SIMULTANEOUS APPROXIMATIONS TO $\zeta(2)$ AND $\zeta(4)$

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Abstract

We exhibit rational approximations to $\zeta(4)$ having the same denominators as in Apéry's rational approximations to $\zeta(2)$. Therefore we provide simultaneous approximations to $\zeta(2)$ and $\zeta(4)$.

1. Main Binomial–harmonic Identities

Identities between linear forms in zeta values constructed from very well–poised hypergeometric series, on one hand, and that occurring in hypergeometric series with fewer parameters, on the other hand, have already been observed (see [7, Section 16]). Related identities, but in different constructions of such linear forms, are noticed in [13], [16], [17], [12] and [6] .

In [7, p.30], the following special case of [7, Proposition 1, p.28] is obtained:

$$-\sum_{j=0}^{n}\frac{\mathrm{d}}{\mathrm{d}j}\Big(\frac{n}{2}-j\Big)\binom{n}{j}^5=(-1)^n\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n},\quad n=0,1,2,\ldots,\tag{1}$$

and it is remarked that formula (1) is equivalent to [8, formula (5)]. The right-hand side of (1) is the coefficient of $\zeta(2)$ in Beukers' expression of Apéry's linear form in 1 and $\zeta(2)$, where ζ is the Riemann zeta function.

With motivations detailed in Section 2, and using methods sketched in Section 3, we prove in Section 4 the following

Theorem 1. *We have*

$$\begin{aligned}&-\sum_{j=0}^{n}\frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}j^3}\Big(\frac{n}{2}-j\Big)\binom{n}{j}^5\\&\qquad=(-1)^n\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\big(H_j^{(2)}+H_j(3H_j-2H_{n-j}-H_{n+j})\big).\end{aligned}\tag{2}$$

In Section 5 we speculate about possible generalizations of (1) and (2), that perhaps deserve further investigations.

Here the symbol $\frac{\mathrm{d}^k}{\mathrm{d}j^k}$ has the following meaning:

$$\frac{\mathrm{d}^k}{\mathrm{d}j^k}\Big(\frac{n}{2}-j\Big)\binom{n}{j}^5 = \frac{\mathrm{d}^k}{\mathrm{d}\varepsilon^k}\Big(\frac{n}{2}-j-\varepsilon\Big)\frac{n!^5}{(1+\varepsilon)_j^5(1-\varepsilon)_{n-j}^5}\bigg|_{\varepsilon=0}, \quad k=1,2,3,\ldots,$$

where

$$(\xi)_n = \frac{\Gamma(\xi+n)}{\Gamma(\xi)} = \xi(\xi+1)\cdots(\xi+n-1) \quad \text{if } n>0, \qquad (\xi)_0 = 1,$$

and Γ is Euler's Gamma function. Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}j}\Big(\frac{n}{2}-j\Big)\binom{n}{j}^5 &= \Big(\frac{n}{2}-j\Big)\binom{n}{j}^5\Big(5H_{n-j}-5H_j-\frac{1}{\frac{n}{2}-j}\Big)\\ &= \frac{n}{2}\binom{n}{j}^5\Big(5H_{n-j}-5H_j\Big)+\binom{n}{j}^5\Big(5jH_j-5jH_{n-j}-1\Big),\end{aligned}$$

with

$$H_j^{(k)} = 1+\cdots+\frac{1}{j^k}, \quad H_0^{(k)}=0 \qquad (k=1,2,\ldots),$$

so that (1) can be restated as in [8]:

$$\sum_{j=0}^{n}\Big(1-5jH_j+5jH_{n-j}\Big)\binom{n}{j}^5 = (-1)^n\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}.$$

The higher derivatives are recursively obtained as follows:

$$\frac{\mathrm{d}}{\mathrm{d}j}\Big(H_{n-j}-H_j\Big) = H_{n-j}^{(2)}+H_j^{(2)}, \qquad \frac{\mathrm{d}}{\mathrm{d}j}\Big(H_{n-j}^{(2)}+H_j^{(2)}\Big) = 2\Big(H_{n-j}^{(3)}-H_j^{(3)}\Big).$$

Therefore (2) becomes the following explicit binomial–harmonic identity:

$$\begin{aligned}&-\frac{1}{3!}\sum_{j=0}^{n}\binom{n}{j}^5\bigg(125\Big(\frac{n}{2}-j\Big)\Big(H_{n-j}-H_j\Big)^3+10\Big(\frac{n}{2}-j\Big)\Big(H_{n-j}^{(3)}-H_j^{(3)}\Big)\\ &+75\Big(\frac{n}{2}-j\Big)\Big(H_{n-j}-H_j\Big)\Big(H_{n-j}^{(2)}+H_j^{(2)}\Big)-75\Big(H_{n-j}-H_j\Big)^2-15\Big(H_{n-j}^{(2)}+H_j^{(2)}\Big)\bigg)\\ &\qquad = (-1)^n\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\bigg(H_j^{(2)}+H_j\Big(3H_j-2H_{n-j}-H_{n+j}\Big)\bigg). \qquad (3)\end{aligned}$$

