ON RELATED BINOMIAL HARMONIC IDENTITIES

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Abstract

We develop new closed form representation of a positive sum of reciprocal binomial coefficients and its alternating counterpart. We also identify new integral and hypergeometric representations for the binomial-harmonic number sums.

1. Introduction

In this paper we are interested in the closed form expressions of the two related binomial sums

$$S(j,k,q)=\sum_{n=1}^{\infty}\frac{1}{\binom{qkn+j}{k}},$$

and its alternating counterpart

$$A(j,k,q)=\sum_{n=1}^{\infty}\frac{(-1)^n}{\binom{qkn+j}{k}}.$$

The results in this paper add to the literature and are sums that are not considered in the recently published research results of [1] and [2]. Moreover, the work in this paper generalizes and extends the work of [10]. First we recall some definitions of some special functions that will be useful throughout this paper. The Gamma function, for $z\in\mathbb{C}$, as given by Euler in integral form, is

$$\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt,\ \Re(z)>0,$$

where the special case for $z\in\mathbb{N}$ reduces to, from the recurrence relation, $\Gamma(n+1)=n\Gamma(n)=n!$. The Pochhammer, or shifted factorial, is defined by $(\lambda)_\nu=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$.

[1] Lovingly dedicated to Anthony Kara

The Beta function, or Euler integral of the first kind, is

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},\ \Re(z) > 0, \Re(w) > 0.$$

Let

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \int_0^1 \frac{1-t^n}{1-t}\,dt = \gamma + \psi(n+1) = \sum_{j=1}^{\infty} \frac{n}{j(j+n)}, \qquad H_0 := 0$$

be the nth harmonic number, where γ denotes the Euler-Mascheroni constant, $H_n^{(m)} = \sum_{r=1}^{n} \frac{1}{r^m}$ is the m^{th} order harmonic number, and $\psi(z)$ is the Digamma (or Psi) function defined by

$$\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \text{ and } \psi(1+z) = \psi(z) + \frac{1}{z}.$$

Moreover

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right).$$

A generalized hypergeometric function is defined by

$$\begin{aligned} {}_pF_q[z] &= {}_pF_q\left[\left.\begin{array}{c} a_1, a_2, , ..., a_p \\ b_1, b_2, , ..., b_q \end{array}\right| z\right] = {}_pF_q\left[(a_p);(b_q) \mid z\right] \\ &= \sum_{n\geq 0} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!} = \sum_{n\geq 0} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{z^n}{n!} \end{aligned} \tag{1}$$

for b_j non-negative integers or zero. When $p \leq q$, ${}_pF_q[z]$ converges for all complex values of z, ${}_pF_q[z]$ is an entire function. When $p > q+1$, ${}_pF_q[z]$ converges for $z = 0$ unless it terminates, which it does when one of the parameters a_j is a negative integer, and hence ${}_pF_q[z]$ is a polynomial in z. When $p = q+1$ the series converges in the unit disc $|z| < 1$, and also for $|z| = 1$ provided that $\Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0$. When $p = 2, q = 1$ we have the familiar Gauss hypergeometric function

$${}_2F_1\left[\left.\begin{array}{c} a, b \\ c \end{array}\right| z\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}dt,$$

where $|z| < 1$, $\Re(c-b) > 0$ and $\Re(b) > 0$. In the subsequent analysis we shall also employ the consecutive derivative operator of the inverse binomial,

$$\frac{d^{(m)}}{dj^{(m)}}\left(\binom{qkn+j}{k}^{-1}\right),$$

and investigate the resulting binomial harmonic sum

$$S^{(m)}(j,k,q)=\sum_{n=1}^{\infty}\frac{d^{(m)}}{dj^{(m)}}\left(\binom{qkn+j}{k}^{-1}\right)$$

and its alternating counterpart

$$A^{(m)}(j,k,q)=\sum_{n=1}^{\infty}(-1)^{n}\frac{d^{(m)}}{dj^{(m)}}\left(\binom{qkn+j}{k}^{-1}\right).$$

The following lemma will be useful in the development of the main theorem.

