ON RELATED BINOMIAL HARMONIC IDENTITIES

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Abstract
We develop new closed form representation of a positive sum of reciprocal binomial coefficients and its alternating counterpart. We also identify new integral and hypergeometric representations for the binomial-harmomic number sums.

1. Introduction
In this paper we are interested in the closed form expressions of the two related binomial sums

\[ S(j, k, q) = \sum_{n=1}^{\infty} \frac{1}{(qkn + j \choose k)}, \]

and its alternating counterpart

\[ A(j, k, q) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(qkn + j \choose k)}. \]

The results in this paper add to the literature and are sums that are not considered in the recently published research results of [1] and [2]. Moreover, the work in this paper generalizes and extends the work of [10]. First we recall some definitions of some special functions that will be useful throughout this paper. The Gamma function, for \( z \in \mathbb{C} \), as given by Euler in integral form, is

\[ \Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} dt, \quad \Re(z) > 0, \]

where the special case for \( z \in \mathbb{N} \) reduces to, from the recurrence relation, \( \Gamma(n + 1) = n\Gamma(n) = n! \). The Pochhammer, or shifted factorial, is defined by \((\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\).
The Beta function, or Euler integral of the first kind, is

$$B(z, w) = \frac{1}{\Gamma(z) \Gamma(w)} \int_0^1 t^{z-1} (1 - t)^{w-1} \, dt, \quad \Re(z) > 0, \Re(w) > 0.$$ 

Let

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \int_0^1 \frac{1 - t^n}{1 - t} \, dt = \gamma + \psi(n + 1) = \sum_{j=1}^{\infty} \frac{n}{j(j+n)}, \quad H_0 := 0$$

be the $n$th harmonic number, where $\gamma$ denotes the Euler-Mascheroni constant, $H_n^{(m)} = \sum_{r=1}^{n} \frac{1}{r^m}$ is the $m$th order harmonic number, and $\psi(z)$ is the Digamma (or Psi) function defined by

$$\psi(z) := \frac{d}{dz} \{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{and} \quad \psi(1 + z) = \psi(z) + \frac{1}{z}.$$ 

Moreover

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n + 1} - \frac{1}{n + z} \right).$$

A generalized hypergeometric function is defined by

$$p_{Fq} [z] = p_{Fq} \left[ \begin{array}{c} a_1, a_2, \ldots, a_p \n b_1, b_2, \ldots, b_q \end{array} \right] \left( z \right) = p_{Fq} [(a_p) ; (b_q) \mid z]$$

$$= \sum_{n \geq 0} (a_1)_n \ldots (a_p)_n \frac{z^n}{(b_1)_n \ldots (b_q)_n \ n!} = \sum_{n \geq 0} \frac{\prod_{j=1}^{p} (a_j)_n \ z^n}{\prod_{j=1}^{q} (b_j)_n \ n!}$$

(1)

for $b_j$ non-negative integers or zero. When $p \leq q$, $p_{Fq} [z]$ converges for all complex values of $z$, $p_{Fq} [z]$ is an entire function. When $p > q + 1$, $p_{Fq} [z]$ converges for $z = 0$ if it terminates, which it does when one of the parameters $a_j$ is a negative integer, and hence $p_{Fq} [z]$ is a polynomial in $z$. When $p = q + 1$ the series converges in the unit disc $|z| < 1$, and also for $|z| = 1$ provided that $\Re \left( \sum_{j=1}^{p} b_j - \sum_{j=1}^{p} a_j \right) > 0.$

When $p = 2, q = 1$ we have the familiar Gauss hypergeometric function

$$2_{F1} \left[ \begin{array}{c} a, b 
 c \end{array} \right] \left( z \right) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - zt)^{-1} \, dt,$$

where $|z| < 1, \Re (c - b) > 0$ and $\Re (b) > 0$. In the subsequent analysis we shall also employ the consecutive derivative operator of the inverse binomial,

$$\frac{d^{(m)}}{dz^{(m)}} \left( \binom{q k n + j}{k} \right)^{-1},$$
and investigate the resulting binomial harmonic sum

\[ S^{(m)}(j,k,q) = \sum_{n=1}^{\infty} \frac{d(j)}{d(j)^{(m)}} \left( \frac{q kn + j}{k} \right)^{-1} \]

and its alternating counterpart

\[ A^{(m)}(j,k,q) = \sum_{n=1}^{\infty} (-1)^n \frac{d(j)}{d(j)^{(m)}} \left( \frac{q kn + j}{k} \right)^{-1} \].

