ON CERTAIN QUATERNARY QUADRATIC FORMS

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Abstract
In this paper, we determine all triplets of positive integers a, b, and c such that every nonnegative integer can be represented as

$$f_c^{a,b}(x,y,z,w) = ax^2 + by^2 + c(z^2 + zw + w^2) \text{ with } x, y, z, w \in \mathbb{Z}.$$

Furthermore, we prove that $f_c^{a,b}$ can represent all the nonnegative integers if it represents $1, 2, 3, 5, 6$, and 10.

1. Introduction

Throughout this paper, we set $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, and $\mathbb{Z}$ denotes the set of rational integers. In addition, the triangular numbers are $t_x = x(x+1)/2,\ (x \in \mathbb{N}_0)$. Furthermore, a, b and c are fixed positive integers with $1 \leq a \leq b$.

More than 200 years ago, Lagrange [4, pp. 279] proved one of the most celebrated theorems in number theory, the Four Squares Theorem, which states that every positive integer can be expressed as a sum of four squares. Since then, the universal representability by a quadratic form has been becoming one of the most interesting problems in number theory. Incredibly, Ramanujan [8] extended Lagrange's theorem to other positive definite quaternary quadratic forms $ax^2 + by^2 + cz^2 + du^2$, and determined all of the positive integers a, b, c and d such that $ax^2 + by^2 + cz^2 + du^2$ can represent positive integers universally, totally 54 cases. Unlike Jacobi's "proof" to Lagrange's theorem, which follows directly from his beautiful formula,

$$|\{(x, y, z, u) \in \mathbb{Z}^4 \,|\, n = x^2 + y^2 + z^2 + u^2\}| = 8\sigma(n) - 32\sigma(n/4) > 0,$$

where $\sigma(n) = \sum_{d|n} d$, Ramanujan's proofs rely on the representability of ternary quadratic forms and their relations to the quaternary quadratic forms under investigation. For example, for Lagrange's Four Squares Theorem, one first shows [3] that a positive integer n can be expressed as a sum of three squares if and only if $n \neq 4^k(8l+7)$ for any nonnegative integers k and l. Then if $n \neq 4^k(8l+7)$, one has $n = x^2+y^2+z^2+0^2$; if $n = 4^k(8l+7)$, then $n = 4^k(8l+6)+(2^k)^2$, where $n = 4^k(8l+6)$ is representable by a sum of three squares.

Inspired by Ramanujan's proofs, in our recent work [7], using a similar strategy, the author succeeded in extending his results to the quaternary quadratic polynomials $at_x + bt_y + c(z^2+zw+w^2)$. In this work, we aim to further extend the previous work of Ramanujan to another type of quaternary quadratic forms, namely, $ax^2+by^2+c(z^2+zw+w^2)$. We conclude this section by summarizing our main results in the following theorem:

Theorem 1. *For positive integers $a, b,$ and c with $a \leq b$, set*

$$f_c^{a,b}(x,y,z,w) = ax^2+by^2+c(z^2+zw+w^2), \ (x,y,z,w \in \mathbb{Z}).$$

(1) *Every nonnegative integer can be represented by $f_c^{a,b}$ if and only if*

$$(a,b,c) = \begin{cases} (1,b,1) & (b = 1,2,3,4,5,6), \\ (2,b,1) & (b = 2,3,4,5,6,7,8,9,10), \\ (1,b,2) & (b = 1,2,3,4,5), \\ (1,2,3), \\ (1,2,4). \end{cases}$$

(2) *If $f_c^{a,b}$ represents $1, 2, 3, 5, 6,$ and 10, then it can represent all the nonnegative integers.*

2. Notations and Preliminaries

To prove Theorem 1, we follow Ramanujan by introducing

$$\varphi(q) = \sum_{n\in\mathbb{Z}} q^{n^2}, \ \psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}, \ a(q) = \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2}, \ (q \in \mathbb{C}, \ |q|<1),$$

and we apply the following identities:

$$a(q) = a(q^4)+6q\psi(q^2)\psi(q^6), \ \varphi(q)\varphi(q^3) = a(q^4)+2q\psi(q^2)\psi(q^6). \tag{1}$$

For the proofs of these formulas, see Berndt [1, pp. 232] and Hirschhorn et al. [5].

