ON THE NESTED LOCAL POSTAGE STAMP PROBLEM

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Abstract

Let h be a fixed positive integer, and let A be any nonempty set of positive integers. Let $n(h, A)$ denote the largest integer n such that $1, \ldots, n$ are all representable as a sum of at most h (not necessarily distinct) elements of A. The local Postage Stamp problem is to determine $n(h, A)$ for a given pair h, A. The nested local Postage Stamp problem is to determine $n(h, S_k)$ and also the extremal bases S_k such that each $S_k = S_{k-1} \cup \{s_k^\star\}$ and $n(h, S_k)$ maximizes $n(h, S)$ over all S obtained by adding one element to S_{k-1}. We solve the nested local Postage Stamp problem for the case $h = 2$.

1. Introduction

Given positive integers h and k, the `Postage Stamp Problem` (PSP) asks to determine the largest positive integer $n = n(h, k)$ such that every integer in $\{1, \ldots, n\}$ can be written as a sum of *at most* h (not necessarily distinct) elements of some k-set . The `Postage Stamp Problem` derives its name from the situation where we require the largest integer $n = n(h, k)$ such that all stamp values from 1 to n may be made up from a collection of k integer-valued stamp denominations with the restriction that an envelope can have no more than h stamps, repetitions being allowed. The problem of determining $n(h, k)$ is apparently due to Rohrbach [3], and has been studied often. A large and extensive bibliography can be found in [1, 2].

Given a set of positive integers A, the `local Postage Stamp Problem` (local PSP) asks to determine the largest positive integer $n = n(h, A)$ such that every

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integer in $\{1,\ldots,n\}$ can be written as a sum of *at most* h (not necessarily distinct) elements of A. Thus

$$n(h,k) = \max\{n(h,A) : |A| = k\}.$$

The local PSP has been resolved for only a few families A, in the sense that an explicit formula for $n(h,A)$ has been given for these families A and for any positive integer h; these include the family of arithmetic progressions [4] and the family of geometric progressions [5].

The purpose of this note is to resolve the "nested" local PSP in the case $h = 2$. This consists of finding all sequences $\{S_k\}_{k\geq 1}$ of sets which are recursively defined as follows:

- Set $S_1 = \{1\}$.
- Given S_{k-1}, $k \geq 2$, determine all $S_k = S_{k-1} \cup \{s^\star\}$, $s^\star \in \mathbb{N}$, such that

$$n(2,S_k) = \max\left\{n(2,A) : A = S_{k-1} \cup \{a\}, a \in \mathbb{N}\right\}.$$

The main result of this note is to list all such extremal sets S_k, and to determine $n(2,S_k)$. We record this as Theorem 1, which includes Tables 1 and 2. Our results show that, for $k \geq 8$,

$$n(2,S_k) = \left\lceil \frac{19k}{3} \right\rceil - 22 + \begin{cases} 0 & \text{if } k \equiv 0,3,4 \pmod 6; \\ 1 & \text{if } k \equiv 2,5 \pmod 6; \\ 2 & \text{if } k \equiv 1 \pmod 6). \end{cases}$$

The numbers $n(2,S_k)$ for $1 \leq k \leq 7$ do not follow this pattern, except for $k = 7$ (nevertheless, we list this separately, since the extremal sets do not fit the general pattern that emerges for $k \geq 8$).

Rohrbach [3] showed that

$$c_1k^2 + O(k) \leq n(2,k) \leq c_2k^2 + O(k),$$

where $c_1 = 0.25$ and $c_2 = 0.4992$. The best known value for c_1 is $\frac{2}{7}$ and for c_2 is 0.4802. The growth of $n(2,S_k)$ is only linear in k, and far from optimum, as is only to be expected from such a restriction as a nested sequence of sets poses.

2. Main Results

In this section, we prove the main result of this note. We show that there is a repetitive pattern in the extremal sets S_k starting at $k = 8$ and with period 6. The nature of the problem suggests an inductive proof, which is the path we follow.

The proof is divided into two parts: (i) the base cases, dealing with $k \in \{1, \ldots, 7\}$, and (ii) the inductive cases, dealing with the remaining cases subdivided by the congruence classes modulo 6. However, to get the inductive cases to start, we again need to resolve the cases where $k \in \{8, \ldots, 13\}$.

In Table 2 and elsewhere, we use $AP(a, d, k)$ to denote the first k terms of the arithmetic progression with first term a and common difference d. Also note that in Table 2, $|S_k| = k$ and $|T_k| = k - 7$.