The following lemma will be useful in the development of the main theorem.

**Lemma 1.** Let \( p(n) \) and \( q(n) \) be polynomials in \( n \) where all the roots of \( q(n) \) are simple. Assume no root of \( q(n) \) is in \( \mathbb{N} \) and let \( \deg(p(n)) \leq \deg(q(n) - 2) \). Let \( v_n = \frac{p(n)}{q(n)} \). Then

\[ \sum_{n=0}^{\infty} v_n = -\sum_{r=1}^{k} \alpha_r \psi(\beta_r) \tag{2} \]

where

\[ v_n = \frac{p(n)}{q(n)} = \sum_{r=1}^{k} \frac{\alpha_r}{n + \beta_r}. \tag{3} \]

**Proof.** From \( v_n = \frac{p(n)}{q(n)} \) we have \( \sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty} \frac{p(n)}{q(n)} \). By partial fraction expansion, \( v_n = \sum_{r=1}^{k} \frac{\alpha_r}{n + \beta_r} \) since all the roots of \( q(n) \) are simple. For the series \( \sum_{n=0}^{\infty} v_n \) to converge it suffices to have \( \lim_{n \to \infty} n v_n = 0 \), in which case \( \sum_{r=1}^{k} \alpha_r = 0 \). Now

\[ \sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty} \sum_{r=1}^{k} \frac{\alpha_r}{n + \beta_r} = \sum_{n=0}^{\infty} \sum_{r=1}^{k} \alpha_r \left( \frac{1}{n + \beta_r} - \frac{1}{n + 1} \right) \]

\[ = \sum_{r=1}^{k} \alpha_r \sum_{n=0}^{\infty} \left( \frac{1}{n + \beta_r} - \frac{1}{n + 1} \right) \]

\[ = -\sum_{r=1}^{k} \alpha_r \left( \gamma + \psi(\beta_r) \right) \]

\[ = -\sum_{r=1}^{k} \alpha_r \psi(\beta_r) \]

and the lemma is proved. \( \square \)
2. Closed Form Summation

We now prove the following theorems.

**Theorem 1.** Let $k \in \mathbb{N} \setminus \{1\}$, $j \in (-1, \infty)$ and $q \in R^+$, then we have the novel representation

\[
S(j, k, q) = \sum_{n=1}^{\infty} \frac{1}{q^{kn} + j} = \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \psi\left(\frac{q^k + 1 - r + j}{qk}\right).
\]

(4)

The case $j = 0$ reduces to

\[
S(0, k, q) = \sum_{n=1}^{\infty} \frac{1}{q^{kn}} = \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \psi\left(\frac{q^k + 1 - r}{qk}\right).
\]

(5)

Also

\[
S(0, k, q) = \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \left(\psi\left(\frac{r-1}{qk}\right) + \pi \cot\left(\frac{\pi(r-1)}{qk}\right)\right).
\]

(6)

**Proof.** Consider the expansion,

\[
S(j, k, q) = \sum_{n=1}^{\infty} \frac{1}{q^{kn} + j} = \frac{1}{q!} \sum_{n=1}^{\infty} \frac{1}{(qn+j)!} \psi\left(\frac{q^k + 1 - r + j}{qk}\right).
\]

(4)

\[
= k! \sum_{n=1}^{\infty} \frac{1}{qn + j + 1 - r} = k! \sum_{n=1}^{\infty} \frac{1}{(qn + j + 1 - k)_k}
\]

where Pochhammer’s symbol $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$. By partial fraction decomposition we have

\[
S(j, k, q) = \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n + \frac{q^k + 1 + j + r}{qk}}\right),
\]

\[
= \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n + \frac{q^{k+1+j+r}}{qk}}\right).
\]

(6)
and applying Lemma 1 we conclude
\[
S(j, k, q) = \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \psi\left(\frac{qk+1+j-r}{qk}\right)
\]
and (4) follows. For \( j = 0 \) (5) follows. By the reflection formula of the Digamma function \( \psi(1-z) = \psi(z) + \pi \cot \pi z \), then (6) follows. Utilizing the definition (1), for the hypergeometric function, we can write
\[
S(j, k, q) = \frac{1}{kq+j} F_{kq} \left[ \begin{array}{c} kq+j+1-k, kq+j+2-k, \ldots, 2qk-j-k \\ kq-kq, kq+1-kq, \ldots, kq+j-kq \end{array} \middle| 1 \right].
\]