Furthermore, in our proof of (2)–(3) we make use, inter alia, of

$$\frac{\mathrm{d}^2}{\mathrm{d}j^2}\Big(\frac{n}{2}-j\Big)\binom{n}{j}^5 = 0,$$

i.e.,

$$\sum_{j=0}^{n}\binom{n}{j}^{5}\Bigg(25\Big(\frac{n}{2}-j\Big)\Big(H_{n-j}-H_j\Big)^2+5\Big(\frac{n}{2}-j\Big)\Big(H^{(2)}_{n-j}+H^{(2)}_j\Big) - 10\Big(H_{n-j}-H_j\Big)\Bigg)=0. \quad (4)$$

All the results in the present paper are obtained by means of classical hypergeometric transformations, though a computer proof of (3) should be also possible, e.g. through the Wilf–Zeilberger method (see [1, Section 3.11]), along the same lines as in the proof of (1) in [8].

2. Motivations: Linear Forms in 1, $\zeta(2)$, and $\zeta(4)$

The quantities in (1) and (2) naturally appear as coefficients in suitable linear forms in zeta values. In order to show this point, let us recall some well known facts. Our exposition closely follows [7].

We first consider Beukers' series yielding Apéry's rational approximations to $\zeta(2)$. By partial fraction decomposition of the summand, we have

$$\Lambda_n := \sum_{k=1}^{\infty}\frac{n!(k-n)_n}{(k)_{n+1}^2} = A_n\zeta(2)-B_n,$$

where A_n and B_n are the right–hand side of (1) and (2), respectively (see e.g. [7, Section 2]).

A special case of [7, formula (2.16)] yields

$$\Theta_n := \sum_{k=1}^{\infty}\Big(k+\frac{n}{2}\Big)\frac{n!^5}{(k)_{n+1}^5}(-1)^k = \alpha_n\widetilde{\zeta}(4)+\beta_n\widetilde{\zeta}(2)+\gamma_n, \quad (5)$$

where α_n, β_n and γ_n are rational numbers. Here

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s} \quad \text{and} \quad \widetilde{\zeta}(s)=\sum_{k=1}^{\infty}\frac{(-1)^k}{k^s}=(2^{1-s}-1)\zeta(s).$$

Also,

$$\alpha_n=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\sum_{j=0}^{n}\Big(-\frac{n}{2}+j+\varepsilon\Big)\frac{n!^5}{(1+\varepsilon)_j^5(1-\varepsilon)_{n-j}^5}\Bigg|_{\varepsilon=0}.$$

In particular, α_n is equal to the left–hand side of (1). Similarly,

$$3!\beta_n=\frac{\mathrm{d}^3}{\mathrm{d}\varepsilon^3}\sum_{j=0}^{n}\Big(-\frac{n}{2}+j+\varepsilon\Big)\frac{n!^5}{(1+\varepsilon)_j^5(1-\varepsilon)_{n-j}^5}\Big|_{\varepsilon=0}.$$

Therefore β_n is equal to the left–hand side of (2). Moreover, the identity (4) is equivalent to

$$\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\sum_{j=0}^{n}\Big(-\frac{n}{2}+j+\varepsilon\Big)\frac{n!^5}{(1+\varepsilon)_j^5(1-\varepsilon)_{n-j}^5}\Big|_{\varepsilon=0}=0, \tag{6}$$

which holds because the coefficient of $\widetilde{\zeta}(3)$ in Θ_n vanishes (see [7, Sections 2.2 and 2.4]).

Using $A_n = \alpha_n$ and assuming $B_n = \beta_n$, after recalling that $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$, we get

$$\frac{\pi^2}{12}\Lambda_n - \Theta_n = \frac{17}{8}\alpha_n\zeta(4) - \gamma_n.$$

Rough calculations show that

$$\frac{1}{3n^4} \le |\Theta_n| \le \frac{2}{n^4}.$$

This easily implies that

$$\lim_{n\to\infty}|\Theta_n|^{1/n} = 1.$$

Moreover (see [9, formula (10)])

$$\lim_{n\to\infty}|\alpha_n|^{1/n} = \lim_{n\to\infty}|A_n|^{1/n} = \left(\frac{1+\sqrt{5}}{2}\right)^5$$

Since $\Theta_n \to 0$ and $\Lambda_n \to 0$ $(n \to \infty)$, we infer that

$$\frac{\beta_n}{\alpha_n} \to \zeta(2) \quad \text{and} \quad \frac{8}{17}\frac{\gamma_n}{\alpha_n} \to \zeta(4) \qquad (n\to\infty).$$

This justifies the title of the present paper.

