ON CERTAIN QUATERNARY QUADRATIC FORMS

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Abstract
In this paper, we determine all triplets of positive integers \(a, b,\) and \(c\) such that every nonnegative integer can be represented as
\[
f^a,b_c(x, y, z, w) = ax^2 + by^2 + c(z^2 + zw + w^2) \quad \text{with} \quad x, y, z, w \in \mathbb{Z}.
\]
Furthermore, we prove that \(f^a,b_c\) can represent all the nonnegative integers if it represents 1, 2, 3, 5, 6, and 10.

1. Introduction
Throughout this paper, we set \(\mathbb{N} = \{1, 2, 3, \ldots\}\) and \(\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\},\) and \(\mathbb{Z}\) denotes the set of rational integers. In addition, the triangular numbers are \(t_x = x(x + 1)/2, \quad (x \in \mathbb{N}_0).\) Furthermore, \(a, b\) and \(c\) are fixed positive integers with \(1 \leq a \leq b.\)

More than 200 years ago, Lagrange [4, pp. 279] proved one of the most celebrated theorems in number theory, the Four Squares Theorem, which states that every positive integer can be expressed as a sum of four squares. Since then, the universal representability by a quadratic form has been becoming one of the most interesting problems in number theory. Incredibly, Ramanujan [8] extended Lagrange’s theorem to other positive definite quaternary quadratic forms \(ax^2 + by^2 + cz^2 + du^2,\) and determined all of the positive integers \(a, b, c\) and \(d\) such that \(ax^2 + by^2 + cz^2 + du^2\) can represent positive integers universally, totally 54 cases. Unlike Jacobi’s “proof” to Lagrange’s theorem, which follows directly from his beautiful formula,
\[
|\{(x, y, z, u) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + u^2\}| = 8\sigma(n) - 32\sigma(n/4) > 0,
\]
where \(\sigma(n) = \sum_{d|n} d,\) Ramanujan’s proofs rely on the representability of ternary quadratic forms and their relations to the quaternary quadratic forms under investigation. For example, for Lagrange’s Four Squares Theorem, one first shows [3]
that a positive integer \( n \) can be expressed as a sum of three squares if and only if \( n \neq 4^k(8l + 7) \) for any nonnegative integers \( k \) and \( l \). Then if \( n \neq 4^k(8l + 7) \), one has \( n = x^2 + y^2 + z^2 + 0^2 \); if \( n = 4^k(8l + 7) \), then \( n = 4^k(8l + 6) + (2^k)^2 \), where \( n = 4^k(8l + 6) \) is representable by a sum of three squares.

Inspired by Ramanujan’s proofs, in our recent work [7], using a similar strategy, the author succeeded in extending his results to the quaternary quadratic polynomials \( at_x + bt_y + c(z^2 + zw + w^2) \). In this work, we aim to further extend the previous work of Ramanujan to another type of quaternary quadratic forms, namely, \( ax^2 + by^2 + c(z^2 + zw + w^2) \). We conclude this section by summarizing our main results in the following theorem:

**Theorem 1.** For positive integers \( a, b, \) and \( c \) with \( a \leq b \), set

\[
 f_{a,b}^c(x, y, z, w) = ax^2 + by^2 + c(z^2 + zw + w^2), \ (x, y, z, w \in \mathbb{Z}).
\]

(1) Every nonnegative integer can be represented by \( f_{a,b}^c \) if and only if \( (a, b, c) = \begin{cases} (1, b, 1) & (b = 1, 2, 3, 4, 5, 6), \\ (2, b, 1) & (b = 2, 3, 4, 5, 6, 7, 8, 9, 10), \\ (1, b, 2) & (b = 1, 2, 3, 4, 5), \\ (1, 2, 3), \\ (1, 2, 4). \end{cases} \)

(2) If \( f_{a,b}^c \) represents \( 1, 2, 3, 5, 6, \) and \( 10 \), then it can represent all the nonnegative integers.

2. Notations and Preliminaries

To prove Theorem 1, we follow Ramanujan by introducing

\[
 \varphi(q) = \sum_{n \in \mathbb{Z}} q^n, \ \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \ a(q) = \sum_{m,n \in \mathbb{Z}} q^{m^2+mn+n^2}, \ (q \in \mathbb{C}, \ |q| < 1),
\]

and we apply the following identities:

\[
 a(q) = a(q^4) + 6q\psi(q^2)\psi(q^6), \ \varphi(q)\varphi(q^3) = a(q^4) + 2q\psi(q^2)\psi(q^6). \quad (1)
\]

For the proofs of these formulas, see Berndt [1, pp. 232] and Hirschhorn et al. [5].