\( \square \)

We remark that in the interesting paper [10], Nimbran considers the representation of
\[
\frac{1}{k!} S(0, k, 1) = \sum_{n=1}^\infty \frac{(nk-k)!}{(nk)!},
\]
for \( k \in \mathbb{N} \setminus \{1\} \), in closed form and evaluates \( S(0, k, 1) \) for \( k = \{2, 3, 4, 5, 6, 8, 10, 12\} \). In particular \( S(0, 2, 1) = \ln 2 \) is listed in [7], \( S(0, 3, 1) = \frac{\sqrt{3}}{12} - \frac{1}{2} \ln 3 \) and \( S(0, 4, 1) = \frac{1}{4} \ln 2 - \frac{\pi}{24} \) are listed in [9]. Nimbran’s search of the literature yields no other evaluation of \( S(0, k, 1) \) for \( k \geq 5 \) and then sets out to evaluate \( S(0, k, 1) \) for \( k = \{5, 6, 8, 10, 12\} \). Nimbran claims \( S(0, 10, 1) \) is difficult to evaluate and finds it impossible to evaluate \( S(0, k, 1) \) for any other values of \( k \). Nimbran’s method of evaluating \( S(0, k, 1) \) is indeed ingenious and relies on the representation
\[
\ln p = \sum_{m=1}^{p} \left( \sum_{r=1}^{m-1} \left( \frac{1}{mp+r-m} - \frac{1}{mp} \right) \right),
\]
which is a generalization of an identity given by Euler in 1734 [6]. We therefore see that the representation (4) gives a general identity for \( S(j, k, q) \) for every \( k \in \mathbb{N} \setminus \{1\} \). Also recently [11], using a generalized binomial theorem in terms of Bell polynomials, evaluates some sums involving inverse binomial coefficients. The same technique is also used to calculate a class of hypergeometric transformation formulas, hence, there is still interest in evaluating binomial sums.

The following corollary applies.

**Corollary 1.** Let \( k \in \mathbb{N} \setminus \{1\}, \ j \in (-1, \infty) \) and \( q \in R^+ \), then we have the representation
\[
T(j, k, q) = \sum_{n=1}^\infty \frac{1}{2qkn+j} = \frac{1}{2q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \psi\left(\frac{2qk+1+j-r}{2qk}\right).
\]
The case \( j = 0 \) reduces to

\[
T(0, k, q) = \sum_{n=1}^{\infty} \frac{1}{2q kn} = \frac{1}{2q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \psi\left(\frac{2qk+1-r}{2qk}\right).
\]

Also

\[
T(0, k, q) = \frac{1}{2q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \begin{pmatrix} \psi\left(\frac{r-1}{2qk}\right) \\ +\pi \cot\left(\frac{\pi(r-1)}{2qk}\right) \end{pmatrix}.
\]

**Proof.** The proof follows directly from Theorem 1. \(\square\)

Now we investigate \( A(j, k, q) \).

**Theorem 2.** Under the assumptions of Theorem 1,

\[
A(j, k, q) = \sum_{n=1}^{\infty} \frac{(-1)^n}{q kn + j} = \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \begin{pmatrix} \psi\left(\frac{2qk+r+j}{2qk}\right) \\ -\psi\left(\frac{qk+r+j}{qk}\right) \end{pmatrix}.
\]

The case \( j = 0 \) reduces to

\[
A(0, k, q) = \sum_{n=1}^{\infty} \frac{(-1)^n}{q kn} = \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \begin{pmatrix} \psi\left(\frac{r-1}{2qk}\right) - \psi\left(\frac{r-1}{qk}\right) \\ +\pi \cot\left(\frac{\pi(r-1)}{2qk}\right) - \pi \cot\left(\frac{\pi(r-1)}{qk}\right) \end{pmatrix}.
\]

Also,

\[
A(0, k, q) = \frac{1}{q} \sum_{r=1}^{k} (-1)^{r+k+1} \binom{k-1}{r-1} \begin{pmatrix} \psi\left(\frac{r-1}{2qk}\right) - \psi\left(\frac{r-1}{qk}\right) \\ +\pi \cot\left(\frac{\pi(r-1)}{2qk}\right) - \pi \cot\left(\frac{\pi(r-1)}{qk}\right) \end{pmatrix}.
\]