For fixed positive integers a and c, and for each $n \in \mathbb{N}_0$, we define

$$A_c^a(n) = \sharp\left\{(x,y,z)\in\mathbb{Z}^3 \,|\, n = f_{0,c}^a(x,0,y,z)\right\}.$$

Moreover, for fixed positive integers a, b, and c, and for each $n \in \mathbb{N}_0$, we set

$$\begin{aligned} r_{a,b}(n) =& \sharp \left\{ (x,y) \in \mathbb{Z}^2 \,|\, n = ax^2 + by^2 \right\}, \\ t_{a,b}(n) =& \sharp \left\{ (x,y) \in \mathbb{N}_0^2 \,|\, n = at_x + bt_y \right\}, \\ r_{a,b,c}(n) =& \sharp \left\{ (x,y,z) \in \mathbb{Z}^3 \,|\, n = ax^2 + by^2 + cz^2 \right\}, \\ m_{a\text{-}b,c}(n) =& \sharp \left\{ (x,y,z) \in \mathbb{Z} \times \mathbb{N}_0^2 \,|\, n = ax^2 + bt_y + ct_z \right\}. \end{aligned}$$

3. The Case Where $c = 1$

Lemma 1. *Suppose that* $1 \le a \le b,\ c = 1$, *and* $f_c^{a,b}$ *can represent all the nonnegative integers. Then,* $a = 1$ *or* 2.

Proof. From the supposition, $n = 2$ can be written as

$$2 = ax^2 + by^2 + (z^2 + zw + w^2),\ (x, y, z, w \in \mathbb{Z}).$$

On the other hand, $n = 2$ cannot be written as $z^2 + zw + w^2$ with $z, w \in \mathbb{Z}$, which implies that $a = 1, 2$. □

3.1. The Case Where $a = 1$

We use the following result, which was introduced by Dickson [3, pp. 112-113]:

Lemma 2. *A nonnegative integer* n *can be written as* $x^2 + y^2 + 3z^2$ *with* $x, y, z \in \mathbb{Z}$ *if and only if* $n \neq 9^k(9l + 6),\ (k, l \in \mathbb{N}_0)$.

As a consequence of Lemma 2, we have the following proposition:

Proposition 1. *A nonnegative integer* n *can be written as* $x^2 + (y^2 + yz + z^2)$ *with* $x, y, z \in \mathbb{Z}$ *if and only if* $n \neq 9^k(9l + 6),\ (k, l \in \mathbb{N}_0)$.

Proof. By multiplying both sides of (1) by $\varphi(q)$, we have that

$$\begin{aligned} \varphi(q)a(q) =& \varphi(q)a(q^4) + 6q\varphi(q)\psi(q^2)\psi(q^6), \\ \varphi(q)^2\varphi(q^3) =& \varphi(q)a(q^4) + 2q\varphi(q)\psi(q^2)\psi(q^6), \end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} A_1^1(n)q^n = \sum_{n=0}^{\infty} A_4^1(n)q^n + 6q \sum_{n=0}^{\infty} m_{1\text{-}2,6}(n)q^n, \tag{2}$$

$$\sum_{n=0}^{\infty} r_{1,1,3}(n)q^n = \sum_{n=0}^{\infty} A_4^1(n)q^n + 2q \sum_{n=0}^{\infty} m_{1\text{-}2,6}(n)q^n. \tag{3}$$

In our present notation, Lemma 2 states that $n \neq 9^k(9l+6), (k, l \in \mathbb{N}_0)$ if and only if $r_{1,1,3}(n) > 0$. By equating coefficients in (3), we see that Lemma 2 is equivalent to $A_4^1(n) > 0$ or $m_{1\text{-}2,6}(n-1) > 0$, which in turn is equivalent to $A_1^1(n) > 0$ from (2). Therefore, as $A_1^1(n)$ is the number of ways n can be represented as $x^2+(y^2+yz+z^2)$, the proposition follows. □

By Proposition 1, we obtain the following theorem:

Theorem 2. *Any nonnegative integer n can be represented by $f_1^{1,b}$ if and only if $b = 1, 2, 3, 4, 5,$ or 6.*

Proof. We first prove the "only if" direction. Therefore, $n = 6$ can be represented by

$$6 = x^2 + by^2 + (z^2 + zw + w^2), \; x, y, z, w \in \mathbb{Z}.$$

On the other hand, Proposition 1 shows that $n = 6$ cannot be written as $x^2 + (z^2 + zw + w^2)$, which implies that $b = 1, 2, 3, 4, 5, 6$.