For fixed positive integers \( a \) and \( c \), and for each \( n \in \mathbb{N}_0 \), we define

\[
 A_{a,c}^n = \{ (x, y, z) \in \mathbb{Z}^3 | n = f_{a,c}^n(x, 0, y, z) \}.
\]
Moreover, for fixed positive integers $a, b, $ and $c$, and for each $n \in \mathbb{N}_0$, we set
\[
\begin{align*}
    r_{a,b}(n) &= |\{(x,y) \in \mathbb{Z}^2 \mid n = ax^2 + by^2\}|, \\
    t_{a,b}(n) &= |\{(x,y) \in \mathbb{N}_0^2 \mid n = at_x + bt_y\}|, \\
    r_{a,b,c}(n) &= |\{(x,y,z) \in \mathbb{Z}^3 \mid n = ax^2 + by^2 + cz^2\}|, \\
    m_{a,b,c}(n) &= |\{(x,y,z) \in \mathbb{Z} \times \mathbb{N}_0^2 \mid n = ax^2 + bt_y + ct_z\}|.
\end{align*}
\]

3. The Case Where $c = 1$

**Lemma 1.** Suppose that $1 \leq a \leq b$, $c = 1$, and $f_{c}^{a,b}$ can represent all the nonnegative integers. Then, $a = 1$ or $2$.

**Proof.** From the supposition, $n = 2$ can be written as
\[
2 = ax^2 + by^2 + (z^2 + zw + w^2), \quad (x,y,z,w) \in \mathbb{Z}.
\]
On the other hand, $n = 2$ cannot be written as $z^2 + zw + w^2$ with $z, w \in \mathbb{Z}$, which implies that $a = 1, 2$. \qed

3.1. The Case Where $a = 1$

We use the following result, which was introduced by Dickson [3, pp. 112-113]:

**Lemma 2.** A nonnegative integer $n$ can be written as $x^2 + y^2 + 3z^2$ with $x, y, z \in \mathbb{Z}$ if and only if $n \neq 9^k(9l + 6)$, $(k,l) \in \mathbb{N}_0$.

As a consequence of Lemma 2, we have the following proposition:

**Proposition 1.** A nonnegative integer $n$ can be written as $x^2 + (y^2 + yz + z^2)$ with $x, y, z \in \mathbb{Z}$ if and only if $n \neq 9^k(9l + 6)$, $(k,l) \in \mathbb{N}_0$.

**Proof.** By multiplying both sides of (1) by $\varphi(q)$, we have that
\[
\begin{align*}
    \varphi(q)\alpha(q) &= \varphi(q)a(q^4) + 6q\varphi(q)\psi(q^2)\psi(q^6), \\
    \varphi(q)^2\varphi(q^3) &= \varphi(q)a(q^4) + 2q\varphi(q)\psi(q^2)\psi(q^6),
\end{align*}
\]
which implies that
\[
\begin{align}
    \sum_{n=0}^{\infty} A_{1}^1(n)q^n &= \sum_{n=0}^{\infty} A_{1}^1(n)q^n + 6q \sum_{n=0}^{\infty} m_{1,2,6}(n)q^n, \quad (2) \\
    \sum_{n=0}^{\infty} r_{1,1,3}(n)q^n &= \sum_{n=0}^{\infty} A_{1}^1(n)q^n + 2q \sum_{n=0}^{\infty} m_{1,2,6}(n)q^n. \quad (3)
\end{align}
\]
In our present notation, Lemma 2 states that \( n \neq 9^k(9l + 6), (k, l \in \mathbb{N}_0) \) if and only if \( r_{1,1,3}(n) > 0 \). By equating coefficients in (3), we see that Lemma 2 is equivalent to \( A_{1}^{1}(n) > 0 \) or \( m_{1,2,6}(n-1) > 0 \), which in turn is equivalent to \( A_{1}^{1}(n) > 0 \) from (2). Therefore, as \( A_{1}^{1}(n) \) is the number of ways \( n \) can be represented as \( x^2 + (y^2 + yz + z^2) \), the proposition follows.

By Proposition 1, we obtain the following theorem:

**Theorem 2.** Any nonnegative integer \( n \) can be represented by \( f_{1}^{1,b} \) if and only if \( b = 1, 2, 3, 4, 5, \) or \( 6 \).