**Proof.** We begin by noticing that

\[
S(j, k, q) + A(j, k, q) = 2T(j, k, q).
\]
from which $A(j, k, q)$ follows directly, as does $A(0, k, q)$. The identity (9) follows from an application of the Digamma reflection formula. Utilizing the definition (1), for the hypergeometric function, we can write

$$A(j, k, q) = -\frac{1}{\binom{kq + j}{k}} k_{q+1} F_{kq} \left[ \begin{array}{c} \frac{kq+j+1-k}{kq} \frac{kq+j+2-k}{kq} \cdots, \frac{2kq+j-k}{kq} \\ \frac{kq+j+1}{kq} \frac{kq+j+2}{kq} \cdots, \frac{2kq+j}{kq} \end{array} \right] - 1 \right].$$

\[\square\]

**Remark 1.** The other notable case is for the situation of $j = qk$, from which we ascertain, in a straightforward manner, the following results:

\[S(qk, k, q) = S(0, k, q) - \frac{1}{\binom{qk}{k}},\]

\[A(qk, k, q) = -A(0, k, q) - \frac{1}{\binom{qk}{k}}\]

and

\[S(qk, k, q) - A(qk, k, q) = S(0, k, q) - A(0, k, q) = 2T(0, k, q).\]

**Example 1.** Some examples follow:

$$S\left(j, k, \frac{1}{k}\right) = \frac{j + 1}{(k - 1) \binom{j + 1}{k}},$$

$$S(2, 2, 6) = \frac{\pi}{6} - 1 - \frac{1}{\sqrt{3}} \ln\left(\sqrt{3} - 1\right) + \frac{\sqrt{3} + 1}{6} \ln 2,$$

$$S\left(\frac{1}{8}, 3, \frac{1}{4}\right) = \frac{496}{55} + 12 \ln 3, \quad T\left(\frac{3}{2}, 4, \frac{1}{4}\right) = 8\pi - \frac{64}{3},$$

$$T\left(0, 2, \frac{3}{4}\right) = \ln 3 - \frac{\sqrt{3} \pi}{9}, \quad T(4, 2, 3) = \frac{1}{2} \ln 2 - \frac{1}{4} \ln 3 + \frac{\sqrt{3} - 1}{12\sqrt{3}} \pi - \frac{1}{6};$$
A(4, 3, 2) = \frac{2 - \sqrt{3}}{2\sqrt{3}} \pi - \frac{1}{4}, \quad A\left(-\frac{1}{2}, 2, 1\right) = -2\sqrt{2} \ln \left(\sqrt{2} + 1\right),

A\left(j, 2, \frac{1}{4p}\right) = 4p \left(H_{jp} - H_{jp-p} - (H_{2jp} - H_{2jp-2p})\right), \quad p \in \mathbb{R}^+.

In the next section we give an extension to Theorem 1 by incorporating harmonic numbers to the sums \( S(j, k, q), \ A(j, k, q) \) and associating the sum with hypergeometric and integral representation.

3. Extension

We begin with the proof of the following Theorem.

**Theorem 3.** Let the assumptions of Theorem 1 apply and let \( m \in \mathbb{N} \). Then,

\[
S^{(m)}(j, k, q) = \sum_{n=1}^{\infty} \frac{d^{(m)}}{dj^{(m)}} \left( \frac{qkn + j}{k} \right)^{-1} = \sum_{n=1}^{\infty} Q^{(m)}(j, k, q)
\]

\[= \frac{1}{k^m q^{m+1}} \sum_{r=1}^{k} (-1)^{r+k+1} \left( \frac{k-1}{r-1} \right) \psi^{(m)} \left( \frac{qk+1+j-r}{qk} \right)\]

\[= \frac{m!}{k^m q^{m+1}} \sum_{r=1}^{k} (-1)^{r+m+k+1} \left( \frac{k-1}{r-1} \right) H^{(m+1)}_{\frac{j-r}{qk}}\]  \hspace{1cm} (10)

where

\[
Q^{(m)}(j, k, q) = \frac{d^{(m)}}{dj^{(m)}} \left( \frac{qkn + j}{k} \right)^{-1}.
\]  \hspace{1cm} (12)