Theorem 1. *Let $S_1 = \{1\}$. For each $k > 1$, let S_k be any k-set such that*

$$n\big(2, S_k\big) = \max\{n(2, S) : S \supset S_{k-1}, |S| = k\}.$$

(i) *For $k \in \{1, \ldots, 7\}$, $n(2, S_k)$ and all sets S for which $n(2, S) = n(2, S_k)$ and $|S \setminus S_{k-1}| = 1$ are given in Table 1.*

k	S_k	$n(2, S_k)$
1	$\{1\}$	2
2	$\{1, 2\}, \{1, 3\}$	4
3	$\{1, 3, 4\}$	8
4	$\{1, 3, 4, 5\}, \{1, 3, 4, 6\}, \{1, 3, 4, 9\}$	10
5	$\{1, 3, 4, 9, 11\}$	15
6	$\{1, 3, 4, 9, 11, 16\}, \{1, 3, 4, 9, 11, 8\}$	20
7	$\{1, 3, 4, 9, 11, 16, 12\}, \{1, 3, 4, 9, 11, 16, 20\}$	25

Table 1: Extremal Sets for nested local PSP for $1 \leq k \leq 7$

(ii) *For $k \geq 8$, $n(2, S_k)$ and all sets S for which $n(2, S) = n(2, S_k)$ and $|S \setminus S_{k-1}| = 1$ are given in Table 2. The result for $n(2, S_k)$ is given by*

$$n(2, S_k) = \left\lceil \frac{19k}{3} \right\rceil - 22 + \begin{cases} 0 & \text{if } k \equiv 0, 3, 4 \pmod 6; \\ 1 & \text{if } k \equiv 2, 5 \pmod 6; \\ 2 & \text{if } k \equiv 1 \pmod 6. \end{cases}$$

Proof. We prove this result by induction on k. There is no discernable pattern corresponding to the cases $k \in \{1, \ldots, 7\}$. We treat these as the base cases, determining not only $n\big(2, S_k\big)$ in these cases, but also the extremal sets A for which $n\big(2, A\big) = n\big(2, S_k\big)$.

The cases corresponding to $k \geq 8$ exhibit a pattern. To prove this pattern, the nature of the problem necessitates that we again determine $n\big(2, S_k\big)$ together with all the extremal sets A, where A is a k-set, $k \in \{8, \ldots, 13\}$. This allows us

k	T_k	$n(2,S_k)$
$6m$ $(m>1)$	$AP(26,38,m-1)\cup AP(30,38,m-1)\cup AP(36,38,m-1)$ $\cup AP(40,38,m-1)\cup AP(50,38,m-1)\cup AP(54,38,m-2)$	$38m-22$
$6m+1$ $(m>1)$	$AP(26,38,m-1)\cup AP(30,38,m-1)\cup AP(36,38,m-1)$ $\cup AP(40,38,m-1)\cup AP(50,38,m-1)\cup AP(54,38,m-1)$	$38m-13$
$6m+2$ $(m>0)$	$AP(26,38,m)\cup AP(30,38,m-1)\cup AP(36,38,m-1)$ $\cup AP(40,38,m-1)\cup AP(50,38,m-1)\cup AP(54,38,m-1)$	$38m-8$
$6m+3$ $(m>0)$	$AP(26,38,m)\cup AP(30,38,m)\cup AP(36,38,m-1)$ $\cup AP(40,38,m-1)\cup AP(50,38,m-1)\cup AP(54,38,m-1)$	$38m-3$
	$AP(26,38,m)\cup AP(30,38,m-1)\cup AP(36,38,m-1)$ $\cup AP(40,38,m-1)\cup AP(50,38,m-1)\cup AP(54,38,m-1)$ $\cup\{38m-16\}$	
$6m+4$ $(m>0)$	$AP(26,38,m)\cup AP(30,38,m)\cup AP(36,38,m)$ $\cup AP(40,38,m-1)\cup AP(50,38,m-1)\cup AP(54,38,m-1)$	$38m+4$
	$AP(26,38,m)\cup AP(30,38,m)\cup AP(36,38,m-1)$ $\cup AP(40,38,m-1)\cup AP(50,38,m-1)\cup AP(54,38,m-1)$ $\cup\{38j-28\}$, $1\le j\le m$	
	$AP(26,38,m)\cup AP(30,38,m)\cup AP(36,38,m-1)$ $\cup AP(40,38,m-1)\cup AP(50,38,m-1)\cup AP(54,38,m-1)$ $\cup\{38j-14\}$, $1\le j\le m$	
	$AP(26,38,m)\cup AP(30,38,m)\cup AP(36,38,m-1)$ $\cup AP(40,38,m-1)\cup AP(50,38,m-1)\cup AP(54,38,m-1)$ $\cup\{38j\}$, $1\le j<m$	
	$AP(26,38,m)\cup AP(30,38,m)\cup AP(36,38,m-1)$ $\cup AP(40,38,m-1)\cup AP(50,38,m-1)\cup AP(54,38,m-1)$ $\cup\{19m+1\}$, m odd	
$6m+5$ $(m>0)$	$AP(26,38,m)\cup AP(30,38,m)\cup AP(36,38,m)$ $\cup AP(40,38,m)\cup AP(50,38,m-1)\cup AP(54,38,m-1)$	$38m+11$

Table 2: Extremal Sets for nested local PSP for $k\geq 8$. Note that $S_k=T_k\cup\{1,3,4,9,11,16,12\}$.

to build the base for the induction hypothesis, and complete the proof by induction.