Unfortunately, the above approximations to $\zeta(4)$ are far from being good enough to yield, together with Apéry's approximations to $\zeta(2)$, a linear independence measure of 1, $\zeta(2)$ and $\zeta(4)$. The conjectural generalizations we mention in Section 5 are not good enough to serve for Diophantine applications, either. However, they apparently suggest the possibility that some other linear forms in $1, \zeta(2), \zeta(4)$ that share with Θ_n an hypergeometric origin (see [11], [4], [3], [2]) have a (for now hidden) companion linear form in $1, \zeta(2)$ which plays the role of Λ_n. This is one motivation for this paper. The second one is the quest for methods towards a proof of the general denominator conjecture in [14].

3. Strategy of the Proof: Application of the Whipple Transformation

Before proving that $B_n = \beta_n$ we recall the proof that $A_n = \alpha_n$, given in [7]. One first writes

$$\alpha_n = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \sum_{j=0}^{n} \Big(-\frac{n}{2} + j + \varepsilon \Big) \frac{n!^5}{(1+\varepsilon)_j^5 (1-\varepsilon)_{n-j}^5} \Big|_{\varepsilon=0}.$$

Since

$$\sum_{j=0}^{n} \Big(-\frac{n}{2} + j + \varepsilon \Big) \frac{n!^5}{(1+\varepsilon)_j^5 (1-\varepsilon)_{n-j}^5} \Big|_{\varepsilon=0} = 0, \tag{7}$$

we have

$$\alpha_n = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \sum_{j=0}^{n} (-1)^j \Big(-\frac{n}{2} + j + \varepsilon \Big) \frac{(-n+\varepsilon)_j^5}{(1+\varepsilon)_j^5} \Big|_{\varepsilon=0},$$

because $(-1)^j(-n+\varepsilon)_j(1-\varepsilon)_{n-j} = (1-\varepsilon)_n$ is independent of j. Moreover

$$\alpha_n = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \Big(-\frac{n}{2} + \varepsilon \Big) \sum_{j=0}^{n} (-1)^j \frac{(-\frac{n}{2}+\varepsilon+1)_j}{(-\frac{n}{2}+\varepsilon)_j} \frac{(-n)_j}{j!} \frac{(-n+2\varepsilon)_j}{(1+2\varepsilon)_j} \frac{(-n+\varepsilon)_j^3}{(1+\varepsilon)_j^3} \Big|_{\varepsilon=0},$$

or, in hypergeometric notation,

$$\alpha_n = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \Bigg[\Big(-\frac{n}{2} + \varepsilon \Big) \times {}_6F_5 \left[\begin{matrix} -n+2\varepsilon, & -\frac{n}{2}+\varepsilon+1, & -n+\varepsilon, & -n+\varepsilon, & -n+\varepsilon, & -n \\ & -\frac{n}{2}+\varepsilon, & 1+\varepsilon, & 1+\varepsilon, & 1+\varepsilon, & 1+2\varepsilon \end{matrix} ; -1 \right] \Bigg]_{\varepsilon=0}.$$

We now apply Whipple's transformation between a very well–poised ${}_6F_5$ with argument -1 and a ${}_3F_2$ with argument 1 (see [5, formula 4.4(2)]):

$$\begin{aligned} &{}_6F_5 \left[\begin{matrix} a, & \frac{a}{2}+1, & b, & c, & d, & e \\ & \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix} ; -1 \right] \\ &\quad = \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)} \, {}_3F_2 \left[\begin{matrix} 1+a-b-c, & d, & e \\ & 1+a-b, & 1+a-c \end{matrix} ; 1 \right]. \end{aligned} \tag{8}$$

We obtain, after a few simplifications,

$$\begin{aligned} &\frac{1}{\varepsilon} \sum_{j=0}^{n} (-1)^j \Big(-\frac{n}{2} + j + \varepsilon \Big) \frac{(-n)_j}{j!} \frac{(-n+2\varepsilon)_j}{(1+2\varepsilon)_j} \frac{(-n+\varepsilon)_j^3}{(1+\varepsilon)_j^3} \\ &\qquad = \frac{(-n+2\varepsilon)_n}{(1+\varepsilon)_n} \sum_{j=0}^{n} \frac{(n+1)_j}{j!} \frac{(-n+\varepsilon)_j}{(1+\varepsilon)_j} \frac{(-n)_j}{(1+\varepsilon)_j}. \end{aligned}$$

On making $\varepsilon \to 0$ we get $\alpha_n = A_n$.