**Proof.** From the identity (4) we differentiate both sides \( "m" \) times with respect to \( j \) so that,

\[
S^{(m)}(j, k, q) = \sum_{n=1}^{\infty} \frac{d^{(m)}}{dj^{(m)}} \left( \frac{qkn + j}{k} \right)^{-1} = \sum_{n=1}^{\infty} Q^{(m)}(j, k, q)
\]

\[= \frac{1}{k^m q^{m+1}} \sum_{r=1}^{k} (-1)^{r+k+1} \left( \frac{k-1}{r-1} \right) \psi^{(m)} \left( \frac{qk+1+j-r}{qk} \right)\]

and (10) follows. From the known identity relating polygamma functions with harmonic numbers,

\[
\psi^{(m)}(1 + z) = (-1)^m m! \left( H^{(m+1)}_z - \zeta(1 + m) \right),
\]  \hspace{1cm} (13)
we have
\[ S^{(m)}(j, k, q) = \frac{m!}{k^m q^{m+1}} \sum_{r=1}^{k} (-1)^{r+m+k+1} \binom{k-1}{r-1} H_{\frac{r}{q}}^{(m+1)} \]

since
\[ \sum_{r=1}^{k} (-1)^{r} \binom{k-1}{r-1} = 0, \text{ for } k \geq 2, \]
and hence (11) follows. For completeness we detail some values of \( Q^{(m)}(j, k, q) \):
\[ Q^{(1)}(j, k) = \frac{1}{\binom{q kn+j}{k}} \left( H_{q kn+j-k} - H_{q kn+j} \right) \]
and
\[ Q^{(2)}(j, k, q) = \frac{1}{\binom{q kn+j}{k}} \left( (H_{q kn+j} - H_{q kn+j-k})^2 + (H_{q kn+j}^{(2)} - H_{q kn+j-k}^{(2)}) \right). \]

Some more details on the function \( Q^{(m)}(j, k, q) \) are given in the paper [13]. □

Next we investigate \( A^{(m)}(j, k, q) \) as described below.

**Theorem 4.** Let the assumptions of Theorem 1 apply and let \( m \in \mathbb{N} \). Then,
\[ A^{(m)}(j, k, q) = \sum_{n=1}^{\infty} (-1)^{n} \frac{d^{(m)}}{dj^{(m)}} \left( \frac{q kn+j}{k} \right)^{-1} = \sum_{n=1}^{\infty} (-1)^{n} Q^{(m)}(j, k, q) \]
\[ = \frac{1}{k^m q^{m+1}} \sum_{r=1}^{k} (-1)^{r+m+k+1} \binom{k-1}{r-1} \left( \frac{1}{2^m} \psi^{(m)}(\frac{2k+1+j-r}{qk}) \right. \]
\[ - \left. \psi^{(m)}(\frac{2k+1+j-r}{qk}) \right) \]
\[ = \frac{m!}{k^m q^{m+1}} \sum_{r=1}^{k} (-1)^{r+m+k+1} \binom{k-1}{r-1} \left( \frac{1}{2^m} H_{\frac{r}{q}+1}^{(m+1)} - H_{\frac{r}{q}}^{(m+1)} \right), \]  
where \( Q^{(m)}(j, k, q) \) is given by (12).
Proof. From the identity (7) we differentiate both sides ”m” times with respect to j so that,
\[ A^{(m)}(j, k, q) = \sum_{n=1}^{\infty} (-1)^n \frac{d^{(m)}}{d(j)^n} \left( \left( \frac{qkn + j}{k} \right)^{-1} \right) = \sum_{n=1}^{\infty} (-1)^n Q^{(m)}(j, k, q) \]

\[ = \frac{1}{km^{q+1}} \sum_{r=1}^{k} (-1)^{r+k+1} \left( \begin{array}{c} k - 1 \\ r - 1 \end{array} \right) \left( \frac{k}{2n} \psi^{(m)} \left( \frac{2qk+1+j-r}{2qk} \right) - \psi^{(m)} \left( \frac{qk+r}{qk} \right) \right), \]

and (15) follows. From the known identity (13), relating polygamma functions with harmonic numbers, then
\[ A^{(m)}(j, k, q) = \frac{m!}{km^{q+1}} \sum_{r=1}^{k} (-1)^{r+m+k+1} \left( \begin{array}{c} k - 1 \\ r - 1 \end{array} \right) \left( \frac{1}{2n} H_{1}^{(m+1)}_{1+j-r} - H_{1}^{(m+1)}_{qk+r} \right) \]

since,
\[ \sum_{r=1}^{k} (-1)^{r} \left( \begin{array}{c} k - 1 \\ r - 1 \end{array} \right) = 0, \text{ for } k \geq 2 \]

and (16) is attained. \( \Box \)

The cases \( j = 0 \) and \( j = kq \) are interesting and the results are given in the next corollary.