Base Cases ($1\le k\le 7$).

(i) (Case $k=1$) Observe that $S_1=\{1\}$ is the unique 1-set for which $n\big(2,S_1\big)=\max\{n\big(2,A\big):|A|=1\}$.

(ii) (Case $k=2$) Note that $n\big(2,\{1,2\}\big)=n\big(2,\{1,3\}\big)=4$. For any $m>3$, $n\big(2,\{1,m\}\big)=2$. Thus $S_2=\{1,2\}$ or $\{1,3\}$.

(iii) (Case $k = 3$) Note that $n\big(2, \{1, 3, 4\}\big) = 8$. Since $n\big(2, S_2\big) = 4$, the integer we must add to S_2 must not exceed 5. To achieve $n\big(2, S\big) \geq 8$, we must have $\max S \geq 4$. Hence the added integer must be one of 4 and 5. We easily verify that $n\big(2, \{1, 2, 4\}\big) = n\big(2, \{1, 3, 5\}\big) = 6$, and $n\big(2, \{1, 2, 5\}\big) = 7$.

(iv) (Case $k = 4$) We verify that $n\big(2, \{1, 3, 4, 5\}\big) = n\big(2, \{1, 3, 4, 6\}\big) = n\big(2, \{1, 3, 4, 9\}\big) = 10$. Since $n\big(2, S_3\big) = 8$, the integer we must add to S_3 must not exceed 9. To achieve $n\big(2, S\big) \geq 10$, we must have $\max S \geq 5$. Hence the added integer must be one of $5, 6, 7, 8, 9$. It may be checked that $n\big(2, \{1, 3, 4, 7\}\big) = 8$ and $n\big(2, \{1, 3, 4, 8\}\big) = 9$.

(v) (Case $k = 5$) We verify that $n\big(2, \{1, 3, 4, 9, 11\}\big) = 15$. Since $n\big(2, S_4\big) = 10$, the integer we must add to S_4 must not exceed 11. To achieve $n\big(2, S\big) \geq 15$, we must have $\max S \geq 8$. Hence the added integer must be one of $8, 9, 10, 11$. Adding the numbers $8, 9, 10, 11$ to the set $\{1, 3, 4, 5\}$ fails to achieve $14, 11, 12, 13$ respectively. Adding the same sequence of numbers to the set $\{1, 3, 4, 6\}$ fails to achieve $13, 11, 15, 13$ respectively. Adding the numbers $8, 10$ to the set $\{1, 3, 4, 9\}$ fails to achieve $14, 15$ respectively.

(vi) (Case $k = 6$) We verify that $n\big(2, \{1, 3, 4, 9, 11, 16\}\big) = n\big(2, \{1, 3, 4, 9, 11, 8\}\big) = 20$. Since $n\big(2, S_5\big) = 15$, the integer we must add to S_5 must not exceed 16. However, there is no limit to how small that added integer must be since $\max S_5$ is already large enough. Adding any one integer other than $8, 16$ to the set $\{1, 3, 4, 9, 11\}$ will fail to achieve at least one of $16, 17, 19$.

(vii) (Case $k = 7$) We verify that $n\big(2, \{1, 3, 4, 9, 11, 16, 12\}\big) = n\big(2, \{1, 3, 4, 9, 11, 16, 20\}\big) = 25$. Since $n\big(2, S_6\big) = 20$, the integer we must add to S_6 must not exceed 21. There is no limit to how small that added integer must be when $\max S_6 = 16$. However, when $\max S_6 = 11$, to achieve $n\big(2, S\big) \geq 25$, we must have $\max S \geq 13$. Adding any one integer other than $12, 20$ to the set $\{1, 3, 4, 9, 11, 16\}$ will fail to achieve at least one of $21, 23$, and adding any one integer to the set $\{1, 3, 4, 9, 11, 8\}$ will fail to achieve at least one of $21, 23, 25$.

Inductive Cases ($k \geq 8$)

The extremal sets S_k follows a pattern when $k > 7$, as given by Table 2. We establish the extremal sets S_k as well as the values $n\big(2, S_k\big)$ by induction on k.

(viii) (Case $k = 8$) Let $S_7^{(1)} = \{1, 3, 4, 9, 11, 12, 16\}$ and $S_7^{(2)} = \{1, 3, 4, 9, 11, 16, 20\}$. We verify that $n\big(2, S_7^{(1)} \cup \{26\}\big) = 30$. Since $n\big(2, S_7\big) = 25$, the integer we must add to $S_7^{(1)}$ or to $S_7^{(2)}$ must not exceed 26. There is no limit to how small that added integer must be. Adding any one integer to the set $S_7^{(2)}$ will fail to achieve at least one of $26, 28, 30$. Adding any one integer except 26 to the set $S_7^{(1)}$ will fail to achieve at least one of $26, 29, 30$.