In view of $\beta_n = B_n$, our first task is to write β_n, i.e., the left–hand side of (3), as the third derivative of a suitable ${}_6F_5$ sum. After applying (8), β_n is written as the second derivative of a ${}_3F_2$ sum. Then we apply two transformations between ${}_3F_2$ sums in order to finalize our proof.

4. Proof of Theorem 1

4.1. Second Application of (8)

Let $x_1, x_2, y_1, y_2, z_1, z_2$ be six complex parameters to be chosen later. We consider the following functions of ε:

$$f_{n,j}(\varepsilon) = \Big(-\frac{n}{2} + j + \varepsilon\Big)\binom{n}{j}\frac{n!}{(1+2\varepsilon)_j(1-2\varepsilon)_{n-j}}\frac{n!}{(1+x_1\varepsilon)_j(1-x_2\varepsilon)_{n-j}}$$
$$\times\frac{n!}{(1+y_1\varepsilon)_j(1-y_2\varepsilon)_{n-j}}\frac{n!}{(1+z_1\varepsilon)_j(1-z_2\varepsilon)_{n-j}}.$$

Easy calculations show that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big(f_{n,j}(\varepsilon)\big)_{\varepsilon=0} = \binom{n}{j}^5$$
$$+\Big(-\frac{n}{2}+j\Big)\binom{n}{j}^5\Big(\big(2+x_2+y_2+z_2\big)H_{n-j} - \big(2+x_1+y_1+z_1\big)H_j\Big),$$

and similar but more complicated expressions, involving squares and cubes of our parameters, can be written down for the second and the third derivative of $f_{n,j}(\varepsilon)$ at $\varepsilon = 0$, where, with respect to similar formulas in Section 1, each element of the type $5^k(H_{n-j}^{(l)} + (-1)^l H_j^{(l)})^k$ is replaced by $\big((2^l+x_2^l+y_2^l+z_2^l)H_{n-j}^{(l)} + (-1)^l(2^l+x_1^l+y_1^l+z_1^l)H_j^{(l)}\big)^k$. With the choice $x_1 = x_2 = 1$, $y_1 = y_2 = 1+i$ and $z_1 = z_2 = 1-i$, where $i = \sqrt{-1}$, we have $2^l + x_1^l + y_1^l + z_1^l = 2^l + x_2^l + y_2^l + z_2^l = 5$ $(l = 1, 2, 3)$, whence

$$3!\beta_n = \sum_{j=0}^{n}\frac{\mathrm{d}^3}{\mathrm{d}\varepsilon^3}\big(f_{n,j}(\varepsilon)\big)_{\varepsilon=0},$$

or, more explicitly,

$$\beta_n = \frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}\varepsilon^3}\Bigg[\sum_{j=0}^{n}\Big(-\frac{n}{2}+j+\varepsilon\Big)\binom{n}{j}$$
$$\times\frac{n!}{(1+2\varepsilon)_j(1-2\varepsilon)_{n-j}}\frac{n!}{(1+(1+i)\varepsilon)_j(1-(1-i)\varepsilon)_{n-j}}$$
$$\times\frac{n!}{(1+(1-i)\varepsilon)_j(1-(1+i)\varepsilon)_{n-j}}\frac{n!}{(1+\varepsilon)_j(1-\varepsilon)_{n-j}}\Bigg]_{\varepsilon=0}.$$

This can be rewritten in the form

$$\beta_n = \frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}\varepsilon^3}\Bigg[\frac{\left(-\frac{n}{2}+\varepsilon\right)n!^4}{(-n+2\varepsilon)_n(-n+(1+i)\varepsilon)_n(-n+(1-i)\varepsilon)_n(-n+\varepsilon)_n}$$

$$\times\, {}_6F_5\left[\begin{matrix}-n+2\varepsilon, & -\frac{n}{2}+\varepsilon+1, & -n+(1-i)\varepsilon, & -n+(1+i)\varepsilon, & -n+\varepsilon, & -n \\ & -\frac{n}{2}+\varepsilon, & 1+(1+i)\varepsilon, & 1+(1-i)\varepsilon, & 1+\varepsilon, & 1+2\varepsilon\end{matrix};-1\right]\Bigg]_{\varepsilon=0}.$$