**Proof.** We first prove the “only if” direction. Therefore, \( n = 6 \) can be represented by
\[
6 = x^2 + by^2 + (z^2 + zw + w^2), \quad x, y, z, w \in \mathbb{Z}.
\]

On the other hand, Proposition 1 shows that \( n = 6 \) cannot be written as \( x^2 + (z^2 + zw + w^2) \), which implies that \( b = 1, 2, 3, 4, 5, 6 \).

In order to establish the “if” direction, we need only prove that \( f_{1}^{1,b} \) represents \( n = 9l + 6 \) with \( l \in \mathbb{N}_0 \).

When \( b = 1, 2, 3, 4 \) or \( 5 \), we can set \( y = 1 \) to obtain that
\[
n - by^2 = 9l + 6 - b \cdot 1^2 \equiv 5, 4, 3, 2, 1 \text{ mod } 9,
\]
which can be written as \( x^2 + (z^2 + zw + w^2) \) with \( x, z, w \in \mathbb{Z} \).

Suppose that \( b = 6 \). Then, if \( n = 6 \) or \( 15 \), or \( l = 2 \) or \( 8 \) mod \( 9 \), we can set \( y = 1 \) to obtain that
\[
n - 6 \cdot 1^2 = 9l, \quad (l = 0, 1 \text{ or } l = 2, 8 \text{ mod } 9),
\]
which can be represented as \( x^2 + (z^2 + zw + w^2) \) with \( x, z, w \in \mathbb{Z} \).

If \( l \geq 2 \) and \( l \not\equiv 2 \) or \( 8 \text{ mod } 9 \), then by taking \( y = 2 \) we have that
\[
n - 6 \cdot 2^2 = 9l + 6 - 24 = 9(l - 2),
\]
which can be written as \( x^2 + (z^2 + zw + w^2) \) with \( x, z, w \in \mathbb{Z} \).

3.2. The Case Where \( a = 2 \)

We use the following result, which was proved by Dickson [2]:

**Lemma 3.**

(1) A nonnegative integer \( n \) can be written as \( x^2 + 2y^2 + 3z^2 \) with \( x, y, z \in \mathbb{Z} \) if and only if \( n \neq 4^k(16l + 10) \) with \( k, l \in \mathbb{N}_0 \).

(2) A nonnegative integer \( n \) can be written as \( x^2 + 2(y^2 + yz + z^2) \) with \( x, y, z \in \mathbb{Z} \) if and only if \( n \neq 4^k(8l + 5) \) with \( k, l \in \mathbb{N}_0 \).
From Lemma 3, we obtain the following proposition:

**Proposition 2.** A nonnegative integer \( n \) can be written as \( 2x^2 + (y^2 + yz + z^2) \) with \( x, y, z \in \mathbb{Z} \) if and only if \( n \neq 4^k(16l + 10) \) with \( k, l \in \mathbb{N}_0 \).

**Proof.** By multiplying both sides of (1) by \( \varphi(q^2) \), we have that

\[
\varphi(q^2)a(q) = \varphi(q^2)\alpha(q^2) + 6q\varphi(q^2)\psi(q^2)\psi(q^6),
\]

which implies that

\[
\sum_{n=0}^{\infty} A_2^2(n)q^n = \sum_{n=0}^{\infty} A_2^1(N)q^{2N} + 6q \sum_{N=0}^{\infty} m_{1,1,3}(N)q^{2N}, \tag{4}
\]

\[
\sum_{n=0}^{\infty} r_{1,2,3}(n)q^n = \sum_{N=0}^{\infty} A_2^1(N)q^{2N} + 2q \sum_{N=0}^{\infty} m_{1,1,3}(N)q^{2N}. \tag{5}
\]

Suppose that \( n \) is even and \( n = 2N \). In our notation, Lemma 3 states that \( n \neq 4^k(16l + 10) \), \((k, l) \in \mathbb{N}_0 \) if and only if \( r_{1,2,3}(n) > 0 \). By equating coefficients in (5), we see that Lemma 3 is equivalent to \( A_2^1(N) > 0 \), which in turn is equivalent to \( A_2^2(n) > 0 \) from (4). Therefore, as \( A_2^2(n) \) is the number of ways \( n \) can be represented by \( 2x^2 + (y^2 + yz + z^2) \), the proposition follows.