Therefore the unique candidate for the set S_8 is $S_7^{(1)} \cup \{26\}$.

(ix) (Case $k = 9$) We verify that $n\big(2, S_8 \cup \{22\}\big) = n\big(2, S_8 \cup \{30\}\big) = 35$. Since $n\big(2, S_8\big) = 30$, the integer we must add to S_8 must not exceed 31. There is no limit to how small that added integer must be. Adding any one integer except $22, 30$ to the set S_8 will fail to achieve at least one of $31, 33$.

Therefore the two candidate for the set S_9 are $S_9^{(1)} = S_8 \cup \{22\}$ and $S_9^{(2)} = S_8 \cup \{30\}$.

(x) (Case $k = 10$) We verify that $n\big(2, S_9^{(2)} \cup \{10\}\big) = n\big(2, S_9^{(2)} \cup \{20\}\big) = n\big(2, S_9^{(2)} \cup \{24\}\big) = n\big(2, S_9^{(2)} \cup \{36\}\big) = 42$. Since $n\big(2, S_9\big) = 35$, the integer we must add to S_9 must not exceed 36. There is no limit to how small that added integer must be. Adding any one integer to the set $S_9^{(1)}$ will fail to achieve at least one of $36, 39, 40, 41$. Adding any one integer except $10, 20, 24, 36$ to the set $S_9^{(2)}$ will fail to achieve at least one of $36, 40$.

Therefore the four candidates for the set S_{10} are $S_{10}^{(1)} = S_9^{(2)} \cup \{10\}$, $S_{10}^{(2)} = S_9^{(2)} \cup \{20\}$, $S_{10}^{(3)} = S_9^{(2)} \cup \{24\}$, and $S_{10}^{(4)} = S_9^{(2)} \cup \{36\}$.

(xi) (Case $k = 11$) We verify that $n\big(2, S_{10}^{(4)} \cup \{40\}\big) = 49$. Since $n\big(2, S_{10}\big) = 42$, the integer we must add to S_{10} must not exceed 43. There is no limit to how small that added integer must be. Adding any one integer to the set $S_{10}^{(j)}$, $j \in \{1, 2, 3\}$, will fail to achieve at least one of $43, 44, 45, 47$. Adding any one integer except 40 to the set $S_{10}^{(4)}$ will fail to achieve at least one of $43, 44, 49$.

Therefore the unique candidate for the set S_{11} is $S_{10}^{(4)} \cup \{40\}$.

(xii) (Case $k = 12$) We verify that $n\big(2, S_{11} \cup \{50\}\big) = 54$. Since $n\big(2, S_{11}\big) = 49$, the integer we must add to S_{11} must not exceed 50. There is no limit to how small that added integer must be. Adding any one integer except 50 to the set S_{11} will fail to achieve at least one of $50, 53, 54$.

Therefore the unique candidate for the set S_{12} is $S_{11} \cup \{50\}$.

(xiii) (Case $k = 13$) We verify that $n\big(2, S_2 \cup \{54\}\big) = 63$. Since $n\big(2, S_{12}\big) = 54$, the integer we must add to S_{12} must not exceed 55. There is no limit to how small that added integer must be. Adding any one integer except 54 to the set S_{12} will fail to achieve at least one of $55, 57, 63$. Note that if 55 and 57 are covered, then 58 is also covered.

Therefore the unique candidate for the set S_{13} is $S_{12} \cup \{54\}$.

Throughout the rest of the proof, we shall let $S_7 = \{1, 3, 4, 9, 11, 12, 16\}$ and let $T_k = S_k \setminus S_7$ for $k \geq 8$.

- `Case 1`: $(k = 6m + 2, m > 0)$
  We claim that $n\big(2, S_k\big)$ increases from $38m - 13$ to $38m - 8$ as k increases from $6m + 1$ to $6m + 2$. We also show this value is uniquely achieved by the addition of $s^\star = 26 + 38(m - 1) = 38m - 12$. Observe that
  $$s^\star, s + 11, s + 12, s^\star + 3, s^\star + 4,$$
  where $s = 54 + 38(m - 2) = 38m - 22$, cover the integers in the interval $[38m - 12, 38m - 8]$.

  We claim that $38m - 7$ cannot be expressed as the sum of two elements of $S_k = S_7 \cup T_k$. Since each element of T_k is even and since $x + y < (38 \cdot 2) - 7 \leq 38m - 7$ for all $x, y \in S_7$, we must have $38m - 7 = x + y$, with $x \in T_k$ and $y \in \{1, 3, 9, 11\}$. However, none of the numbers $38m - 8, 38m - 10, 38m - 16, 38m - 18$ belong to T_k, as may be easily verified. Hence our claim that $38m - 7$ cannot be expressed as the sum of two elements of S_k is verified.