Let us apply (8) with $a=-n+2\varepsilon$, $b=-n+(1-i)\varepsilon$, $c=-n+(1+i)\varepsilon$, $d=-n+\varepsilon$, $e=-n$. After a few simplifications we get

$$\begin{aligned}\beta_n &= \frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}\varepsilon^3}\Bigg[\varepsilon\,\frac{n!^4}{(1+\varepsilon)_n(-n+(1+i)\varepsilon)_n(-n+(1-i)\varepsilon)_n(-n+\varepsilon)_n} \\ &\qquad\times\, {}_3F_2\left[\begin{matrix}1+n, & -n+\varepsilon, & -n \\ & 1+(1+i)\varepsilon, & 1+(1-i)\varepsilon\end{matrix};1\right]\Bigg]_{\varepsilon=0} \\ &= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Bigg[\frac{n!^4}{(1+\varepsilon)_n(-n+(1+i)\varepsilon)_n(-n+(1-i)\varepsilon)_n(-n+\varepsilon)_n} \\ &\qquad\times\sum_{j=0}^{n}(-1)^j\binom{n+j}{n}\binom{n}{j}\frac{j!(-n+\varepsilon)_j}{(1+(1+i)\varepsilon)_j(1+(1-i)\varepsilon)_j}\Bigg]_{\varepsilon=0}.\end{aligned}$$

Similarly, again by (8),

$$\begin{aligned}0 &= \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Bigg[\sum_{j=0}^{n}\left(-\frac{n}{2}+j+\varepsilon\right)\binom{n}{j} \\ &\qquad\times\frac{n!}{(1+2\varepsilon)_j(1-2\varepsilon)_{n-j}}\frac{n!}{(1+(1+i)\varepsilon)_j(1-(1-i)\varepsilon)_{n-j}} \\ &\qquad\times\frac{n!}{(1+(1-i)\varepsilon)_j(1-(1+i)\varepsilon)_{n-j}}\frac{n!}{(1+\varepsilon)_j(1-\varepsilon)_{n-j}}\Bigg]_{\varepsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Bigg[\frac{n!^4}{(1+\varepsilon)_n(-n+(1+i)\varepsilon)_n(-n+(1-i)\varepsilon)_n(-n+\varepsilon)_n} \\ &\qquad\times\sum_{j=0}^{n}(-1)^j\binom{n+j}{n}\binom{n}{j}\frac{j!(-n+\varepsilon)_j}{(1+(1+i)\varepsilon)_j(1+(1-i)\varepsilon)_j}\Bigg]_{\varepsilon=0}.\end{aligned}$$

Let

$$g_{n,j}(\varepsilon) = \frac{(-1)^n\,n!^4}{(1+\varepsilon)_n(-n+(1+i)\varepsilon)_n(-n+(1-i)\varepsilon)_n(-n+\varepsilon)_n} \times\frac{(-1)^j j!(-n+\varepsilon)_j}{(1+(1+i)\varepsilon)_j(1+(1-i)\varepsilon)_j}.$$

We have
$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big(g_{n,j}(\varepsilon)\big)_{\varepsilon=0} = \binom{n}{j}\big(H_n + H_{n-j} - 2H_j\big).$$
On multiplying by $\binom{n}{j}\binom{n+j}{n}$ and summing up on j,
$$\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\big(H_n + H_{n-j} - 2H_j\big) = 0. \tag{9}$$

We can summarize this by saying that (9) is obtained by (4)–(6), through (8). Furthermore, a straightforward computation of $\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\big(g_{n,j}(\varepsilon)\big)_{\varepsilon=0}$ yields, upon summation,

$$(-1)^n\beta_n = \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\Big(4H_n^2 + 2H_n^{(2)} + \Big(H_{n-j} - H_n - 2H_j\Big)^2 \\ + H_{n-j}^{(2)} - H_n^{(2)} + 4H_n\Big(H_{n-j} - H_n - 2H_j\Big)\Big).$$

On multiplying (9) by H_n and using that in the last sum, we arrive at
$$(-1)^n\beta_n = \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\Big(H_n^{(2)} + H_{n-j}^{(2)} + \Big(H_{n-j} - 2H_j\Big)^2 - H_n^2\Big). \tag{10}$$