Suppose that \( n \) is odd and \( n = 2N + 1 \). In the same way, Lemma 3 states that \( n \neq 4^k(16l + 10) \), \((k, l) \in \mathbb{N}_0 \) if and only if \( r_{1,2,3}(n) > 0 \). By equating coefficients in (5), we see that Lemma 3 is equivalent to \( m_{1,1,3}(N) > 0 \), which in turn is equivalent to \( A_2^2(n) > 0 \) from (4). Therefore, the proposition follows.

From Proposition 2, we obtain the following theorem:

**Theorem 3.** Suppose that \( 2 \leq b \). Then, \( f_1^{2,b} \) can represent all the nonnegative integers if and only if \( b = 2, 3, 4, 5, 6, 7, 8, 9, \) or \( 10 \).

**Proof.** We first treat the “only if” direction. From the supposition, \( n = 10 \) can be represented by

\[
10 = 2x^2 + by^2 + (z^2 + zw + w^2), \quad x, y, z, w \in \mathbb{Z}.
\]

Proposition 2 shows that \( n = 10 \) cannot be written as \( 2x^2 + (z^2 + zw + w^2) \), which implies that \( 2 \leq b \leq 10 \).

In order to establish the “if” direction, we need only prove that \( f_1^{2,b} \) represents \( n = 16l + 10 \) with \( l \in \mathbb{N}_0 \).

When \( b \neq 2 \) or \( 10 \), we can take \( y = 1 \) to obtain that

\[
n - by^2 = 16l + 10 - b \cdot 1^2 \neq 0, 8, 10 \mod 16,
\]
which can be written as $2x^2 + (z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

When $b = 2$, we take $y = 2$ to obtain that
\[ n - 2 \cdot 2^2 = 16l + 10 - 8 = 16l + 2, \]
which can be represented as $2x^2 + (z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

Finally, suppose that $b = 10$. If $n = 10$ or 26, then setting $y = 1$ yields that
\[ n - 10 \cdot 1^2 = 0 \text{ or } 16, \]
which can be written as $2x^2 + (z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

If $l \geq 2$, then by taking $y = 2$ we have that
\[ n - 10 \cdot 2^2 = 16l + 10 - 40 = 16(l - 2) + 2, \]
which can be represented as $2x^2 + (z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. \hfill \Box

3.3. Summary

From Theorems 2 and 3 and their proofs, we obtain the following theorem:

**Theorem 4.** Let $a$ and $b$ be positive integers with $a \leq b$.

1. Any nonnegative integer can be represented by $f_{1}^{a,b}$ if and only if the pair $(a, b)$ is given by one of the following:
   \[ (a, b) = \begin{cases} (1, b) & (b = 1, 2, 3, 4, 5, 6), \\ (2, b) & (b = 2, 3, 4, 5, 6, 7, 8, 9, 10). \end{cases} \]

2. If $f_{1}^{a,b}$ represents 1, 2, 6, and 10, then it can represent all the nonnegative integers.

Furthermore, we obtain the following theorem:

**Theorem 5.** For fixed positive integers $a$ and $c$, set
\[ f_{c}^{a}(x, y, z) = ax^2 + c(y^2 + yz + z^2) \text{ with } x, y, z \in \mathbb{Z}. \]

Then, there exist no positive integers $a$ and $c$ such that $f_{c}^{a}$ can represent all the nonnegative integers.

**Proof.** Suppose that there exist such positive integers $a$ and $c$. By taking $n = 1$, we have that $a = 1$ or $c = 1$.

Suppose that $a = 1$. Then, the choice of $n = 2$ implies that $c = 1$ or 2. On the other hand, if $(a, c) = (1, 1) \text{ or } (1, 2)$, then it follows from Proposition 1 and Lemma
3 that there exist positive integers that cannot be expressed by \( f_1^1 \) or \( f_2^1 \), which is contradiction.

Suppose that \( c = 1 \). Then, the choice of \( n = 2 \) implies that \( a = 1 \) or \( 2 \). If \((a, c) = (1, 1), (2, 1)\), then it follows from Propositions 1 and 2 that there exist positive integers that cannot be expressed by \( f_1^1 \) or \( f_2^1 \), which is contradiction. \( \square \)

**Remark.** In [3, pp.104], Dickson proved that there exist no positive integers \( a, b \) and \( c \) such that \( ax^2 + by^2 + cz^2 \), with \( x, y, z \in \mathbb{Z} \), can represent all the nonnegative integers.