In order to establish the "if" direction, we need only prove that $f_1^{1,b}$ represents $n = 9l + 6$ with $l \in \mathbb{N}_0$.

When $b = 1, 2, 3, 4$ or 5, we can set $y = 1$ to obtain that

$$n - by^2 = 9l + 6 - b \cdot 1^2 \equiv 5, 4, 3, 2, 1 \bmod 9,$$

which can be written as $x^2 + (z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

Suppose that $b = 6$. Then, if $n = 6$ or 15, or $l \equiv 2$ or $8 \bmod 9$, we can set $y = 1$ to obtain that

$$n - 6 \cdot 1^2 = 9l, \; (l = 0, 1 \text{ or } l \equiv 2, 8 \bmod 9),$$

which can be represented as $x^2 + (z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

If $l \geq 2$ and $l \not\equiv 2$ or $8 \bmod 9$, then by taking $y = 2$ we have that

$$n - 6 \cdot 2^2 = 9l + 6 - 24 = 9(l - 2),$$

which can be written as $x^2 + (z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. □

3.2. The Case Where $a = 2$

We use the following result, which was proved by Dickson [2]:

Lemma 3.

(1) *A nonnegative integer n can be written as $x^2 + 2y^2 + 3z^2$ with $x, y, z \in \mathbb{Z}$ if and only if $n \neq 4^k(16l + 10)$ with $k, l \in \mathbb{N}_0$.*

(2) *A nonnegative integer n can be written as $x^2 + 2(y^2 + yz + z^2)$ with $x, y, z \in \mathbb{Z}$ if and only if $n \neq 4^k(8l + 5)$ with $k, l \in \mathbb{N}_0$.*

From Lemma 3, we obtain the following proposition:

Proposition 2. *A nonnegative integer n can be written as $2x^2 + (y^2 + yz + z^2)$ with $x, y, z \in \mathbb{Z}$ if and only if $n \neq 4^k(16l + 10)$ with $k, l \in \mathbb{N}_0$.*

Proof. By multiplying both sides of (1) by $\varphi(q^2)$, we have that

$$\begin{aligned}\varphi(q^2)a(q) =& \varphi(q^2)a(q^4) + 6q\varphi(q^2)\psi(q^2)\psi(q^6),\\ \varphi(q)\varphi(q^2)\varphi(q^3) =& \varphi(q^2)a(q^4) + 2q\varphi(q^2)\psi(q^2)\psi(q^6),\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} A_1^2(n)q^n = \sum_{n=0}^{\infty} A_2^1(N)q^{2N} + 6q\sum_{N=0}^{\infty} m_{1\text{-}1,3}(N)q^{2N}, \tag{4}$$

$$\sum_{n=0}^{\infty} r_{1,2,3}(n)q^n = \sum_{N=0}^{\infty} A_2^1(N)q^{2N} + 2q\sum_{N=0}^{\infty} m_{1\text{-}1,3}(N)q^{2N}. \tag{5}$$

Suppose that n is even and $n = 2N$. In our notation, Lemma 3 states that $n \neq 4^k(16l+10),\ (k, l \in \mathbb{N}_0)$ if and only if $r_{1,2,3}(n) > 0$. By equating coefficients in (5), we see that Lemma 3 is equivalent to $A_2^1(N) > 0$, which in turn is equivalent to $A_1^2(n) > 0$ from (4). Therefore, as $A_1^2(n)$ is the number of ways n can be represented by $2x^2 + (y^2 + yz + z^2)$, the proposition follows.