Corollary 2. For \( j = 0 \),
\[ S^{(m)}(0, k, q) = \sum_{n=1}^{\infty} \lim_{j \to 0} \left( \frac{d^{(m)}}{d(j)^n} \left( \left( \frac{kqn + j}{k} \right)^{-1} \right) \right) = \sum_{n=1}^{\infty} Q^{(m)}(0, k, q) \]

\[ = \frac{(-1)^{k+m+1} m!}{km^{q+1} \zeta(m+1)} \]

\[ + \frac{1}{km^{q+1}} \sum_{r=1}^{k-1} (-1)^{r+k} \left( \begin{array}{c} k - 1 \\ r \end{array} \right) \psi^{(m)} \left( \frac{qk-r}{qk} \right). \] \hspace{1cm} (17)

For \( j = kq \),
\[ S^{(m)}(kq, k, q) = \sum_{n=1}^{\infty} \lim_{j \to kq} \left( \frac{d^{(m)}}{d(j)^n} \left( \left( \frac{kqn + j}{k} \right)^{-1} \right) \right) \]

\[ = \frac{(-1)^{k+m} m!}{km^{q+1}} (1 - \zeta(m+1)) \]

\[ + \frac{1}{km^{q+1}} \sum_{r=1}^{k-1} (-1)^{r+k} \left( \begin{array}{c} k - 1 \\ r \end{array} \right) \psi^{(m)} \left( \frac{2qk-r}{qk} \right) \] \hspace{1cm} (18)
Proof. From (10) we have
\[
S^{(m)}(0,k,q) = \sum_{n=1}^{\infty} \lim_{j \to 0} \left( \frac{d^{(m)}}{d^{j(m)}} \left( (-kqn + j) \right) \right) \\
= \sum_{r=1}^{k} \sum_{r=1}^{k} (-1)^{r+k+1} \left( \frac{k-1}{r-1} \right) \psi^{(m)}(1 + \frac{1-r}{qk}).
\]

At \( r = 1 \), we notice that \( \psi^{(m)}(1) = (-1)^{m+1} m! \zeta(m+1) \) and hence
\[
S^{(m)}(0,k,q) = \frac{(-1)^{k+m+1} m!}{k^{m}q^{m+1}} \zeta(m+1) \\
+ \frac{1}{k^{m}q^{m+1}} \sum_{r=2}^{k} (-1)^{r+k+1} \left( \frac{k-1}{r-1} \right) \psi^{(m)}(1 + \frac{1-r}{qk}).
\]

and, making a change in the summation index, then (17) follows. For the case \( j = qk \),
\[
S^{(m)}(kq,k,q) = \sum_{n=1}^{\infty} \lim_{j \to kq} \left( \frac{d^{(m)}}{d^{j(m)}} \left( (-kqn + j) \right) \right) \\
= \sum_{r=1}^{k} \sum_{r=1}^{k} (-1)^{r+k+1} \left( \frac{k-1}{r-1} \right) \psi^{(m)}(2qk + 1 - r).
\]

At \( r = 1 \), we notice that \( \psi^{(m)}(2) = (-1)^{m} m!(1 - \zeta(m+1)) \) Hence
\[
S^{(m)}(kq,k,q) = \frac{(-1)^{k+m} m!}{k^{m}q^{m+1}} (1 - \zeta(m+1)) \\
+ \frac{1}{k^{m}q^{m+1}} \sum_{r=2}^{k} (-1)^{r+k+1} \left( \frac{k-1}{r-1} \right) \psi^{(m)}(2 + \frac{1-r}{qk}),
\]

and making a change in the summation index, then (18) follows. \( \square \)

The following corollary refers to the \( A^{(m)}(j,k,q) \), for the two special cases of \( j = 0 \) and \( j = qk \).