### 4. The Case Where \( c = 2 \)

From Lemma 3 (2), we can obtain the following theorem:

**Theorem 6.** Suppose that \( 1 \leq a \leq b \). Then, \( f_2^{a,b} \) can represent all the nonnegative integers if and only if \( a = 1 \) and \( b = 1, 2, 3, 4, \) or \( 5 \).

**Proof.** We first deal with the “only if” direction. By choosing \( n = 1 \), we see that \( a = 1 \). By taking \( n = 5 \), we find that

\[
5 = x^2 + by^2 + 2(z^2 + zw + w^2), \quad (x, y, z, w \in \mathbb{Z}).
\]

Lemma 3 (2) states that \( n = 5 \) cannot be written as \( x^2 + 2(z^2 + zw + w^2) \), which implies that \( 1 \leq b \leq 5 \).

In order to establish the “if” direction, we need only prove that \( f_2^{1,b} \) represents \( n = 8l + 5 \), with \( l \in \mathbb{N}_0 \).

We first consider the case that \( b = 1 \). By taking \( y = 2 \), we obtain that

\[
n - 1 \cdot 2^2 = 8l + 5 - 4 = 8l + 1,
\]

which can be represented as \( x^2 + 2(z^2 + zw + w^2) \) with \( x, z, w \in \mathbb{Z} \).

When \( b = 2, 3, \) or \( 4 \), we set \( y = 1 \) to obtain that

\[
n - by^2 = 8l + 5 - b \cdot 1^2 = 3, 2, \) or \( 1 \) mod \( 8 \),
\]

which can be written as \( x^2 + 2(z^2 + zw + w^2) \) with \( x, z, w \in \mathbb{Z} \).

Finally, we suppose that \( b = 5 \). If \( l = 0 \) or \( 1 \), then by taking \( y = 1 \) we obtain that

\[
n - 5 \cdot 1^2 = 8l + 5 - 5 = 0 \) or \( 8 \),
\]

which can be represented as \( x^2 + 2(z^2 + zw + w^2) \) with \( x, z, w \in \mathbb{Z} \).

If \( l \geq 2 \), then by taking \( y = 2 \) we have that

\[
n - 5 \cdot 2^2 = 8l + 5 - 20 = 8(l - 2) + 1,
\]

which can be written as \( x^2 + 2(z^2 + zw + w^2) \) with \( x, z, w \in \mathbb{Z} \). \( \square \)
From the proof of Theorem 6, we obtain the following theorem:

**Theorem 7.** If \( f_2^{a,b} \) represents 1 and 5, then it can represent all the nonnegative integers.

5. The Case Where \( c = 3 \)

We use the following result, which was given by Dickson [3, pp. 112-113]:

**Lemma 4.** A nonnegative integer \( n \) can be written as \( x^2 + 3y^2 + 9z^2 \) with \( x, y, z \in \mathbb{Z} \) if and only if \( n \neq 3l + 2 \) or \( 9^k(9l + 1) \), with \( k, l \in \mathbb{N}_0 \).

Using Lemma 4, we obtain the following proposition:

**Proposition 3.** A nonnegative integer \( n \) can be written as \( x^2 + 3(y^2 + yz + z^2) \) with \( x, y, z \in \mathbb{Z} \) if and only if \( n \neq 3l + 2 \) or \( 9^k(9l + 6) \), with \( k, l \in \mathbb{N}_0 \).

**Proof.** By replacing \( q \) with \( q^3 \) in (1), we have that

\[
\begin{align*}
\alpha(q^3) &= \alpha(q^{12}) + 6q^6\psi(q^6)\psi(q^{18}), \\
\varphi(q^3)\varphi(q^3) &= \alpha(q^{12}) + 2q^3\psi(q^6)\psi(q^{18}).
\end{align*}
\]

By multiplying both sides of these equations by \( \varphi(q) \), we obtain that

\[
\begin{align*}
\varphi(q)\alpha(q^3) &= \varphi(q)\alpha(q^{12}) + 6q^3\varphi(q)\psi(q^6)\psi(q^{18}), \\
\varphi(q)\varphi(q^3)\varphi(q^3) &= \varphi(q)\alpha(q^{12}) + 2q^3\varphi(q)\psi(q^6)\psi(q^{18}),
\end{align*}
\]

which implies that

\[
\begin{align*}
\sum_{n=0}^{\infty} A_3^1(n)q^n &= \sum_{n=0}^{\infty} A_{12}^1(n)q^n + 6q^3\sum_{n=0}^{\infty} m_{1,618}(n)q^n, \quad (6) \\
\sum_{n=0}^{\infty} r_{1,3,9}(n)q^n &= \sum_{n=0}^{\infty} A_{12}^1(n)q^n + 2q^3\sum_{n=0}^{\infty} m_{1,618}(n)q^n. \quad (7)
\end{align*}
\]