Suppose that n is odd and $n = 2N + 1$. In the same way, Lemma 3 states that $n \neq 4^k(16l+10),\ (k, l \in \mathbb{N}_0)$ if and only if $r_{1,2,3}(n) > 0$. By equating coefficients in (5), we see that Lemma 3 is equivalent to $m_{1\text{-}1,3}(N) > 0$, which in turn is equivalent to $A_1^2(n) > 0$ from (4). Therefore, the proposition follows. □

From Proposition 2, we obtain the following theorem:

Theorem 3. *Suppose that $2 \leq b$. Then, $f_1^{2,b}$ can represent all the nonnegative integers if and only if $b = 2, 3, 4, 5, 6, 7, 8, 9,$ or 10.*

Proof. We first treat the "only if" direction. From the supposition, $n = 10$ can be represented by

$$10 = 2x^2 + by^2 + (z^2 + zw + w^2),\ \ x, y, z, w \in \mathbb{Z}.$$

Proposition 2 shows that $n = 10$ cannot be written as $2x^2 + (z^2 + zw + w^2)$, which implies that $2 \leq b \leq 10$.

In order to establish the "if" direction, we need only prove that $f_1^{2,b}$ represents $n = 16l + 10$ with $l \in \mathbb{N}_0$.

When $b \neq 2$ or 10, we can take $y = 1$ to obtain that

$$n - by^2 = 16l + 10 - b \cdot 1^2 \not\equiv 0, 8, 10 \bmod 16,$$

which can be written as $2x^2 + (z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

When $b = 2$, we take $y = 2$ to obtain that

$$n - 2 \cdot 2^2 = 16l + 10 - 8 = 16l + 2,$$

which can be represented as $2x^2 + (z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

Finally, suppose that $b = 10$. If $n = 10$ or 26, then setting $y = 1$ yields that

$$n - 10 \cdot 1^2 = 0 \text{ or } 16,$$

which can be written as $2x^2 + (z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

If $l \geq 2$, then by taking $y = 2$ we have that

$$n - 10 \cdot 2^2 = 16l + 10 - 40 = 16(l - 2) + 2,$$

which can be represented as $2x^2 + (z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. □

3.3. Summary

From Theorems 2 and 3 and their proofs, we obtain the following theorem:

Theorem 4. *Let a and b be positive integers with $a \leq b$.*

(1) *Any nonnegative integer can be represented by $f_1^{a,b}$ if and only if the pair (a, b) is given by one of the following:*

$$(a,b) = \begin{cases} (1,b) & (b = 1,2,3,4,5,6), \\ (2,b) & (b = 2,3,4,5,6,7,8,9,10). \end{cases}$$

(2) *If $f_1^{a,b}$ represents $1, 2, 6,$ and 10, then it can represent all the nonnegative integers.*

Furthermore, we obtain the following theorem:

Theorem 5. *For fixed positive integers a and c, set*

$$f_c^a(x, y, z) = ax^2 + c(y^2 + yz + z^2) \text{ with } x, y, z \in \mathbb{Z}.$$

Then, there exist no positive integers a and c such that f_c^a can represent all the nonnegative integers.

Proof. Suppose that there exist such positive integers a and c. By taking $n = 1$, we have that $a = 1$ or $c = 1$.

Suppose that $a = 1$. Then, the choice of $n = 2$ implies that $c = 1$ or 2. On the other hand, if $(a, c) = (1, 1)$ or $(1, 2)$, then it follows from Proposition 1 and Lemma

3 that there exist positive integers that cannot be expressed by f_1^1 or f_2^1, which is contradiction.

Suppose that $c = 1$. Then, the choice of $n = 2$ implies that $a = 1$ or 2. If $(a, c) = (1, 1), (2, 1)$, then it follows from Propositions 1 and 2 that there exist positive integers that cannot be expressed by f_1^1 or f_2^1, which is contradiction. □

Remark. In [3, pp.104], Dickson proved that there exist no positive integers a, b and c such that $ax^2 + by^2 + cz^2$, with $x, y, z \in \mathbb{Z}$, can represent all the nonnegative integers.

4. The Case Where $c = 2$

From Lemma 3 (2), we can obtain the following theorem:

Theorem 6. *Suppose that* $1 \leq a \leq b$. *Then,* $f_2^{a,b}$ *can represent all the nonnegative integers if and only if* $a = 1$ *and* $b = 1, 2, 3, 4,$ *or* 5.