Lemma 1. *Let $p(n)$ and $q(n)$ be polynomials in n where all the roots of $q(n)$ are simple. Assume no root of $q(n)$ is in $\mathbb{N}$ and let $\deg(p(n))\le\deg(q(n)-2)$. Let $v_n=\frac{p(n)}{q(n)}$. Then*

$$\sum_{n=0}^{\infty}v_n=-\sum_{r=1}^{k}\alpha_r\psi(\beta_r) \tag{2}$$

where

$$v_n=\frac{p(n)}{q(n)}=\sum_{r=1}^{k}\frac{\alpha_r}{n+\beta_r}. \tag{3}$$

Proof. From $v_n=\frac{p(n)}{q(n)}$ we have $\sum\limits_{n=0}^{\infty}v_n=\sum\limits_{n=0}^{\infty}\frac{p(n)}{q(n)}$. By partial fraction expansion, $v_n=\sum_{r=1}^{k}\frac{\alpha_r}{n+\beta_r}$ since all the roots of $q(n)$ are simple. For the series $\sum\limits_{n=0}^{\infty}v_n$ to converge it suffices to have $\lim\limits_{n\to\infty}nv_n=0$, in which case $\sum_{r=1}^{k}\alpha_r=0$. Now

$$\begin{aligned}\sum_{n=0}^{\infty}v_n=\sum_{n=0}^{\infty}\sum_{r=1}^{k}\frac{\alpha_r}{n+\beta_r} &= \sum_{n=0}^{\infty}\sum_{r=1}^{k}\alpha_r\left(\frac{1}{n+\beta_r}-\frac{1}{n+1}\right)\\ &= \sum_{r=1}^{k}\alpha_r\sum_{n=0}^{\infty}\left(\frac{1}{n+\beta_r}-\frac{1}{n+1}\right)\\ &= -\sum_{r=1}^{k}\alpha_r\left(\gamma+\psi(\beta_r)\right)\\ &= -\sum_{r=1}^{k}\alpha_r\psi(\beta_r)\end{aligned}$$

and the lemma is proved. □

2. Closed Form Summation

We now prove the following theorems.

Theorem 1. *Let* $k \in \mathbb{N}\backslash\{1\}$, $j \in (-1,\infty)$ *and* $q \in R^{+}$, *then we have the novel representation*

$$S(j,k,q)=\sum_{n=1}^{\infty}\frac{1}{\binom{qkn+j}{k}}=\frac{1}{q}\sum_{r=1}^{k}(-1)^{r+k+1}\binom{k-1}{r-1}\psi(\frac{qk+1-r+j}{qk}). \tag{4}$$

The case $j=0$ *reduces to*

$$S(0,k,q)=\sum_{n=1}^{\infty}\frac{1}{\binom{qkn}{k}}=\frac{1}{q}\sum_{r=1}^{k}(-1)^{r+k+1}\binom{k-1}{r-1}\psi(\frac{qk+1-r}{qk}). \tag{5}$$

Also

$$S(0,k,q)=\frac{1}{q}\sum_{r=1}^{k}(-1)^{r+k+1}\binom{k-1}{r-1}\begin{pmatrix}\psi(\frac{r-1}{qk})\\ +\pi\cot\left(\frac{\pi(r-1)}{qk}\right)\end{pmatrix}. \tag{6}$$

Proof. Consider the expansion,

$$\begin{aligned}S(j,k,q)&=\sum_{n=1}^{\infty}\frac{1}{\binom{qkn+j}{k}}=\sum_{n=1}^{\infty}\frac{k!\ (qkn-k+j)!}{(qkn+j)!}\\ &= k!\sum_{n=1}^{\infty}\frac{1}{\prod_{r=1}^{k}(qkn+j+1-r)}=k!\sum_{n=1}^{\infty}\frac{1}{(qkn+j+1-k)_k}\\ &= \sum_{n=0}^{\infty}\frac{1}{\prod_{r=1}^{k}\left(n+1+\frac{qk+j+1-r}{qk}\right)},\end{aligned}$$

where Pochhammer's symbol $(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)}$. By partial fraction decomposition we have

$$S(j,k,q)=\frac{1}{q}\sum_{r=1}^{k}(-1)^{r+k+1}\binom{k-1}{r-1}\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+\frac{qk+1+j-r}{qk}}\right),$$

and applying Lemma 1 we conclude

$$S(j,k,q) = \frac{1}{q}\sum_{r=1}^{k}(-1)^{r+k+1}\binom{k-1}{r-1}\psi\left(\frac{qk+1+j-r}{qk}\right)$$

and (4) follows. For $j=0$ (5) follows. By the reflection formula of the Digamma function $\psi(1-z)=\psi(z)+\pi\cot\pi z$, then (6) follows. Utilizing the definition (1), for the hypergeometric function, we can write

$$S(j,k,q) = \frac{1}{\binom{kq+j}{k}}\ {}_{kq+1}F_{kq}\left[\begin{matrix}\frac{kq+j+1-k}{kq},\frac{kq+j+2-k}{kq},...,\frac{2kq+j-k}{kq},1\\ \frac{kq+j+1}{kq},\frac{kq+j+2}{kq},...,\frac{2kq+j}{kq}\end{matrix}\middle| 1\right].$$