In our notation, Lemma 4 states that \( n \neq 3l + 2, 9^k(9l + 6), (k, l \in \mathbb{N}_0) \) is equivalent to \( r_{1,3,9}(n) > 0 \). By equating coefficients in (7), we see that Lemma 4 is equivalent to \( A_{12}^1(n) > 0 \) or \( m_{1,6,18}(n - 3) > 0 \), which in turn is equivalent to \( A_3^1(n) > 0 \) from (6). Therefore, as \( A_3^1(n) \) is the number of ways \( n \) can be represented by \( x^2 + 3(y^2 + yz + z^2) \), the proposition follows.

Noting that \( a = 1 \) if \( f_3^{a,b} \) can represent all the nonnegative integers, we obtain the following theorem:
Theorem 8. Suppose that $1 \leq a \leq b$. Then, $f_{3}^{a,b}$ can represent all the nonnegative integers if and only if $a = 1$ and $b = 2$.

Proof. First, let us prove the “only if” direction. The choice of $n = 1$ implies that $a = 1$. Proposition 3 implies that $b = 1$ or 2. Assume that $b = 1$, and

$$6 = x^2 + y^2 + 3(z^2 + zw + w^2)$$

with $x, y, z, w \in \mathbb{Z}$,

which implies that $z^2 + zw + w^2 = 1, \neq 0, 2$. Then, it follows that $x^2 + y^2 = 3$, which is impossible.

In order to establish the “if” direction, we need only prove that $f_{3}^{1,b}$ can represent $n = 3l + 2$ or $9l + 6$, with $l \in \mathbb{N}_0$.

Suppose that $n = 3l + 2$. We first consider the case where $l \equiv 0 \mod 3$. It is obvious that $f_{3}^{1,2}$ can represent $n = 2$. When $l \geq 1$ and $l \equiv 0 \mod 3$, we obtain that

$$n - 2 \cdot 2^2 = 3l + 2 - 8 = 3(l - 2),$$

which can be written as $x^2 + 3(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

When $l \equiv 1 \mod 3$, by taking $y = 1$, we have that

$$n - 2y^2 = 3l + 2 - 2 = 3l,$$

which can be written as $x^2 + 3(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

Assume that $l \equiv 2 \mod 3$. For $l = 2, 5$, or 8, we set $y = 2$ to obtain that

$$n - 2 \cdot 2^2 = 3l + 2 - 2 \cdot 2^2 = 0, 9, 9 \cdot 2,$$

which can be represented as $x^2 + 3(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. If $l \geq 11$ and $l = 3L + 2$ with $L \in \mathbb{N}_0$, then by taking $y = 4$ we have that

$$n - 2 \cdot 4^2 = 3l + 2 - 2 \cdot 4^2 = 3(L + 1),$$

which can be written as $x^2 + 3(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

Finally, we suppose that $n = 9l + 6$. By taking $y = 1$, we have that

$$n - 2y^2 = 9l + 6 - 2 = 9l + 4 \equiv 1 \mod 3,$$

which can be represented as $x^2 + 3(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. 

From the proof of Theorem 8, we obtain the following theorem:

Theorem 9. If $f_{3}^{a,b}$ represents 1, 2, and 6, then it can represent all the nonnegative integers.
6. The Case Where $c = 4$

We use the following result that was proved by Dickson [3, pp. 112-113]:

**Lemma 5.** A nonnegative integer $n$ can be written as $x^2 + 4y^2 + 12z^2$ with $x, y, z \in \mathbb{Z}$ if and only if $n \neq 4l + 2, 4l + 3, \text{ or } 9^k(9l + 6)$, with $k, l \in \mathbb{N}_0$.

Using Lemma 5, we obtain the following proposition:

**Proposition 4.** A nonnegative integer $n$ can be written as $x^2 + 4(y^2 + yz + z^2)$ with $x, y, z \in \mathbb{Z}$ if and only if $n \neq 4l + 2, 4l + 3, \text{ or } 9^k(9l + 6)$, with $k, l \in \mathbb{N}_0$.