Proof. We first deal with the "only if" direction. By choosing $n = 1$, we see that $a = 1$. By taking $n = 5$, we find that

$$5 = x^2 + by^2 + 2(z^2 + zw + w^2),\ (x, y, z, w \in \mathbb{Z}).$$

Lemma 3 (2) states that $n = 5$ cannot be written as $x^2 + 2(z^2 + zw + w^2)$, which implies that $1 \leq b \leq 5$.

In order to establish the "if" direction, we need only prove that $f_2^{1,b}$ represents $n = 8l + 5$, with $l \in \mathbb{N}_0$.

We first consider the case that $b = 1$. By taking $y = 2$, we obtain that

$$n - 1 \cdot 2^2 = 8l + 5 - 4 = 8l + 1,$$

which can be represented as $x^2 + 2(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

When $b = 2, 3,$ or 4, we set $y = 1$ to obtain that

$$n - by^2 = 8l + 5 - b \cdot 1^2 \equiv 3, 2, \text{ or } 1 \bmod 8,$$

which can be written as $x^2 + 2(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

Finally, we suppose that $b = 5$. If $l = 0$ or 1, then by taking $y = 1$ we obtain that

$$n - 5 \cdot 1^2 = 8l + 5 - 5 = 0 \text{ or } 8,$$

which can be represented as $x^2 + 2(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

If $l \geq 2$, then by taking $y = 2$ we have that

$$n - 5 \cdot 2^2 = 8l + 5 - 20 = 8(l - 2) + 1,$$

which can be written as $x^2 + 2(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. □

From the proof of Theorem 6, we obtain the following theorem:

Theorem 7. *If $f_2^{a,b}$ represents* 1 *and* 5*, then it can represent all the nonnegative integers.*

5. The Case Where $c = 3$

We use the following result, which was given by Dickson [3, pp. 112-113]:

Lemma 4. *A nonnegative integer n can be written as $x^2+3y^2+9z^2$ with $x, y, z \in \mathbb{Z}$ if and only if $n \neq 3l+2$ or $9^k(9l+6)$, with $k, l \in \mathbb{N}_0$.*

Using Lemma 4, we obtain the following proposition:

Proposition 3. *A nonnegative integer n can be written as $x^2 + 3(y^2 + yz + z^2)$ with $x, y, z \in \mathbb{Z}$ if and only if $n \neq 3l + 2$ or $9^k(9l + 6)$, with $k, l \in \mathbb{N}_0$.*

Proof. By replacing q with q^3 in (1), we have that

$$\begin{aligned} a(q^3) =& a(q^{12}) + 6q^3\psi(q^6)\psi(q^{18}), \\ \varphi(q^3)\varphi(q^9) =& a(q^{12}) + 2q^3\psi(q^6)\psi(q^{18}). \end{aligned}$$

By multiplying both sides of these equations by $\varphi(q)$, we obtain that

$$\begin{aligned} \varphi(q)a(q^3) =& \varphi(q)a(q^{12}) + 6q^3\varphi(q)\psi(q^6)\psi(q^{18}), \\ \varphi(q)\varphi(q^3)\varphi(q^9) =& \varphi(q)a(q^{12}) + 2q^3\varphi(q)\psi(q^6)\psi(q^{18}), \end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} A_3^1(n)q^n = \sum_{n=0}^{\infty} A_{12}^1(n)q^n + 6q^3 \sum_{n=0}^{\infty} m_{1\text{-}6,18}(n)q^n, \tag{6}$$

$$\sum_{n=0}^{\infty} r_{1,3,9}(n)q^n = \sum_{n=0}^{\infty} A_{12}^1(n)q^n + 2q^3 \sum_{n=0}^{\infty} m_{1\text{-}6,18}(n)q^n. \tag{7}$$

In our notation, Lemma 4 states that $n \neq 3l + 2$, $9^k(9l + 6)$, $(k, l \in \mathbb{N}_0)$ is equivalent to $r_{1,3,9}(n) > 0$. By equating coefficients in (7), we see that Lemma 4 is equivalent to $A_{12}^1(n) > 0$ or $m_{1\text{-}6,18}(n-3) > 0$, which in turn is equivalent to $A_3^1(n) > 0$ from (6). Therefore, as $A_3^1(n)$ is the number of ways n can be represented by $x^2 + 3(y^2 + yz + z^2)$, the proposition follows. □

Noting that $a = 1$ if $f_3^{a,b}$ can represent all the nonnegative integers, we obtain the following theorem:

Theorem 8. *Suppose that* $1 \le a \le b$. *Then,* $f_3^{a,b}$ *can represent all the nonnegative integers if and only if* $a = 1$ *and* $b = 2$.