Corollary 3. For \( j = 0 \),
\[
A^{(m)}(0,k,q) = \sum_{n=1}^{\infty} (-1)^{n} \lim_{j \to 0} \left( \frac{d^{(m)}}{d^{j(m)}} \left( (-kqn + j) \right) \right) = \sum_{n=1}^{\infty} Q^{(m)}(0,k,q) \\
= \frac{(-1)^{k+m+1} m!}{q(2kq)^{m}} (1 - 2^{m}) \zeta(m+1) 
\]
\[ + \frac{1}{k^m q^{m+1}} \sum_{r=1}^{k-1} (-1)^{r+k} \left( \frac{k - 1}{r} \right) \left( \frac{1}{2m} \psi^{(m)} (\frac{2qk - r}{2qk}) - \psi^{(m)} (\frac{qk - r}{qk}) \right). \]

For \( j = qk \),
\[ A^{(m)} (kq, k, q) = \sum_{n=1}^{\infty} (-1)^n \lim_{j \to kq} \left( \frac{d^{(m)}}{dj^{(m)}} \left( \left( \frac{kqn + j}{k} \right)^{-1} \right) \right) \]
\[ = \frac{(-1)^{k+m}}{k^m q^{m+1}} (1 + (2^{-m} - 1) \zeta (m + 1)) \]
\[ + \frac{1}{k^m q^{m+1}} \sum_{r=1}^{k-1} (-1)^{r+k} \left( \frac{k - 1}{r} \right) \left( \frac{1}{2m} \psi^{(m)} (\frac{3qk - r}{2qk}) - \psi^{(m)} (\frac{2qk - r}{qk}) \right). \]

Proof. The proof follows the same pattern as the proof in the previous corollary. \( \square \)

Example 2. Some illustrative examples follow:
\[ S^{(2)} (j, 3, q) = \sum_{n=1}^{\infty} \frac{(H_{3qn+j} - H_{3qn+j-3})^2 + (H_{3qn+j}^{(2)} - H_{3qn+j-3}^{(2)})}{3qn + j} \]
\[ = -\frac{1}{9q^2} \left( \psi^{(2)} \left( \frac{j - 2 + 3q}{3q} \right) - 2 \psi^{(2)} \left( \frac{j - 1 + 3q}{3q} \right) + \psi^{(2)} \left( \frac{j + 3q}{3q} \right) \right), \]
\[ S^{(m)} (j, 2, q) = \sum_{n=1}^{\infty} \frac{d^{(m)}}{dj^{(m)}} \left( \frac{2qn + j}{2} \right)^{-1} \]
\[ = \frac{1}{2m q^{m+1}} \left( \psi^{(m)} \left( \frac{2q + j}{2q} \right) - \psi^{(m)} \left( \frac{2q + j - 1}{2q} \right) \right). \]

By the Leibniz rule of differentiation, we can also write,
\[ S^{(m)} (j, 2, q) = 2 \sum_{n=1}^{\infty} \sum_{t=0}^{m} \frac{(-1)^m m!}{(2qn + j)^{m+1}} \frac{(-1)^m}{(2qn + j - 1)^{m-t+1}}. \]

Using the recurrence relation of the polygamma function,
\[ \psi^{(m)} (1 + z) = \psi^{(m)} (z) + \frac{(-1)^m m!}{z^{m+1}}, \]
from then (21) we have
\[
S^{(m)}(j, 2, q) = 2 \sum_{n=1}^{\infty} \sum_{t=0}^{m} \frac{(-1)^m m!}{(2qn + j)^{t+1}(2qn + j - 1)^{m-t+1}}
\]
\[
= \frac{1}{2mq^{m+1}} \left( \psi^{(m)} \left( \frac{j}{2q} \right) - \psi^{(m)} \left( \frac{j-1}{2q} \right) \right) + 2 (-1)^m m! \left( \frac{1}{j+1} - \frac{1}{(j-1)^{m+1}} \right),
\]
from which
\[
S^{(4)}(4, 2, 6) = 2 \sum_{n=1}^{\infty} \sum_{t=0}^{4} \frac{4!}{(12n + 4)^{t+1}(12n + 3)^{5-t}}
\]
\[
= \frac{125}{1728} \zeta(5) + \frac{(15 - 2\sqrt{3})}{4656} \pi^5 - \frac{781}{5184}.
\]
We also have
\[
S^{(m)}(2q, 2, q) = \sum_{n=1}^{\infty} \lim_{j \to 2q} \left( \frac{d^{(m)}}{d_j^{(m)}} \left( \begin{array}{c} 2qn + j \nonumber \end{array} \right)^{-1} \right)
\]
\[
= \frac{(-1)^m m!}{2mq^{m+1}} (1 - \zeta(m+1)) - \frac{1}{2mq^{m+1}} \psi^{(m)} \left( \frac{4q - 1}{2q} \right)
\]
\[
= 2 \sum_{n=1}^{\infty} \sum_{t=0}^{m} \frac{(-1)^m m!}{(2qn + 2q)^{t+1}(2qn + 2q - 1)^{m-t+1}}.
\]
\[
A^{(m)}(0, 2, q) = \sum_{n=1}^{\infty} (-1)^n \lim_{j \to 0} \left( \frac{d^{(m)}}{d_j^{(m)}} \left( \begin{array}{c} 2qn + j \nonumber \end{array} \right)^{-1} \right)
\]
\[
= \frac{(-1)^{m+1} m!}{4mq^{m+1}} (1 - 2^m - \zeta(m+1)) - \frac{1}{4mq^{m+1}} \psi^{(m)} \left( \frac{4q - 1}{4q} \right)
\]
\[
+ \frac{1}{2mq^{m+1}} \psi^{(m)} \left( \frac{2q - 1}{2q} \right)
\]
\[
A^{(m)}(2q, 2, q) = \frac{(-1)^m m! (1 + (2^m - 1) \zeta(m+1))}{2mq^{m+1}} - \frac{1}{4mq^{m+1}} \psi^{(m)} \left( \frac{6q - 1}{4q} \right)
\]
\[
+ \frac{1}{2mq^{m+1}} \psi^{(m)} \left( \frac{4q - 1}{2q} \right)
\]
\[
= 2 \sum_{t=0}^{m} \sum_{n=1}^{\infty} \frac{(-1)^n + m}{(2qn + 2q)^{t+1}(2qn + 2q - 1)^{m-t+1}}.
\]
The expression \( S^{(m)}(j, k, q) \) and \( A^{(m)}(j, k, q) \) can also be represented in integral and hypergeometric form and for completeness the following is recorded.