  We now claim that if $t^\star \neq s^\star$, then at least one of $38m - 12, 38m - 9, 38m - 8$ cannot be written as the sum of two elements of $S_k = S_{k-1} \cup \{t^\star\}$. Let, if possible, $t^\star + a = 38m - 12$, $t^\star + b = 38m - 9$, $t^\star + c = 38m - 8$, where $a, b, c \in S_k$.

  We claim that $t^\star \neq a, b, c$. Note that $t^\star \neq b$ (since $t^\star + b = 38m - 9$ is odd), so $b \in S_{k-1}$. If $t^\star = a$, then $t^\star = 19m - 6$, and $b = 19m - 3$, $c = 19m - 2$ are consecutive integers in S_{k-1}. This is impossible, since the largest pair of consecutive integers in S_{k-1} is $11, 12$. If $t^\star = c$, then $t^\star = 19m - 4$, and $a = 19m - 8$, $b = 19m - 5$ are integers in S_{k-1}. This too is impossible, since the largest pair of integers in S_{k-1} in which the elements differ by 3 is $9, 12$. This proves our claim that $t^\star \neq a, b, c$.

  Thus, $a, b = a + 3, c = a + 4 \in S_{k-1}$, and this is easily seen to be impossible. This proves our claim that at least one of $38m - 12, 38m - 9, 38m - 8$ cannot be written as the sum of two elements of $S_k = S_{k-1} \cup \{t^\star\}$ when $t^\star \neq s^\star$.

  This completes the proof of Case 1.

- `Case 2`: $(k = 6m + 3, m > 0)$
  We claim that $n\big(2, S_k\big)$ increases from $38m - 8$ to $38m - 3$ as k increases from $6m + 2$ to $6m + 3$. We also show this value is achieved by the addition of either $s_1^\star = 30 + 38(m - 1) = 38m - 8$ or $s_2^\star = 38m - 16$. Observe that
  $$s_1^\star + 1, s_1 + 40, s_1^\star + 3, s_1^\star + 4, s_2 + 9,$$
  and
  $$s_2^\star + 9, s_1 + 40, s_2^\star + 11, s_2^\star + 12, s_2 + 9$$
  where $s_1 = 30 + 38(m - 2) = 38m - 46$ and $s_2 = 26 + 38(m - 1) = 38m - 12$ cover the integers in the interval $[38m - 7, 38m - 3]$.

We claim that $38m-2$ cannot be expressed as the sum of two elements of $S_k = S_7 \cup T_k$. Suppose $38m-2 = x+y$ with $x, y \in S_k$. Since $x+y < (38 \cdot 2) - 2 \leq 38m-2$ for all $x, y \in S_7$, we have $x \in T_k$ and either $y \in T_k$ or $y \in \{4, 12, 16\}$. The first case is ruled out by considering residue classes modulo 38, while the second case is eliminated because none of the numbers $38m-6, 38m-14, 38m-18$ belong to T_k. Hence the claim that $38m-2$ cannot be expressed as the sum of two elements of $S_k = S_7 \cup T_k$ is verified.

We now claim that if $t^\star \neq s_i^\star$, $i = 1, 2$, then at least one of $38m-7, 38m-5$ cannot be written as the sum of two elements of $S_k = S_{k-1} \cup \{t^\star\}$. Let, if possible, $t^\star + a = 38m-7$, $t^\star + b = 38m-5$, where $a, b \in S_k$. Note that $t^\star \neq a, b$ since $t^\star + a$ and $t^\star + b$ are both odd. Hence $a, b = a+2 \in S_{k-1}$. This is possible only when $a = 1$ or $a = 9$, so that $t^\star = 38m-8 = s_1^\star$ or $38m-16 = s_2^\star$, proving our claim that if $t^\star \neq s_i^\star$, $i = 1, 2$, then at least one of $38m-7, 38m-5$ cannot be written as the sum of two elements of $S_k = S_{k-1} \cup \{t^\star\}$.

This completes the proof of Case 2.

- `Case 3`: $(k = 6m+4, m > 0)$

From Case 2, there are two extremal sets S_{k-1}. Let $S_{k-1}^{(1)}$ be that extremal set which contains $38m-8$ (listed first in Table 2 under $k = 6m+3$), and let $S_{k-1}^{(2)}$ be that extremal set which contains $38m-16$ (listed second in Table 2 under $k = 6m+3$).