Proof. First, let us prove the "only if" direction. The choice of $n = 1$ implies that $a = 1$. Proposition 3 implies that $b = 1$ or 2. Assume that $b = 1$, and

$$6 = x^2 + y^2 + 3(z^2 + zw + w^2) \text{ with } x, y, z, w \in \mathbb{Z},$$

which implies that $z^2 + zw + w^2 = 1, \neq 0, 2$. Then, it follows that $x^2 + y^2 = 3$, which is impossible.

In order to establish the "if" direction, we need only prove that $f_3^{1,b}$ can represent $n = 3l + 2$ or $9l + 6$, with $l \in \mathbb{N}_0$.

Suppose that $n = 3l + 2$. We first consider the case where $l \equiv 0 \bmod 3$. It is obvious that $f_3^{1,2}$ can represent $n = 2$. When $l \ge 1$ and $l \equiv 0 \bmod 3$, we obtain that

$$n - 2 \cdot 2^2 = 3l + 2 - 8 = 3(l - 2),$$

which can be written as $x^2 + 3(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

When $l \equiv 1 \bmod 3$, by taking $y = 1$, we have that

$$n - 2y^2 = 3l + 2 - 2 = 3l,$$

which can be written as $x^2 + 3(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

Assume that $l \equiv 2 \bmod 3$. For $l = 2, 5$, or 8, we set $y = 2$ to obtain that

$$n - 2 \cdot 2^2 = 3l + 2 - 2 \cdot 2^2 = 0, 9, 9 \cdot 2,$$

which can be represented as $x^2 + 3(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. If $l \ge 11$ and $l = 3L + 2$ with $L \in \mathbb{N}_0$, then by taking $y = 4$ we have that

$$n - 2 \cdot 4^2 = 3l + 2 - 2 \cdot 4^2 = 3\{3(L - 3) + 1\},$$

which can be written as $x^2 + 3(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

Finally, we suppose that $n = 9l + 6$. By taking $y = 1$, we have that

$$n - 2y^2 = 9l + 6 - 2 = 9l + 4 \equiv 1 \bmod 3,$$

which can be represented as $x^2 + 3(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. □

From the proof of Theorem 8, we obtain the following theorem:

Theorem 9. *If* $f_3^{a,b}$ *represents* $1, 2,$ *and* 6, *then it can represent all the nonnegative integers.*

6. The Case Where $c = 4$

We use the following result that was proved by Dickson [3, pp. 112-113]:

Lemma 5. *A nonnegative integer n can be written as $x^2+4y^2+12z^2$ with $x, y, z \in \mathbb{Z}$ if and only if $n \neq 4l+2,\ 4l+3,$ or $9^k(9l+6)$, with $k, l \in \mathbb{N}_0$.*

Using Lemma 5, we obtain the following proposition:

Proposition 4. *A nonnegative integer n can be written as $x^2 + 4(y^2 + yz + z^2)$ with $x, y, z \in \mathbb{Z}$ if and only if $n \neq 4l+2,\ 4l+3,$ or $9^k(9l+6)$, with $k, l \in \mathbb{N}_0$.*

Proof. By replacing q by q^4 in (1), we have that

$$\begin{aligned} a(q^4) =& a(q^{16}) + 6q^4\psi(q^8)\psi(q^{24}), \\ \varphi(q^4)\varphi(q^{12}) =& a(q^{16}) + 2q^4\psi(q^8)\psi(q^{24}). \end{aligned}$$