□

We remark that in the interesting paper [10], Nimbran considers the representation of

$$\frac{1}{k!}S(0,k,1) = \sum_{n=1}^{\infty}\frac{(nk-k)!}{(nk)!},$$

for $k\in\mathbb{N}\backslash\{1\}$, in closed form and evaluates $S(0,k,1)$ for $k=\{2,3,4,5,6,8,10,12\}$. In particular $S(0,2,1)=\ln 2$ is listed in [7], $S(0,3,1)=\frac{\sqrt{3}\pi}{12}-\frac{1}{4}\ln 3$ and $S(0,4,1)=\frac{1}{4}\ln 2-\frac{\pi}{24}$ are listed in [9]. Nimbran's search of the literature yields no other evaluation of $S(0,k,1)$ for $k\geq 5$ and then sets out to evaluate $S(0,k,1)$ for $k=\{5,6,8,10,12\}$. Nimbran claims $S(0,10,1)$ is difficult to evaluate and finds it impossible to evaluate $S(0,k,1)$ for any other values of k. Nimbran's method of evaluating $S(0,k,1)$ is indeed ingenious and relies on the representation

$$\ln p = \sum_{m\geq 1}\left(\sum_{r=1}^{p-1}\left(\frac{1}{mp+r-m}-\frac{1}{mp}\right)\right),$$

which is a generalization of an identity given by Euler in 1734 [6]. We therefore see that the representation (4) gives a general identity for $S(j,k,q)$ for every $k\in\mathbb{N}\backslash\{1\}$. Also recently [11], using a generalized binomial theorem in terms of Bell polynomials, evaluates some sums involving inverse binomial coefficients. The same technique is also used to calculate a class of hypergeometric transformation formulas, hence, there is still interest in evaluating binomial sums.

The following corollary applies.

Corollary 1. *Let $k\in\mathbb{N}\backslash\{1\}$, $j\in(-1,\infty)$ and $q\in R^{+}$, then we have the representation*

$$T(j,k,q) = \sum_{n=1}^{\infty}\frac{1}{\binom{2qkn+j}{k}} = \frac{1}{2q}\sum_{r=1}^{k}(-1)^{r+k+1}\binom{k-1}{r-1}\psi\left(\frac{2qk+1-r+j}{2qk}\right).$$

The case $j = 0$ reduces to

$$T(0,k,q) = \sum_{n=1}^{\infty} \frac{1}{\binom{2qkn}{k}} = \frac{1}{2q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \psi\left(\frac{2qk+1-r}{2qk}\right).$$

Also

$$T(0,k,q) = \frac{1}{2q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \left(\begin{array}{c} \psi\left(\frac{r-1}{2qk}\right) \\ +\pi \cot\left(\frac{\pi(r-1)}{2qk}\right) \end{array} \right).$$

Proof. The proof follows directly from Theorem 1. □

Now we investigate $A(j,k,q)$.

Theorem 2. *Under the assumptions of Theorem 1,*

$$\begin{aligned} A(j,k,q) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{\binom{qkn+j}{k}} \\ &= \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \left(\begin{array}{c} \psi\left(\frac{2qk+1-r+j}{2qk}\right) \\ -\psi\left(\frac{qk+1-r+j}{qk}\right) \end{array} \right). \end{aligned} \tag{7}$$

The case $j = 0$ reduces to

$$A(0,k,q) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\binom{qkn}{k}} = \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \left(\begin{array}{c} \psi\left(\frac{2qk+1-r}{2qk}\right) \\ -\psi\left(\frac{qk+1-r}{qk}\right) \end{array} \right). \tag{8}$$

Also,

$$A(0,k,q) = \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \left(\begin{array}{c} \psi\left(\frac{r-1}{2qk}\right) - \psi\left(\frac{r-1}{qk}\right) \\ +\pi \cot\left(\frac{\pi(r-1)}{2qk}\right) - \pi \cot\left(\frac{\pi(r-1)}{qk}\right) \end{array} \right). \tag{9}$$