We claim that $n\big(2, S_k\big)$ increases from $38m-3$ to $38m+4$ as k increases from $6m+3$ to $6m+4$. We also show this value is achieved by the addition of $s_1^\star = 36 + 38(m-1) = 38m-2$, or $s_{2,j}^\star = 38j - 28$, $(1 \leq j \leq m)$, or $s_{3,j}^\star = 38j - 14$ $(1 \leq j \leq m)$, or $s_{4,j}^\star = 38j$ $(1 \leq j < m)$, or $s_5^\star = 19m+1$ if m is odd, each to the set $S_{k-1}^{(1)}$.

`Subcase` (i) $\big(S_k \supset S_{k-1}^{(1)}\big)$ Observe that:

$$s_1 + 11, s_2 + 36, s_3 + 9, s_3 + 11, s_2 + 40$$

where $s_1 = 26+38(m-1) = 38m-12$, $s_2 = 40+38(m-2) = 38m-36$ and $s_3 = 30+38(m-1) = 38m-8$ cover the integers in the interval $[38m-2, 38m+4]$ except for $38m-2$ and $38m+2$. These two are covered by

$$\begin{aligned} 38m-2 &= s_1^\star = s_{2,j}^\star + 38(m-j) + 26 = s_{3,j}^\star + 38(m-j) + 12 \\ &= s_{4,j}^\star + 38(m-j) - 2 = s_5^\star + 19m - 3, 38m+2 = s_1^\star + 4 \\ &= s_{2,j}^\star + 38(m-j) + 30 = s_{3,j}^\star + 38(m-j) + 16 \\ &= s_{4,j}^\star + 38(m-j) + 2 = 2s_5^\star. \end{aligned}$$

We claim that $38m+5$ cannot be expressed as the sum of two elements of $S_k = S_7 \cup T_k$. Suppose, to the contrary, that $38m+5 = x+y$, with $x, y \in S_k$.

Since each element of T_k is even, and since $x + y \leq 16 + 12 < (38 \cdot 1) + 5 \leq 38m + 5$ for $x, y \in S_7$, we must have $x \in T_k$ and $y \in \{1, 3, 9, 11\}$. However, none of the numbers $38m - 6, 38m - 4, 38m + 2 = 38(m - 1) + 40, 38m + 4$ belong to T_k. Hence the claim that $38m + 5$ cannot be expressed as the sum of two elements of $S_k = S_7 \cup T_k$ is verified.

We now claim that if $t^\star \neq s^\star$, where $s^\star$ is one of the exceptional integers listed in the claim above, then at least one of $38m-2, 38m+2$ cannot be written as the sum of two elements of $S_k = S^{(1)}_{k-1} \cup \{t^\star\}$. Let, if possible, $t^\star + a = 38m-2$, $t^\star + b = 38m + 2$, where $a, b \in S_k$. Thus, $b - a = 4$, and so if $a, b \in S^{(1)}_{k-1}$, we can only have $a \in \{12\} \cup \mathrm{AP}(26, 38, m) \cup \mathrm{AP}(36, 38, m-1) \cup \mathrm{AP}(50, 38, m-1)$. Note that $\{12\} \cup \mathrm{AP}(50, 38, m - 1) = \mathrm{AP}(12, 38, m)$.

If $t^\star = a$, then $a = 19m - 1$. If $m = 2\ell$, then $a = 38\ell - 1$, and so $b = 38\ell + 3 \notin S^{(1)}_{k-1}$. If $m = 2\ell + 1$, then $a = 38\ell + 18$, and so $b = 38\ell + 22 \notin S^{(1)}_{k-1}$. If $t^\star = b$, then $b = 19m + 1$. If $m = 2\ell$, then $b = 38\ell + 1$, and so $a = 38\ell - 3 \notin S^{(1)}_{k-1}$. If $m = 2\ell + 1$, then $b = 38\ell + 20$, and so $a = 38\ell + 16 \in S^{(1)}_{k-1}$. Hence, the only possibility for $t^\star$, with $t^\star = a$ or b is $t^\star = 19m + 1$, with m odd.

This completes our claim that if $t^\star \neq s^\star$, where $s^\star$ is one of the exceptional integers listed in the claim above, then at least one of $38m-2, 38m+2$ cannot be written as the sum of two elements of $S_k = S^{(1)}_{k-1} \cup \{t^\star\}$.

`Subcase` (ii) $\left(S_k \supset S^{(2)}_{k-1}\right)$ We first claim that neither $38m + 1$ nor $38m + 3$ can be written as the sum of two elements of $S^{(2)}_{k-1} = T^{(2)}_{k-1} \cup S_7$, where $T^{(2)}_{k-1} = T_{k-2} \cup \{38m - 16\}$.