By multiplying both sides of these equations by $\varphi(q)$, we obtain that

$$\begin{aligned} \varphi(q)a(q^4) =& \varphi(q)a(q^{16}) + 6q^4\varphi(q)\psi(q^8)\psi(q^{24}), \\ \varphi(q)\varphi(q^4)\varphi(q^{12}) =& \varphi(q)a(q^{16}) + 2q^4\varphi(q)\psi(q^8)\psi(q^{24}), \end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} A_4^1(n)q^n = \sum_{n=0}^{\infty} A_{16}^1(n)q^n + 6q^4 \sum_{N=0}^{\infty} m_{1\text{-}8,24}(N)q^N, \tag{8}$$

$$\sum_{n=0}^{\infty} r_{1,4,12}(n)q^n = \sum_{n=0}^{\infty} A_{16}^1(n)q^n + 2q^4 \sum_{N=0}^{\infty} m_{1\text{-}8,24}(N)q^N. \tag{9}$$

In our notation, Lemma 5 states that $n \neq 4l+2,\ 4l+3,\ 9^k(9l+6),\ (k, l \in \mathbb{N}_0)$ if and only if $r_{1,4,12}(n) > 0$. By equating coefficients in (9), we see that Lemma 5 is equivalent to $A_{16}^1(n) > 0$ or $m_{1\text{-}8,24}(n-4) > 0$, which in turn is equivalent to $A_4^1(n) > 0$ from (8). Therefore, as $A_4^1(n)$ is the number of ways n can be represented by $x^2 + 4(y^2 + yz + z^2)$, the proposition follows. □

Noting that $a = 1$ and $b = 2$ if $f_4^{a,b}$ can represent all the nonnegative integers, we obtain the following theorem:

Theorem 10. *Any nonnegative integer n can be represented by $f_4^{a,b}$ if and only if $a = 1$ and $b = 2$.*

Proof. We first treat the "only if" direction. By choosing $n = 1$, we see that $a = 1$. By taking $n = 2$, we find that $b = 1, 2$. Moreover, by choosing $n = 3$, we see that $b = 2$.

In order to establish the "if" direction, we need only prove that $f_4^{1,2}$ represents $n = 4l + 2$, $4l + 3$, and $9l + 6$, with $l \in \mathbb{N}_0$.

Suppose that $n = 4l + 2$. If $l \not\equiv 0$ or $6 \bmod 9$, then by taking $y = 1$ we have that

$$n - 2y^2 = 4l + 2 - 2 = 4l,$$

which can be represented as $x^2 + 4(z^2 + zw + w^2)$, with $x, z, w \in \mathbb{Z}$. If $l \equiv 0$ or $6 \bmod 9$, then by taking $y = 3$ we obtain that

$$n - 2y^2 = 4l + 2 - 2 \cdot 3^2 = 4(l - 4),$$

which can be represented as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

Suppose that $n = 4l + 3$. If $l \not\equiv 2$ or $8 \bmod 9$, then by taking $y = 1$ we have that

$$n - 2y^2 = 4l + 3 - 2 = 4l + 1 \not\equiv 0, 6 \bmod 9,$$

which can be represented as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. If $l \equiv 2$ or $8 \bmod 9$, then by taking $y = 3$ we obtain that

$$n - 2y^2 = 4l + 3 - 2 \cdot 3^2 = 4(l - 4) + 1 \equiv 2, 8 \bmod 9,$$

which can be represented as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. It is easy to check that $f_4^{1,2}$ can represent $n = 11$.

Suppose that $n = 9l + 6$. If $l \equiv 0 \bmod 4$ and $l = 4L$ with $L \in \mathbb{N}_0$, then by taking $y = 5$ we have that

$$n - 2y^2 = 9l + 6 - 2 \cdot 5^2 = 4\{9(L - 2) + 7\},$$

which can be written as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. It is easy to check that $f_4^{1,2}$ can represent $n = 6$ or 42.

If $l \equiv 1 \bmod 4$, then by taking $y = 1$ we have that

$$n - 2y^2 = 9l + 6 - 2 \cdot 1^2 = 9l + 4 \equiv 1 \bmod 4,$$

which can be represented as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

If $l \equiv 2 \bmod 4$ and $l = 4L + 2$ with $L \in \mathbb{N}_0$, then by taking $y = 4$ we have that

$$n - 2y^2 = 9l + 6 - 2 \cdot 4^2 = 4(9L - 2),$$

which can be written as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. It is easy to check that $f_4^{1,2}$ can represent $n = 24$.