Proof. We begin by noticing that

$$S(j,k,q) + A(j,k,q) = 2T(j,k,q),$$

from which $A(j,k,q)$ follows directly, as does $A(0,k,q)$. The identity (9) follows from an application of the Digamma reflection formula. Utilizing the definition (1), for the hypergeometric function, we can write

$$A(j,k,q) = -\frac{1}{\binom{kq+j}{k}}\ {}_{kq+1}F_{kq}\left[\left.\begin{array}{c} \frac{kq+j+1-k}{kq}, \frac{kq+j+2-k}{kq}, \ldots, \frac{2kq+j-k}{kq}, 1 \\ \frac{kq+j+1}{kq}, \frac{kq+j+2}{kq}, \ldots, \frac{2kq+j}{kq} \end{array}\right| -1\right].$$

□

Remark 1. The other notable case is for the situation of $j = qk$, from which we ascertain, in a straightforward manner, the following results:

$$S(qk,k,q) = S(0,k,q) - \frac{1}{\binom{qk}{k}},$$

$$A(qk,k,q) = -A(0,k,q) - \frac{1}{\binom{qk}{k}}$$

and

$$\begin{aligned} S(qk,k,q) - A(qk,k,q) &= S(0,k,q) - A(0,k,q) \\ &= 2T(0,k,q). \end{aligned}$$

Example 1. Some examples follow:

$$S\left(j,k,\frac{1}{k}\right) = \frac{j+1}{(k-1)\binom{j+1}{k}},$$

$$S(2,2,6) = \frac{\pi}{6} - 1 - \frac{1}{\sqrt{3}}\ln\left(\sqrt{3}-1\right) + \frac{\sqrt{3}+1}{6}\ln 2,$$

$$\begin{aligned} S\left(-\frac{1}{8},3,\frac{1}{4}\right) &= \frac{496}{55} + 12\ln 3,\ T\left(\frac{3}{2},4,\frac{1}{4}\right) = 8\pi - \frac{64}{3}, \\ T\left(0,2,\frac{3}{4}\right) &= \ln 3 - \frac{\sqrt{3}\pi}{9},\ T(4,2,3) = \frac{1}{2}\ln 2 - \frac{1}{4}\ln 3 + \frac{\sqrt{3}-1}{12\sqrt{3}}\pi - \frac{1}{6}; \end{aligned}$$

$$
\begin{aligned}
A\left(4,3,2\right) &= \frac{2-\sqrt{3}}{2\sqrt{3}}\pi-\frac{1}{4},\ A\left(-\frac{1}{2},2,1\right)=-2\sqrt{2}\ln\left(\sqrt{2}+1\right),\\
A\left(j,2,\frac{1}{4p}\right) &= 4p\left(H_{jp}-H_{jp-p}-\left(H_{2jp}-H_{2jp-2p}\right)\right),\ p\in\mathbb{R}^{+}.
\end{aligned}
$$

In the next section we give an extension to Theorem 1 by incorporating harmonic numbers to the sums $S\left(j,k,q\right)$, $A\left(j,k,q\right)$ and associating the sum with hypergeometric and integral representation.

3. Extension

We begin with the proof of the following Theorem.

Theorem 3. *Let the assumptions of Theorem 1 apply and let $m\in\mathbb{N}$. Then,*

$$
\begin{aligned}
S^{(m)}\left(j,k,q\right) &= \sum_{n=1}^{\infty}\frac{d^{(m)}}{dj^{(m)}}\left(\binom{qkn+j}{k}^{-1}\right)=\sum_{n=1}^{\infty}Q^{(m)}\left(j,k,q\right)\\
&= \frac{1}{k^{m}q^{m+1}}\sum_{r=1}^{k}(-1)^{r+k+1}\binom{k-1}{r-1}\psi^{(m)}\left(\frac{qk+1+j-r}{qk}\right)
\end{aligned}
\tag{10}
$$

$$
=\frac{m!}{k^{m}q^{m+1}}\sum_{r=1}^{k}(-1)^{r+m+k+1}\binom{k-1}{r-1}H^{(m+1)}_{\frac{1+j-r}{qk}} \tag{11}
$$

where

$$
Q^{(m)}\left(j,k,q\right)=\frac{d^{(m)}}{dj^{(m)}}\left(\binom{qkn+j}{k}^{-1}\right). \tag{12}
$$