Suppose, to the contrary, $38m + 1 = x + y$, with $x, y \in S^{(2)}_{k-1}$. Since each element of $T^{(2)}_{k-1}$ is even, and since $16 + 11 < (38 \cdot 1) + 1 \leq 38m + 1$, we must have $38m + 1 = x + y$, with $x \in T^{(2)}_{k-1}$ and $y \in \{1, 3, 9, 11\}$. However, none of the numbers $38m-10, 38m-8 = 30+38(m-1), 38m-2 = 36+38(m-1), 38m$ belong to $T^{(2)}_{k-1}$. A similar argument shows that $38m+3$ also cannot be written as the sum of two elements of $S^{(2)}_{k-1}$, proving the claim that neither $38m + 1$ nor $38m + 3$ can be written as the sum of two elements of $S^{(2)}_{k-1}$.

We now claim that at least one of $38m - 2, 38m + 1, 38m + 2, 38m + 3$ cannot be written as the sum of two elements of $S_k = S^{(2)}_{k-1} \cup \{t^\star\}$. Let, if possible, $t^\star + a = 38m - 2$, $t^\star + b = 38m + 1$, $t^\star + c = 38m + 2$, $t^\star + d = 38m + 3$ where $a, b, c, d \in S_k$. Note that $t^\star \neq b, d$, since $t^\star + b$ and $t^\star + d$ are both odd. Now if $t^\star \neq c$, then $b, c = b + 1, d = b + 2$ must be consecutive integers in $S^{(2)}_{k-1}$, which is impossible. Thus, $t^\star = c$, and so $a, b = a + 3, d = a + 5$ must belong to $S^{(2)}_{k-1}$, and this too is impossible. This proves our claim that at least one of $38m - 2, 38m + 1, 38m + 2, 38m + 3$ cannot be written as the sum of two elements of $S_k = S^{(2)}_{k-1} \cup \{t^\star\}$.

This completes the proof of Case 3.

- Case 4: $(k = 6m + 5, m > 0)$
We claim that $n\big(2, S_k\big)$ increases from $38m + 4$ to $38m + 11$, and this is achieved by the addition of $s^\star = 40 + 38(m-1) = 38m + 2$ to the first of the extremal sets from Case 3, the unique set among the extremal sets containing $36 + 38(m-1) = 38m - 2$; refer to Table 2. Observe that

$$s^\star + 3, s^\star + 4, s_3 + 9, s_1 + 54, s_3 + 11, s_2 + 50, s^\star + 9,$$

where $s_1 = 30+38(m-2) = 38m-46$, $s_2 = 36+38(m-2) = 38m-40$ and $s_3 = 36+38(m-1) = 38m-2$ cover the integers in the interval $[38m+5, 38m+11]$.

We claim that $38m + 12$ cannot be expressed as the sum of two elements of $S_k = S_7 \cup T_k$. Suppose $38m + 12 = x + y$ with $x, y \in S_k$. Since $x + y < (38 \cdot 1) + 12 \leq 38m + 12$ for all $x, y \in S_7$, we have $x \in T_k$ and either $y \in T_k$ or $y \in \{4, 12, 16\}$. The first case is ruled out by considering residue classes modulo 38, while the second case is eliminated because none of the numbers $38m - 4, 38m, 38m + 8$ belong to T_k. Hence our claim that $38m + 12$ cannot be expressed as the sum of two elements of $S_k = S_7 \cup T_k$ is verified.

Suppose $t^\star \neq s^\star$. We claim that at least one of $38m + 5, 38m + 6, 38m + 11$ cannot be written as the sum of two elements of $S_k = S_{k-1} \cup \{t^\star\}$ if the starting set is the first extremal set, and at least one of $38m+5, 38m+6, 38m+7, 38m+11$ cannot be written as the sum of two elements of $S_k = S_{k-1} \cup \{t^\star\}$ if the starting set is not the first extremal set; refer to Table 2.

First consider the case when the base set is the first extremal set above. Let, if possible, $t^\star + a = 38m + 5$, $t^\star + b = 38m + 6$, $t^\star + c = 38m + 11$, where $a, b, c \in S_k$. Note that $t^\star \neq a$ (since $t^\star + a = 38m + 5$ is odd) and $t^\star \neq c$ (since $t^\star + c = 38m + 11$ is odd). Hence $a, c = a + 6 \in S_{k-1}$. However in T_{k-1} the only pairs of integers that differ by 6 are the ones of the form $38\ell - 8$ and $38\ell - 2$. If that is the case, then $b = 38\ell - 7$, which clearly is not in S_{k-1}, forcing $t^\star = b$. But in that case $t^\star + b = 38\ell - 14 \not\equiv 6 \pmod{38}$. This contradiction implies $a, c \notin T_{k-1}$. Hence $a, c \in S_7$, which is only possible if $a = 3$. But then $t^\star = s^\star$. Hence, at least one of $38m + 5, 38m + 6, 38m + 11$ cannot be written as the sum of two elements of S_k, if the starting set is the first extremal set.