**Proof.** By replacing $q$ by $q^4$ in (1), we have that

$$a(q^4) = a(q^{16}) + 6q^4 \psi(q^8)\psi(q^{24}),$$

$$\varphi(q^4)\varphi(q^{12}) = a(q^{16}) + 2q^4 \psi(q^8)\psi(q^{24}).$$

By multiplying both sides of these equations by $\varphi(q)$, we obtain that

$$\varphi(q)a(q^4) = \varphi(q)a(q^{16}) + 6q^4 \varphi(q)\psi(q^8)\psi(q^{24}),$$

$$\varphi(q)\varphi(q^4)\varphi(q^{12}) = \varphi(q)a(q^{16}) + 2q^4 \varphi(q)\psi(q^8)\psi(q^{24}),$$

which implies that

$$\sum_{n=0}^{\infty} A_1^1(n)q^n = \sum_{n=0}^{\infty} A_1^1(n)q^n + 6q^4 \sum_{N=0}^{\infty} m_{1,8,24}(N)q^N, \quad (8)$$

$$\sum_{n=0}^{\infty} r_{1,4,12}(n)q^n = \sum_{n=0}^{\infty} A_1^1(n)q^n + 2q^4 \sum_{N=0}^{\infty} m_{1,8,24}(N)q^N. \quad (9)$$

In our notation, Lemma 5 states that $n \neq 4l + 2, 4l + 3, 9^k(9l + 6)$, $(k, l \in \mathbb{N}_0)$ if and only if $r_{1,4,12}(n) > 0$. By equating coefficients in (9), we see that Lemma 5 is equivalent to $A_1^1(n) > 0$ or $m_{1,8,24}(n - 4) > 0$, which in turn is equivalent to $A_1^1(n) > 0$ from (8). Therefore, as $A_1^1(n)$ is the number of ways $n$ can be represented by $x^2 + 4(y^2 + yz + z^2)$, the proposition follows. \qed

Noting that $a = 1$ and $b = 2$ if $f_{4}^{a,b}$ can represent all the nonnegative integers, we obtain the following theorem:

**Theorem 10.** Any nonnegative integer $n$ can be represented by $f_{4}^{a,b}$ if and only if $a = 1$ and $b = 2$.

**Proof.** We first treat the “only if” direction. By choosing $n = 1$, we see that $a = 1$. By taking $n = 2$, we find that $b = 1, 2$. Moreover, by choosing $n = 3$, we see that $b = 2$. 
In order to establish the “if” direction, we need only prove that $f_{4}^{1,2}$ represents $n = 4l + 2$, $4l + 3$, and $9l + 6$, with $l \in \mathbb{N}_0$.

Suppose that $n = 4l + 2$. If $l \not\equiv 0$ or $6$ mod $9$, then by taking $y = 1$ we have that
\[ n - 2y^2 = 4l + 2 - 2 = 4l, \]
which can be represented as $x^2 + 4(z^2 + zw + w^2)$, with $x, z, w \in \mathbb{Z}$. If $l \equiv 0$ or $6$ mod $9$, then by taking $y = 3$ we obtain that
\[ n - 2y^2 = 4l + 2 - 2 \cdot 3^2 = 4(l - 4), \]
which can be represented as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

Suppose that $n = 4l + 3$. If $l \not\equiv 2$ or $8$ mod $9$, then by taking $y = 1$ we have that
\[ n - 2y^2 = 4l + 3 - 2 = 4l + 1 \not\equiv 0, 6$ mod $9, \]
which can be represented as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. If $l \equiv 2$ or $8$ mod $9$, then by taking $y = 3$ we obtain that
\[ n - 2y^2 = 4l + 3 - 2 \cdot 3^2 = 4(l - 4) + 1 \equiv 2, 8$ mod $9, \]
which can be represented as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. It is easy to check that $f_{4}^{1,2}$ can represent $n = 11$.

Suppose that $n = 9l + 6$. If $l \equiv 0$ mod $4$ and $l = 4L$ with $L \in \mathbb{N}_0$, then by taking $y = 5$ we have that
\[ n - 2y^2 = 9l + 6 - 2 \cdot 5^2 = 4\{9(L - 2) + 7\}, \]
which can be written as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. It is easy to check that $f_{4}^{1,2}$ can represent $n = 6$ or $42$.

If $l \equiv 1$ mod $4$, then by taking $y = 1$ we have that
\[ n - 2y^2 = 9l + 6 - 2 \cdot 1^2 = 9l + 4 \equiv 1$ mod $4, \]
which can be represented as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$.