Proof. From the identity (4) we differentiate both sides ”m” times with respect to j so that,

$$
\begin{aligned}
S^{(m)}\left(j,k,q\right) &= \sum_{n=1}^{\infty}\frac{d^{(m)}}{dj^{(m)}}\left(\binom{qkn+j}{k}^{-1}\right)=\sum_{n=1}^{\infty}Q^{(m)}\left(j,k,q\right)\\
&= \frac{1}{k^{m}q^{m+1}}\sum_{r=1}^{k}(-1)^{r+k+1}\binom{k-1}{r-1}\psi^{(m)}\left(\frac{qk+1+j-r}{qk}\right)
\end{aligned}
$$

and (10) follows. From the known identity relating polygamma functions with harmonic numbers,

$$
\psi^{(m)}(1+z)=(-1)^{m}\,m!\left(H^{(m+1)}_{z}-\zeta\left(1+m\right)\right), \tag{13}
$$

we have

$$S^{(m)}(j,k,q)=\frac{m!}{k^m q^{m+1}}\sum_{r=1}^{k}(-1)^{r+m+k+1}\binom{k-1}{r-1}H^{(m+1)}_{\frac{1+j-r}{qk}}$$

since

$$\sum_{r=1}^{k}(-1)^{r}\binom{k-1}{r-1}=0,\text{ for }k\geq 2,$$

and hence (11) follows. For completeness we detail some values of $Q^{(m)}(j,k,q)$:

$$Q^{(1)}(j,k) = \frac{1}{\binom{qkn+j}{k}}\left(H_{qkn+j-k}-H_{qkn+j}\right)$$

and

$$Q^{(2)}(j,k,q)=\frac{1}{\binom{qkn+j}{k}}\left(\left(H_{qkn+j}-H_{qkn+j-k}\right)^2+\left(H^{(2)}_{qkn+j}-H^{(2)}_{qkn+j-k}\right)\right).$$

Some more details on the function $Q^{(m)}(j,k,q)$ are given in the paper [13]. □

Next we investigate $A^{(m)}(j,k,q)$ as described below.

Theorem 4. *Let the assumptions of Theorem 1 apply and let $m\in\mathbb{N}$. Then,*

$$A^{(m)}(j,k,q) = \sum_{n=1}^{\infty}(-1)^n\frac{d^{(m)}}{dj^{(m)}}\left(\binom{qkn+j}{k}^{-1}\right)=\sum_{n=1}^{\infty}(-1)^n Q^{(m)}(j,k,q) \tag{14}$$

$$= \frac{1}{k^m q^{m+1}}\sum_{r=1}^{k}(-1)^{r+k+1}\binom{k-1}{r-1}\left(\begin{array}{c}\frac{1}{2^m}\psi^{(m)}\left(\frac{2qk+1+j-r}{2qk}\right)\\ -\psi^{(m)}\left(\frac{qk+1+j-r}{qk}\right)\end{array}\right) \tag{15}$$

$$=\frac{m!}{k^m q^{m+1}}\sum_{r=1}^{k}(-1)^{r+m+k+1}\binom{k-1}{r-1}\left(\frac{1}{2^m}H^{(m+1)}_{\frac{1+j-r}{2qk}}-H^{(m+1)}_{\frac{1+j-r}{qk}}\right), \tag{16}$$

where $Q^{(m)}(j,k,q)$ is given by (12).

Proof. From the identity (7) we differentiate both sides "m" times with respect to j so that,

$$\begin{aligned} A^{(m)}(j,k,q) &= \sum_{n=1}^{\infty}(-1)^n \frac{d^{(m)}}{dj^{(m)}}\left(\left(\begin{array}{c} qkn+j \\ k \end{array}\right)^{-1}\right) = \sum_{n=1}^{\infty}(-1)^n Q^{(m)}(j,k,q) \\ &= \frac{1}{k^m q^{m+1}}\sum_{r=1}^{k}(-1)^{r+k+1}\left(\begin{array}{c} k-1 \\ r-1 \end{array}\right)\left(\begin{array}{c} \frac{1}{2^m}\psi^{(m)}(\frac{2qk+1+j-r}{2qk}) \\ -\psi^{(m)}(\frac{qk+1+j-r}{qk}) \end{array}\right), \end{aligned}$$

and (15) follows. From the known identity (13), relating polygamma functions with harmonic numbers, then

$$A^{(m)}(j,k,q) = \frac{m!}{k^m q^{m+1}}\sum_{r=1}^{k}(-1)^{r+m+k+1}\left(\begin{array}{c} k-1 \\ r-1 \end{array}\right)\left(\frac{1}{2^m}H^{(m+1)}_{\frac{1+j-r}{2qk}} - H^{(m+1)}_{\frac{1+j-r}{qk}})\right)$$

since,

$$\sum_{r=1}^{k}(-1)^r\left(\begin{array}{c} k-1 \\ r-1 \end{array}\right) = 0, \text{ for } k \geq 2$$

and (16) is attained. ☐

The cases $j=0$ and $j=kq$ are interesting and the results are given in the next corollary.