4.2. Two More Applications of (8)

We now proceed in the spirit of the proof of [7, Lemmas 16–17, pp.67–68, 72]. Let x, y, z be three real numbers to be chosen later, and let us consider the following function of two variables ε and ω:
$$h_n(\varepsilon,\omega) := \frac{(1+z\varepsilon)_n}{n!}\,{}_3F_2\left[\begin{matrix}1+n+x\varepsilon, & -n+y\varepsilon, & -n \\ & 1+(x+y+1)\varepsilon+\omega, & 1+z\varepsilon\end{matrix};1\right].$$
We have
$$\frac{\partial}{\partial\omega}\big(h_n(\varepsilon,\omega)\big)_{(\varepsilon,\omega)=(0,0)} = -\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}H_j \tag{11}$$
and
$$\frac{\partial^2}{\partial\varepsilon\partial\omega}\big(h_n(\varepsilon,\omega)\big)_{(\varepsilon,\omega)=(0,0)} = \sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\Big((x+y+1)H_j^{(2)} \\ - H_j\Big(x\big(H_{n+j} - H_n\big) + y\big(H_{n-j} - H_n\big) + zH_n - (x+y+z+1)H_j\Big)\Big). \tag{12}$$

An application of Kummer's transformation to $h_n(\varepsilon,\omega)$, or equivalently, two applications of (8), firstly with $a = -n + (y+z+1)\varepsilon + \omega$, $b = -n + (z-x)\varepsilon$,

$c = -n + (y+1)\varepsilon + \omega$, $d = -n + y\varepsilon$, $e = -n$, secondly with c and d interchanged, yield

$$h_n(\varepsilon,\omega) = \frac{\big(1+(z+1)\varepsilon+\omega\big)_n}{n!} \times {}_3F_2\left[\begin{matrix}1+n+(x+1)\varepsilon+\omega, & -n+(y+1)\varepsilon+\omega, & -n \\ & 1+(x+y+1)\varepsilon+\omega, & 1+(z+1)\varepsilon+\omega\end{matrix};1\right].$$

Therefore

$$\frac{\partial}{\partial\omega}\big(h_n(\varepsilon,\omega)\big)_{(\varepsilon,\omega)=(0,0)} = \sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\Big(H_{n+j}+H_{n-j}-H_n-2H_j\Big) \tag{13}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial\varepsilon\partial\omega}\big(h_n(\varepsilon,\omega)\big)_{(\varepsilon,\omega)=(0,0)} &= \sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\Bigg((x+y+z+2)H_j^{(2)} \\ &- (x+1)\big(H_{n+j}^{(2)}-H_n^{(2)}\big)+(y+1)\big(H_{n-j}^{(2)}-H_n^{(2)}\big)-(z+1)H_n^{(2)} \\ &+ \Big(H_{n+j}+H_{n-j}-H_n-2H_j\Big)\times \\ &\times\Big((x+1)\big(H_{n+j}-H_n\big)+(y+1)\big(H_{n-j}-H_n\big)+(z+1)H_n-(x+y+z+2)H_j\Big)\Bigg).\end{aligned} \tag{14}$$

Comparison of (11) and (13) gives

$$\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\Big(H_{n+j}+H_{n-j}-H_n-H_j\Big)=0. \tag{15}$$

On multiplying (15) by H_n and using it in (14), comparing with (12) and after a few simplifications we get

$$\begin{aligned}\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\Bigg(&\Big(H_{n+j}+H_{n-j}-H_n-H_j\Big)\times \\ &\times\Big((x+1)\big(H_{n+j}-H_j\big)+(y+1)\big(H_{n-j}-H_j\big)-(z+1)H_j\Big) \\ &-(x+1)\big(H_{n+j}^{(2)}-H_n^{(2)}\big)+(y+1)\big(H_{n-j}^{(2)}-H_n^{(2)}\big)+(z+1)\big(H_j^{(2)}-H_n^{(2)}\big)\Bigg)=0.\end{aligned}$$

We choose $x=-1$, $y=0$ and $z=-3$, and obtain

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\Big(H_{n-j}^{(2)}+H_n^{(2)}\Big) \\ &= \sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\Big(2H_j^{(2)}-\Big(H_{n+j}+H_{n-j}-H_n-H_j\Big)\Big(H_{n-j}+H_j\Big)\Big).\end{aligned}$$

Substituting this into (10) yields

$$(-1)^n\beta_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n} \Big(2H_j^{(2)} + \Big(H_{n-j} - 2H_j\Big)^2 - H_n^2 \\ - \Big(H_{n+j} + H_{n-j} - H_n - H_j\Big)\Big(H_{n-j} + H_j\Big)\Big).$$

As a result,

$$(-1)^n\big(\beta_n - B_n\big) \\ = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n} \Big(H_n H_{n-j} + H_n H_j + H_j H_{n+j} - H_n^2 - H_j^2 - H_{n+j}H_{n-j}\Big).$$