If $l \equiv 3 \bmod 4$, then by taking $y = 2$ we have that

$$n - 2y^2 = 9l + 6 - 2 \cdot 2^2 = 9l - 2 \equiv 1 \bmod 4,$$

which can be represented as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. $\square$

From the proof of Theorem 10, we obtain the following result:

Theorem 11. *If $f_4^{a,b}$ represents $1, 2,$ and 3, then it can represent all the nonnegative integers.*

7. The Case Where $c \geq 5$

7.1. The Case Where $c = 5$

Theorem 12. *There exist no positive integers a and b such that $1 \leq a \leq b$ and $f_5^{a,b}$ can represent all the nonnegative integers, where*

$$f_5^{a,b}(x, y, z, w) = ax^2 + by^2 + 5(z^2 + zw + w^2).$$

Proof. Suppose that there exist such positive integers a and b. By considering $n = 1$, we have that $a = 1$. Taking $n = 2$, we obtain that $b = 1$ or 2. Considering $n = 3$, we have that $(a, b) = (1, 2)$.

Take $n = 10$. Then, $n = 10$ cannot be written as $x^2 + 2y^2$ with $x, y \in \mathbb{Z}$, which implies that $z^2 + zw + w^2 = 1$, because 2 cannot be represented as $z^2 + zw + w^2$ with $z, w \in \mathbb{Z}$. Therefore, it follows that $5 = 10 - 5 \cdot 1$ can be written as $x^2 + 2y^2$ with $x, y \in \mathbb{Z}$, which is impossible. $\square$

7.2. The Case Where $c \geq 6$

Theorem 13. *There exist no positive integers a, b, and c such that $1 \leq a \leq b$, $c \geq 6$, and $f_c^{a,b}$ can represent all the nonnegative integers.*

Proof. Suppose that there exist such positive integers a, b, and c. By considering $n = 1$, we have that $a = 1$. Taking $n = 2$, we obtain that $b = 1$ or 2. Considering $n = 3$, we have that $(a, b) = (1, 2)$. On the other hand, $n = 5$ cannot be written as $x^2 + 2y^2$ with $x, y \in \mathbb{Z}$. $\square$

8. Proof of Theorem 1

Proof. Suppose that $f_c^{a,b}$ can represent all the nonnegative integers. Then, Theorems 12 and 13 show that $c = 1, 2, 3, 4$. If $c = 1$, Theorem 4 implies that

$$(a, b) = \begin{cases} (1, b) & (b = 1, 2, 3, 4, 5, 6), \\ (2, b) & (b = 2, 3, 4, 5, 6, 7, 8, 9, 10). \end{cases}$$

If $c = 2$, Theorem 6 shows that

$$(a, b) = (1, 1), (1, 2), (1, 3), (1, 4), (1, 5).$$

If $c = 3$, Theorem 8 implies that $(a, b) = (1, 2)$. If $c = 4$, Theorem 10 implies that $(a, b) = (1, 2)$. Therefore, the "only if" direction of Theorem 1 (1) follows. The "if" direction follows from Theorems 4, 6, 8, and 10.

We next prove Theorem 1 (2). From the proofs of Theorems 12, and 13, taking $n = 1, 2, 3, 5, 10$, we see that $1 \leq c \leq 4$.

If $c = 1$, from Theorem 4, choosing $n = 1, 2, 6, 10$ implies that $f_1^{a,b}$ represents all the nonngative integers. If $c = 2$, from Theorem 7, taking $n = 1, 5$ shows that $f_2^{a,b}$ represents all the nonngative integers. If $c = 3$, from Theorem 9, choosing $n = 1, 2, 6$ implies that $f_3^{a,b}$ represents all the nonngative integers. If $c = 4$, from Theorem 11, taking $n = 1, 2, 3$ shows that $f_4^{a,b}$ represents all the nonngative integers. Therefore, Theorem 1 (2) follows. □

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