If $l \equiv 2$ mod $4$ and $l = 4L + 2$ with $L \in \mathbb{N}_0$, then by taking $y = 4$ we have that
\[ n - 2y^2 = 9l + 6 - 2 \cdot 4^2 = 4(9L - 2), \]
which can be written as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. It is easy to check that $f_{4}^{1,2}$ can represent $n = 24$.

If $l \equiv 3$ mod $4$, then by taking $y = 2$ we have that
\[ n - 2y^2 = 9l + 6 - 2 \cdot 2^2 = 9l - 2 \equiv 1$ mod $4, \]
which can be represented as $x^2 + 4(z^2 + zw + w^2)$ with $x, z, w \in \mathbb{Z}$. \hfill \Box

From the proof of Theorem 10, we obtain the following result:

**Theorem 11.** If $f_{4}^{a,b}$ represents $1, 2,$ and $3$, then it can represent all the nonnegative integers.
7. The Case Where $c \geq 5$

7.1. The Case Where $c = 5$

**Theorem 12.** There exist no positive integers $a$ and $b$ such that $1 \leq a \leq b$ and 
\[
    f_{5}^{a,b}(x, y, z, w) = ax^2 + by^2 + 5(z^2 + zw + w^2).
\]

*Proof.* Suppose that there exist such positive integers $a$ and $b$. By considering $n = 1$, we have that $a = 1$. Taking $n = 2$, we obtain that $b = 1$ or 2. Considering $n = 3$, we have that $(a, b) = (1, 2)$.

Take $n = 10$. Then, $n = 10$ cannot be written as $x^2 + 2y^2$ with $x, y \in \mathbb{Z}$, which implies that $z^2 + zw + w^2 = 1$, because 2 cannot be represented as $x^2 + 2y^2$ with $x, y \in \mathbb{Z}$. Therefore, it follows that $5 = 10 - 5 \cdot 1$ can be written as $x^2 + 2y^2$ with $x, y \in \mathbb{Z}$, which is impossible. \hfill \Box

7.2. The Case Where $c \geq 6$

**Theorem 13.** There exist no positive integers $a, b,$ and $c$ such that $1 \leq a \leq b$, $c \geq 6$, and $f_{c}^{a,b}$ can represent all the nonnegative integers.

*Proof.* Suppose that there exist such positive integers $a, b,$ and $c$. By considering $n = 1$, we have that $a = 1$. Taking $n = 2$, we obtain that $b = 1$ or 2. Considering $n = 3$, we have that $(a, b) = (1, 2)$. On the other hand, $n = 5$ cannot be written as $x^2 + 2y^2$ with $x, y \in \mathbb{Z}$. \hfill \Box

8. Proof of Theorem 1

*Proof.* Suppose that $f_{c}^{a,b}$ can represent all the nonnegative integers. Then, Theorems 12 and 13 show that $c = 1, 2, 3, 4$. If $c = 1$, Theorem 4 implies that

\[
    (a, b) = \begin{cases} 
    (1, b) & (b = 1, 2, 3, 4, 5, 6), \\
    (2, b) & (b = 2, 3, 4, 5, 6, 7, 8, 9, 10).
    \end{cases}
\]

If $c = 2$, Theorem 6 shows that

\[
    (a, b) = (1, 1), (1, 2), (1, 3), (1, 4), (1, 5).
\]

If $c = 3$, Theorem 8 implies that $(a, b) = (1, 2)$. If $c = 4$, Theorem 10 implies that $(a, b) = (1, 2)$. Therefore, the “only if” direction of Theorem 1 (1) follows. The “if” direction follows from Theorems 4, 6, 8, and 10.

We next prove Theorem 1 (2). From the proofs of Theorems 12, and 13, taking $n = 1, 2, 3, 5, 10$, we see that $1 \leq c \leq 4$. 
If \( c = 1 \), from Theorem 4, choosing \( n = 1, 2, 6, 10 \) implies that \( f_{1}^{a,b} \) represents all the nonnegative integers. If \( c = 2 \), from Theorem 7, taking \( n = 1, 5 \) shows that \( f_{2}^{a,b} \) represents all the nonnegative integers. If \( c = 3 \), from Theorem 9, choosing \( n = 1, 2, 6 \) implies that \( f_{3}^{a,b} \) represents all the nonnegative integers. If \( c = 4 \), from Theorem 11, taking \( n = 1, 2, 3 \) shows that \( f_{4}^{a,b} \) represents all the nonnegative integers. Therefore, Theorem 1 (2) follows.

\[ \square \]

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**References**