**Theorem 5.** Let the assumptions of Theorem 1 apply. Then,

\[
S^{(m)}(j, k, q) = k \int_0^1 \frac{x^{kq+j-k} (1-x)^{k-1} \ln^m x}{1-x^{kq}} dx, \tag{22}
\]

\[
A^{(m)}(j, k, q) = -k \int_0^1 \frac{x^{kq+j-k} (1-x)^{k-1} \ln^m x}{1+x^{kq}} dx. \tag{23}
\]

**Proof.** Consider

\[
S(j, k, q) = \sum_{n=1}^{\infty} \frac{1}{kqn+j} = \sum_{n=1}^{\infty} \frac{\Gamma(kqn+j-k+1) \Gamma(k+1)}{\Gamma(kqn+j+1)}
\]

\[
= k \sum_{n=1}^{\infty} B(k, kqn-k+j+1),
\]

where \( \Gamma(\cdot) \) is the gamma function and \( B(\cdot, \cdot) \) is the beta function. Now

\[
S(j, k, q) = k \int_0^1 \frac{x^j (1-x)^{k-1}}{x^k} \sum_{n=1}^{\infty} (x^{kq})^n dx.
\]

Differentiating \( m \) times with respect to \( j \) results in

\[
S^{(m)}(j, k, q) = k \int_0^1 \frac{x^{kq+j-k} (1-x)^{k-1} \ln^m x}{1-x^{kq}} dx,
\]

and hence (22). The result (23) follows by the same analysis. \( \square \)

**Remark 2.** It is straightforward to see, from (17), (22) and (23), that

\[
S^{(m)}(2q, 2, q) = \frac{(-1)^m m!}{2^m q^{m+1}} (1 - \zeta(m+1)) - \frac{1}{2^m q^{m+1} \psi^{(m)}(4q-1)} \frac{4q-1}{2q}
\]

\[
= 2 \int_0^1 \frac{x^{2q-2} (1-x) \ln^m x}{1-x^{2q}} dx,
\]

\[
A^{(m)}(2q, 2, q) = \frac{(-1)^m m! (1 + (2m-1) \zeta(m+1))}{2^m q^{m+1}} - \frac{1}{4^m q^{m+1} \psi^{(m)}(4q-1)} \frac{6q-1}{4q}
\]

\[
+ \frac{1}{2^m q^{m+1} \psi^{(m)}(4q-1)} \frac{4q-1}{2q}
\]

\[
= -2 \int_0^1 \frac{x^{2q-2} (1-x) \ln^m x}{1+x^{2q}} dx.
\]
Many other examples of binomial sums, harmonic number sums, integral representations and hypergeometric summation are available in [3], [4], [5], [8], [12], [14], [15], [16], [17], [18] and [19].

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**References**