Corollary 2. *For* $j=0$,

$$\begin{aligned} S^{(m)}(0,k,q) &= \sum_{n=1}^{\infty}\lim_{j\to 0}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\left(\begin{array}{c} kqn+j \\ k \end{array}\right)^{-1}\right)\right) = \sum_{n=1}^{\infty} Q^{(m)}(0,k,q) \\ &= \frac{(-1)^{k+m+1}m!}{k^m q^{m+1}}\zeta(m+1) \\ &\quad + \frac{1}{k^m q^{m+1}}\sum_{r=1}^{k-1}(-1)^{r+k}\left(\begin{array}{c} k-1 \\ r \end{array}\right)\psi^{(m)}(\frac{qk-r}{qk}). \end{aligned} \tag{17}$$

For $j=qk$,

$$S^{(m)}(kq,k,q) = \sum_{n=1}^{\infty}\lim_{j\to kq}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\left(\begin{array}{c} kqn+j \\ k \end{array}\right)^{-1}\right)\right)$$

$$\begin{aligned} &= \frac{(-1)^{k+m}m!}{k^m q^{m+1}}(1-\zeta(m+1)) \\ &\quad + \frac{1}{k^m q^{m+1}}\sum_{r=1}^{k-1}(-1)^{r+k}\left(\begin{array}{c} k-1 \\ r \end{array}\right)\psi^{(m)}(\frac{2qk-r}{qk}) \end{aligned} \tag{18}$$

Proof. From (10) we have

$$\begin{aligned} S^{(m)}(0,k,q) &= \sum_{n=1}^{\infty} \lim_{j\to 0}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\binom{kqn+j}{k}^{-1}\right)\right) \\ &= \frac{1}{k^m q^{m+1}} \sum_{r=1}^{k} (-1)^{r+k+1}\binom{k-1}{r-1}\psi^{(m)}(1+\frac{1-r}{qk}). \end{aligned}$$

At $r=1$, we notice that $\psi^{(m)}(1) = (-1)^{m+1} m!\zeta(m+1)$ and hence

$$\begin{aligned} S^{(m)}(0,k,q) &= \frac{(-1)^{k+m+1} m!}{k^m q^{m+1}}\zeta(m+1) \\ &\quad + \frac{1}{k^m q^{m+1}} \sum_{r=2}^{k} (-1)^{r+k+1}\binom{k-1}{r-1}\psi^{(m)}(1+\frac{1-r}{qk}) \end{aligned}$$

and, making a change in the summation index, then (17) follows. For the case $j=qk$,

$$\begin{aligned} S^{(m)}(kq,k,q) &= \sum_{n=1}^{\infty} \lim_{j\to kq}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\binom{kqn+j}{k}^{-1}\right)\right) \\ &= \frac{1}{k^m q^{m+1}} \sum_{r=1}^{k} (-1)^{r+k+1}\binom{k-1}{r-1}\psi^{(m)}(\frac{2qk+1-r}{qk}). \end{aligned}$$

At $r=1$, we notice that $\psi^{(m)}(2) = (-1)^m m!(1-\zeta(m+1))$. Hence

$$\begin{aligned} S^{(m)}(kq,k,q) &= \frac{(-1)^{k+m} m!}{k^m q^{m+1}}(1-\zeta(m+1)) \\ &\quad + \frac{1}{k^m q^{m+1}} \sum_{r=2}^{k} (-1)^{r+k+1}\binom{k-1}{r-1}\psi^{(m)}(2+\frac{1-r}{qk}), \end{aligned}$$

and making a change in the summation index, then (18) follows. □

The following corollary refers to the $A^{(m)}(j,k,q)$, for the two special cases of $j=0$ and $j=qk$.