4.3. End of the Proof

In order to prove that $\beta_n - B_n$ vanishes for any n, let us consider the function

$$k_n(\varepsilon,\omega) = \frac{(1+\varepsilon)_n}{n!}\, {}_3F_2\left[\begin{matrix} -n, & n+1+\varepsilon, & -n-\omega \\ & 1-\omega & 1+\varepsilon \end{matrix}; 1\right].$$

Then

$$\frac{\partial}{\partial\omega}\big(k_n(\varepsilon,\omega)\big)_{(\varepsilon,\omega)=(0,0)} = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n} \Big(H_n - H_{n-j} + H_j\Big)$$

and

$$\frac{\partial^2}{\partial\varepsilon\partial\omega}\big(k_n(\varepsilon,\omega)\big)_{(\varepsilon,\omega)=(0,0)} = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n} \Big(H_{n+j} - H_j\Big)\Big(H_j + H_n - H_{n-j}\Big).$$

We apply to $k_n(\varepsilon,\omega)$ the following formula (see [1, p.142] and [5, Section 3.9]):

$${}_3F_2\left[\begin{matrix} -n, & x, & y \\ & z, & t \end{matrix}; 1\right] = \frac{(t-x)_n}{(t)_n}\, {}_3F_2\left[\begin{matrix} -n, & x, & z-y \\ & z, & x+1-n-t \end{matrix}; 1\right], \tag{16}$$

and get

$$k_n(\varepsilon,\omega) = (-1)^n\, {}_3F_2\left[\begin{matrix} -n, & n+1, & n+1+\varepsilon \\ & 1, & 1-\omega \end{matrix}; 1\right].$$

On applying (16) again,

$$k_n(\varepsilon,\omega) = \frac{(1+\omega)_n}{(1-\omega)_n}\, {}_3F_2\left[\begin{matrix} -n, & n+1, & -n-\varepsilon \\ & 1, & 1+\omega \end{matrix}; 1\right].$$

Thereby

$$\frac{\partial}{\partial\omega}\big(k_n(\varepsilon,\omega)\big)_{(\varepsilon,\omega)=(0,0)} = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n} \Big(2H_n - H_j\Big)$$

and

$$\frac{\partial^2}{\partial\varepsilon\partial\omega}\big(k_n(\varepsilon,\omega)\big)_{(\varepsilon,\omega)=(0,0)} = \sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\Big(2H_n - H_j\Big)\Big(H_n - H_{n-j}\Big).$$

Comparing the two expressions of the first- (respectively, the second-) order (mixed) partial derivatives of $k_n(\varepsilon,\omega)$, we obtain another proof of (9), and establish that $\beta_n - B_n = 0$, as required.

5. Open Questions

5.1. First Generalization

Let $p, q, r \geq 1$ be intergers. We consider the following generalization of (5):

$$\begin{aligned}&\sum_{t\geq 1+\max\{0,(p-1)n,(q-1)n,(r-1)n\}} (-1)^t\Big(t+\frac{n}{2}\Big)\frac{n!^2}{(t)_{n+1}^2}\\ &\times\frac{\big((r+p-1)n\big)!}{\big(t-(p-1)n\big)_{(r+p-1)n+1}}\frac{\big((p+q-1)n\big)!}{\big(t-(q-1)n\big)_{(p+q-1)n+1}}\frac{\big((q+r-1)n\big)!}{\big(t-(r-1)n\big)_{(q+r-1)n+1}}\\ &=\lambda_n\widetilde{\zeta}(4)+\mu_n\widetilde{\zeta}(2)+\omega_n,\end{aligned}$$

for some rational numbers λ_n, μ_n and ω_n. More precisely,

$$\begin{aligned}\lambda_n = (-1)^{(p+q+r+1)n}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Bigg[&\sum_{j=0}^{n}\Big(-\frac{n}{2}+j+\varepsilon\Big)\frac{n!^2}{(1+\varepsilon)_j^2(1-\varepsilon)_{n-j}^2}\\ &\times\frac{\big((r+p-1)n\big)!}{\big(1+\varepsilon\big)_{(p-1)n+j}\big(1-\varepsilon\big)_{rn-j}}\frac{\big((p+q-1)n\big)!}{\big(1+\varepsilon\big)_{(q-1)n+j}\big(1-\varepsilon\big)_{pn-j}}\\ &\times\frac{\big((q+r-1)n\big)!}{\big(1+\varepsilon\big)_{(r-1)n+j}\big(1-\varepsilon\big)_{qn-j}}\Bigg]_{\varepsilon=0}\\ = (-1)^n&\frac{\big((r+p-1)n\big)!\big((p+q-1)n\big)!\big((q+r-1)n\big)!}{(pn)!(qn)!(rn)!\big((p-1)n\big)!\big((q-1)n\big)!\big((r-1)n\big)!}\\ &\times\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Bigg[\Big(-\frac{n}{2}+\varepsilon\Big)\\ \times{}_6F_5&\left[\begin{matrix}-n+2\varepsilon, & -\frac{n}{2}+\varepsilon+1, & -pn+\varepsilon, & -qn+\varepsilon, & -rn+\varepsilon, & -n\\ & -\frac{n}{2}+\varepsilon, & 1+(p-1)n+\varepsilon, & 1+(q-1)n+\varepsilon, & 1+(r-1)n+\varepsilon, & 1+2\varepsilon\end{matrix};-1\right]\Bigg]_{\varepsilon=0}.\end{aligned}$$

