^{19}$. Further, we use [5, Theorem 2] to get $\pi(t) > \mathrm{li}(t) - 2.1204\sqrt{t}/\log t$ for every t such that $5.94 \cdot 10^{11} \leq t \leq 10^{19}$. Combining this with (5.1), we obtain

$$\pi\left(\frac{x}{e}\right) - \frac{\pi(x)^2 \log x}{ex} > \mathrm{Ram}(x),$$

where

$$\mathrm{Ram}(x) = \mathrm{li}\left(\frac{x}{e}\right) - \frac{2.1204\sqrt{x/e}}{\log(x/e)} - \mathrm{li}(x^2)\frac{\log x}{ex}.$$

We prove that $\mathrm{Ram}(x)$ is positive. In order to do this, we first show that the derivative of $\mathrm{Ram}(t)$ is positive for every t satisfying $1.06 \cdot 10^{12} \leq t \leq 10^{19}$. We can show by a straightforward calculation that

$$\mathrm{Ram}'(t) = \frac{(\mathrm{li}(t)\log(t/e) - t)^2}{et^2 \log(t/e)} - \frac{1.0602(\log t - 3)}{e\log^2(t/e)\sqrt{t/e}}. \tag{5.2}$$

From (5.1) and the lower bound for the prime counting function given in (1.7), it follows that $\mathrm{li}(t)\log(t/e) - t > t/(\log t \log(t/e))$ for every t such that $468\,049 \leq t \leq 10^{19}$. We combine this with (5.2) to see that

$$\mathrm{Ram}'(t) > \frac{1}{e\log^2 t \log^3(t/e)} - \frac{1.0602(\log t - 3)}{e\log^2(t/e)\sqrt{t/e}}$$

for every t such that $468\,049 \leq t \leq 10^{19}$. Since $\sqrt{y} \geq 1.0602\sqrt{e}\log^4 y$ for every $y \geq 1.06 \cdot 10^{12}$, we conclude that the derivative of $\mathrm{Ram}(t)$ is positive for every t with $1.06 \cdot 10^{12} \leq t \leq 10^{19}$. A direct computation shows that $\mathrm{Ram}(1.62 \cdot 10^{12}) > 85.86$. So $\mathrm{Ram}(x)$ is positive which implies that Ramanujan's prime counting inequality (1.11) holds unconditionally for every x with $1.62 \cdot 10^{12} \leq x \leq 10^{19}$.

It remains to show that Inequality (1.11) holds for every x with $38\,358\,837\,683 \leq x \leq 1.62 \cdot 10^{12}$. Dudek and Platt [6, Theorem 1.3] showed by computation that $x = 38\,358\,837\,682$ is the largest integer counterexample below 10^{11} and that there are no more failures at integer values before $1.15 \cdot 10^{16}$. Since $t/\log t$ is a strictly increasing function on the interval (e, ∞), we see that the inequality (1.11) holds for every x such that $38\,358\,837\,683 \leq x \leq 1.62 \cdot 10^{12}$. This completes the proof. $\square$

6. Proof of Theorem 4

Now we use Proposition 5 to prove our second main result concerning Ramanujan's prime counting inequality, which is stated in Theorem 4.

Proof of Theorem 4. As in Section 4, let n be a positive integer, $R = 5.573412$, and $c_0 = 3\sqrt{2}/\sqrt{\pi\sqrt{R}}$. Further let λ be a positive real number. Since there is a positive integer $A_1(n, \lambda)$ so that

$$e^{\sqrt{\log x/R}} \geq \left(\frac{\log x}{\lambda}\right)^{n+1/4}$$

for every $x \geq A_1(n, \lambda)$, Proposition 5 implies that

$$\pi(x) > \sum_{k=1}^{n} \frac{(k-1)!x}{\log^k x} + \frac{(n! - c_0\lambda^{n+1/4})x}{\log^{n+1} x} \tag{6.1}$$

for every $x \geq \max\{27, A_0(n), A_1(n, \lambda)\}$ and

$$\pi(x) < \sum_{k=1}^{n} \frac{(k-1)!x}{\log^k x} + \frac{x}{\log^{n+1} x}\left(\frac{n! \log^{n+1} x}{\sqrt{x}\log^{n+1} 2} + n!2^{n+1} + c_0\lambda^{n+1/4}\right) \tag{6.2}$$

for every $x \geq \max\{4, A_1(n, \lambda)\}$.

Now let $n = 6$ and $x_0 = e^{9031}$. It is easy to show that $A_0(6) = 1\,657\,493\,059\,174$ is a suitable choice for $A_0(6)$. Further, we set $\lambda = 14.4086$. Then the function $t - R(6 + 1/4)^2 \log^2(t/\lambda)$ is positive for every $t \geq 9031$, and we can choose $A_1(6, 14.4086) = x_0$. Using (6.1) and (6.2), we get

$$\sum_{k=1}^{6} \frac{(k-1)!x}{\log^k x} - \frac{27\,158\,494x}{\log^7 x} < \pi(x) < \sum_{k=1}^{6} \frac{(k-1)!x}{\log^k x} + \frac{27\,251\,374x}{\log^7 x}$$

for every $x \geq x_0$. Applying these inequalities, we conclude that the inequality

$$\frac{ex}{\log x}\pi\left(\frac{x}{e}\right) - \pi(x)^2 > \frac{x^2 f(\log x)}{\log^{14} x(\log x - 1)^7} \tag{6.3}$$

holds for every $x \geq ex_0$, where the polynomial $f(y)$ is given by

$$\begin{aligned} f(y) = y^{15} &+ 7y^{14} - 81\,660\,454y^{13} + 327\,013\,544y^{12} - 872\,039\,437y^{11} + 1\,199\,056\,017y^{10} \\ &- 1\,308\,062\,388y^9 - 1\,199\,031\,244y^8 - 742\,610\,678\,698\,880y^7 + 5\,198\,360\,646\,460\,072y^6 \\ &- 15\,595\,195\,794\,997\,976y^5 + 25\,992\,104\,849\,073\,228y^4 - 25\,992\,179\,953\,690\,916y^3 \\ &+ 15\,595\,340\,608\,417\,428y^2 - 5\,198\,455\,153\,885\,372y + 742\,637\,384\,887\,876. \end{aligned}$$

Now it is easy to verify that $f(y) > 0$ for every $y \geq 9032$. Applying this to (6.3), we see that Ramanujan's prime counting inequality (1.11) holds unconditionally for every $x \geq ex_0 = e^{9032}$, as desired. $\square$

Remark. Recently, Platt and Trudgian announced that they have fixed the error in the proof of (1.13) and even managed to improve the result in Theorem 4 by showing

$$H_0 \leq e^{8801.037}.$$

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