Corollary 3. *For $j=0$,*

$$\begin{aligned} A^{(m)}(0,k,q) &= \sum_{n=1}^{\infty} (-1)^n \lim_{j\to 0}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\binom{kqn+j}{k}^{-1}\right)\right) = \sum_{n=1}^{\infty} Q^{(m)}(0,k,q) \\ &= \frac{(-1)^{k+m+1} m!(1-2^m)}{q(2kq)^m}\zeta(m+1) \qquad (19) \end{aligned}$$

$$+\frac{1}{k^m q^{m+1}}\sum_{r=1}^{k-1}(-1)^{r+k}\binom{k-1}{r}\left(\frac{1}{2^m}\psi^{(m)}(\frac{2qk-r}{2qk})-\psi^{(m)}(\frac{qk-r}{qk})\right).$$

For $j = qk$,

$$A^{(m)}(kq,k,q)=\sum_{n=1}^{\infty}(-1)^n\lim_{j\to kq}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\binom{kqn+j}{k}^{-1}\right)\right)$$

$$=\frac{(-1)^{k+m}m!}{k^m q^{m+1}}\left(1+\left(2^{-m}-1\right)\zeta(m+1)\right) \tag{20}$$

$$+\frac{1}{k^m q^{m+1}}\sum_{r=1}^{k-1}(-1)^{r+k}\binom{k-1}{r}\left(\frac{1}{2^m}\psi^{(m)}(\frac{3qk-r}{2qk})-\psi^{(m)}(\frac{2qk-r}{qk})\right).$$

Proof. The proof follows the same pattern as the proof in the previous corollary. □

Example 2. Some illustrative examples follow:

$$S^{(2)}(j,3,q) = \sum_{n=1}^{\infty}\frac{\left(H_{3qn+j}-H_{3qn+j-3}\right)^2+\left(H^{(2)}_{3qn+j}-H^{(2)}_{3qn+j-3}\right)}{\binom{3qn+j}{3}}$$

$$= -\frac{1}{9q^3}\left(\psi^{(2)}(\frac{j-2+3q}{3q})-2\psi^{(2)}(\frac{j-1+3q}{3q})+\psi^{(2)}(\frac{j+3q}{3q})\right),$$

$$S^{(m)}(j,2,q) = \sum_{n=1}^{\infty}\frac{d^{(m)}}{dj^{(m)}}\binom{2qn+j}{2}^{-1}$$

$$= \frac{1}{2^m q^{m+1}}\left(\psi^{(m)}\left(\frac{2q+j}{2q}\right)-\psi^{(m)}\left(\frac{2q+j-1}{2q}\right)\right). \tag{21}$$

By the Leibniz rule of differentiation, we can also write,

$$S^{(m)}(j,2,q) = 2\sum_{n=1}^{\infty}\sum_{t=0}^{m}\frac{(-1)^m m!}{(2qn+j)^{t+1}(2qn+j-1)^{m-t+1}}.$$

Using the recurrence relation of the polygamma function,

$$\psi^{(m)}(1+z)=\psi^{(m)}(z)+\frac{(-1)^m m!}{z^{m+1}},$$

then from (21) we have

$$S^{(m)}(j,2,q) = 2\sum_{n=1}^{\infty}\sum_{t=0}^{m}\frac{(-1)^m m!}{(2qn+j)^{t+1}(2qn+j-1)^{m-t+1}}$$

$$= \frac{1}{2^m q^{m+1}}\left(\psi^{(m)}\left(\frac{j}{2q}\right) - \psi^{(m)}\left(\frac{j-1}{2q}\right)\right) + 2(-1)^m m!\left(\frac{1}{j^{m+1}} - \frac{1}{(j-1)^{m+1}}\right),$$

from which

$$\begin{aligned} S^{(4)}(4,2,6) &= 2\sum_{n=1}^{\infty}\sum_{t=0}^{4}\frac{4!}{(12n+4)^{t+1}(12n+3)^{5-t}} \\ &= \frac{125}{1728}\zeta(5) + \frac{\left(15-2\sqrt{3}\right)}{46656}\pi^5 - \frac{781}{5184}. \end{aligned}$$