One may apply Whipple's transformation (8) and obtain, after a few simplifications,

$$\lambda_n = \frac{\big((r+p-1)n\big)!\big((p+q-1)n\big)!\big((q+r-1)n\big)!n!}{(pn)!(qn)!(rn)!\big((p-1)n\big)!\big((q-1)n\big)!(rn)!} \times {}_3F_2\left[\begin{matrix}1+(p+q-1)n, & -rn, & -n \\ & 1+(p-1)n, & 1+(q-1)n\end{matrix}; 1\right].$$

On the other hand,

$$\sum_{t\geq 1+(p+q-1)n} \frac{\big(t-(p+q-1)n\big)_{pn}}{(pn)!} \frac{n!}{(t)_{n+1}} \frac{\big((r+p-1)n\big)!}{\big(t-(p-1)n\big)_{(r+p-1)n+1}} = L_n\zeta(2) - M_n,$$

where L_n and M_n are rationals, and in particular

$$\begin{aligned} L_n &= (-1)^n \sum_{j=0}^{n} \binom{(p+q-1)n+j}{pn}\binom{n}{j}\binom{(r+p-1)n}{rn-j} \\ &= (-1)^n \frac{\big((p+q-1)n\big)!}{(pn)!\big((q-1)n\big)!} \frac{\big((r+p-1)n\big)!}{(rn)!\big((p-1)n\big)!} \\ &\qquad \times {}_3F_2\left[\begin{matrix}1+(p+q-1)n, & -rn, & -n \\ & 1+(p-1)n, & 1+(q-1)n\end{matrix}; 1\right]. \end{aligned}$$

Therefore

$$\lambda_n = (-1)^n \frac{n!\big((q+r-1)n\big)!}{(qn)!(rn)!} L_n,$$

and, in analogy with (2), we also expect that

$$\mu_n = (-1)^n \frac{n!\big((q+r-1)n\big)!}{(qn)!(rn)!} M_n,$$

but the argument that led us to (2) does not obviously extend to a proof of the last (conjectural) identity. Perhaps it would be required to write μ_n as a sum of several ${}_6F_5$ sums.

5.2. Further Generalizations

Let $p, q, r, s \geq 1$ be integers. A further generalization of (5) is represented by

$$\begin{aligned} &\sum_{t\geq 1+\max\{0,(p-1)n,(q-1)n,(r-1)n,(s-1)n\}} (-1)^t \Big(t+\frac{n}{2}\Big)\frac{n!}{(t)_{n+1}} \\ &\qquad \times \frac{\big((s+p-1)n\big)!}{\big(t-(p-1)n\big)_{(s+p-1)n+1}} \frac{\big((p+q-1)n\big)!}{\big(t-(q-1)n\big)_{(p+q-1)n+1}} \\ &\qquad \times \frac{\big((q+r-1)n\big)!}{\big(t-(r-1)n\big)_{(q+r-1)n+1}} \frac{\big((r+s-1)n\big)!}{\big(t-(s-1)n\big)_{(r+s-1)n+1}} \\ &\qquad\qquad = \sigma_n\widetilde{\zeta}(4) + \tau_n\widetilde{\zeta}(2) + \varphi_n, \end{aligned}$$

for some rational numbers σ_n, τ_n and φ_n. In this case the expression of σ_n is more complicated than the one of λ_n in the above subsection. Also, presumably, the associated linear form $\sigma_n\zeta(2) - \tau_n$ is essentially the same considered in [10] and [15]. However, by easy calculations

$$\lim_{n\to\infty} |\lambda_n\widetilde{\zeta}(4) + \mu_n\widetilde{\zeta}(2) + \omega_n|^{1/n} = \lim_{n\to\infty} |\sigma_n\widetilde{\zeta}(4) + \tau_n\widetilde{\zeta}(2) + \varphi_n|^{1/n} = 1.$$

Hence, in view of possible diophantine consequences of the above results and conjectures, further generalizations are required. Some of them have been mentioned in Section 2.

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