We also have

$$\begin{aligned} S^{(m)}(2q,2,q) &= \sum_{n=1}^{\infty}\lim_{j\to 2q}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\binom{2qn+j}{2}^{-1}\right)\right) \\ &= \frac{(-1)^m m!}{2^m q^{m+1}}(1-\zeta(m+1)) - \frac{1}{2^m q^{m+1}}\psi^{(m)}(\frac{4q-1}{2q}) \end{aligned}$$

$$= 2\sum_{n=1}^{\infty}\sum_{t=0}^{m}\frac{(-1)^m m!}{(2qn+2q)^{t+1}(2qn+2q-1)^{m-t+1}}.$$

$$\begin{aligned} A^{(m)}(0,2,q) &= \sum_{n=1}^{\infty}(-1)^n\lim_{j\to 0}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\binom{2qn+j}{2}^{-1}\right)\right) \\ &= \frac{(-1)^{m+1} m!(1-2^m)}{4^m q^{m+1}}\zeta(m+1) - \frac{1}{4^m q^{m+1}}\psi^{(m)}(\frac{4q-1}{4q}) \\ &\quad + \frac{1}{2^m q^{m+1}}\psi^{(m)}(\frac{2q-1}{2q}) \end{aligned}$$

$$\begin{aligned} A^{(m)}(2q,2,q) &= \frac{(-1)^m m!(1+(2^m-1)\zeta(m+1))}{2^m q^{m+1}} - \frac{1}{4^m q^{m+1}}\psi^{(m)}(\frac{6q-1}{4q}) \\ &\quad + \frac{1}{2^m q^{m+1}}\psi^{(m)}(\frac{4q-1}{2q}) \end{aligned}$$

$$= 2\sum_{t=0}^{m}\sum_{n=1}^{\infty}\frac{(-1)^{n+m} m!}{(2qn+2q)^{t+1}(2qn+2q-1)^{m-t+1}}.$$

The expression $S^{(m)}(j,k,q)$ and $A^{(m)}(j,k,q)$ can also be represented in integral and hypergeometric form and for completeness the following is recorded.

Theorem 5. *Let the assumptions of Theorem 1 apply. Then,*

$$S^{(m)}(j,k,q) = k\int_0^1 \frac{x^{kq+j-k}(1-x)^{k-1}\ln^m x}{1-x^{kq}}dx, \tag{22}$$

$$A^{(m)}(j,k,q) = -k\int_0^1 \frac{x^{kq+j-k}(1-x)^{k-1}\ln^m x}{1+x^{kq}}dx. \tag{23}$$

Proof. Consider

$$\begin{aligned} S(j,k,q) &= \sum_{n=1}^{\infty}\frac{1}{\binom{kqn+j}{k}} = \sum_{n=1}^{\infty}\frac{\Gamma(kqn+j-k+1)\Gamma(k+1)}{\Gamma(kqn+j+1)} \\ &= k\sum_{n=1}^{\infty} B(k, kqn-k+j+1), \end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function and $B(\cdot,\cdot)$ is the beta function. Now

$$S(j,k,q) = k\int_0^1 \frac{x^j(1-x)^{k-1}}{x^k}\sum_{n=1}^{\infty}\left(x^{kq}\right)^n \,dx.$$

Differentiating m times with respect to j results in

$$S^{(m)}(j,k,q) = k\int_0^1 \frac{x^{kq+j-k}(1-x)^{k-1}\ln^m x}{1-x^{kq}}dx,$$

and hence (22). The result (23) follows by the same analysis. □

Remark 2. It is straightforward to see, from (17), (22) and (23), that

$$\begin{aligned} S^{(m)}(2q,2,q) &= \frac{(-1)^m m!}{2^m q^{m+1}}(1-\zeta(m+1)) - \frac{1}{2^m q^{m+1}}\psi^{(m)}\left(\frac{4q-1}{2q}\right) \\ &= 2\int_0^1 \frac{x^{4q-2}(1-x)\ln^m x}{1-x^{2q}}dx, \end{aligned}$$

$$\begin{aligned} A^{(m)}(2q,2,q) &= \frac{(-1)^m m!(1+(2^m-1)\zeta(m+1))}{2^m q^{m+1}} - \frac{1}{4^m q^{m+1}}\psi^{(m)}\left(\frac{6q-1}{4q}\right) \\ &\quad + \frac{1}{2^m q^{m+1}}\psi^{(m)}\left(\frac{4q-1}{2q}\right) \\ &= -2\int_0^1 \frac{x^{4q-2}(1-x)\ln^m x}{1+x^{2q}}dx. \end{aligned}$$

Many other examples of binomial sums, harmonic number sums, integral representations and hypergeometric summation are available in [3], [4], [5], [8], [12], [14], [15], [16], [17], [18] and [19].

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