Now suppose the base set is not the first extremal set above. By considering congruence classes modulo 38, we may verify that $38m + 7$ cannot be written as the sum of two elements of S_{k-1}. Therefore we need $t^\star$ such that $t^\star + a = 38m + 5$, $t^\star + b = 38m + 6$, $t^\star + c = 38m + 7$, and $t^\star + d = 38m + 11$. As in the argument above, due to parity considerations, $t^\star \neq a, c, d$. Hence $a, c = a + 2, d = a + 6 \in S_{k-1}$. However, in T_{k-1} the only triples of integers of the form $x, x+2, x+6$ are the ones of the forms (i) $38\ell-28, 38\ell-26, 38\ell-22$, and (ii) $38\ell-14, 38\ell-12, 38\ell-8$. Hence b must be one of $38\ell-27, 38\ell-13$; none of these are in S_{k-1}, forcing $t^\star = b$. But in each of these cases $t^\star + b \not\equiv 6 \pmod{38}$. This contradiction implies $a, c = a+2, d = a+6 \notin T_{k-1}$. Hence $a, a+2, a+6 \in S_7$, and this is impossible. Hence, at least one of $38m+5, 38m+6, 38m+7, 38m+11$ cannot be written as the sum of two elements of S_k, if the starting set is not the first extremal set.

This completes the proof of Case 4.

- `Case 5`: $(k = 6m, m > 1)$
We claim that $n\big(2, S_k\big)$ increases from $38m+11$ to $38m+16$ as k increases from $6m+5$ to $6(m+1)$. We also show this value is uniquely achieved by the addition of $s^\star = 50 + 38(m-1) = 38m+12$. Observe that

$$s^\star, s+11, s+12, s^\star+3, s^\star+4,$$

where $s = 40 + 38(m-1) = 38m+2$, cover the integers in the interval $[38m+12, 38m+16]$.

We claim that $38m+17$ cannot be expressed as the sum of two elements of $S_k = S_7 \cup T_k$. Since each element of T_k is even and since $x+y < (38\cdot 2)+17 \leq 38m+17$ for all $x, y \in S_7$, we must have $38m+17 = x+y$, with $x \in T_k$ and $y \in \{1, 3, 9, 11\}$. However, none of the numbers $38m+6, 38m+8, 38m+14, 38m+16$ belong to T_k. Hence our claim that $38m+17$ cannot be expressed as the sum of two elements of $S_k = S_7 \cup T_k$ is verified.

The remainder of the proof in Case 5 follows along the same lines as in Case 1.

We now claim that if $t^\star \neq s^\star$, then at least one of $38m+12, 38m+15, 38m+16$ cannot be written as the sum of two elements of $S_k = S_{k-1} \cup \{t^\star\}$. Let, if possible, $t^\star + a = 38m+12$, $t^\star + b = 38m+15$, $t^\star + c = 38m+16$, where $a, b, c \in S_k$.

We claim that $t^\star \neq a, b, c$. Note that $t^\star \neq b$ (since $t^\star + b = 38m+15$ is odd), so $b \in S_{k-1}$. If $t^\star = a$, then $t^\star = 19m+6$, and $b = 19m+9$, $c = 19m+10$ are consecutive integers in S_{k-1}. This is impossible, since the largest pair of consecutive integers in S_{k-1} is $11, 12$. If $t^\star = c$, then $t^\star = 19m+8$, and $a = 19m+4$, $b = 19m+7$ are integers in S_{k-1}. This too is impossible, since the elements in the largest pair of integers in S_{k-1} that differ by 3 are $9, 12$. This proves our claim that $t^\star \neq a, b, c$.

Thus, $a, b = a+3, c = a+4 \in S_{k-1}$, and this is easily seen to be impossible. This proves our claim that at least one of $38m+12, 38m+15, 38m+16$ cannot be written as the sum of two elements of $S_k = S_{k-1} \cup \{t^\star\}$ when $t^\star \neq s^\star$.

This completes the proof of Case 5.

- `Case 6`: $(k = 6m + 1, m > 1)$
  We claim that $n\big(2, S_k\big)$ increases from $38m + 16$ to $38m + 25$ as k increases from $6(m + 1)$ to $6(m + 1) + 1$. We also show this value is uniquely achieved by the addition of $s^\star = 54 + 38(m - 1) = 38m + 16$. Observe that

  $$s^\star + 1, s_2 + 16, s^\star + 3, s^\star + 4, s_3 + 9, s_1 + 30, s_3 + 11, s_3 + 12, s^\star + 9,$$

  where $s_1 = 30 + 38(m - 1) = 38m - 8$, $s_2 = 40 + 38(m - 1) = 38m + 2$ and $s_3 = 50 + 38(m - 1) = 38m + 12$ cover the integers in the interval $[38m + 17, 38m + 25]$.

  We claim that $38m + 26$ cannot be expressed as the sum of two elements of $S_k = S_7 \cup T_k$. Suppose $38m + 26 = x + y$ with $x, y \in S_k$. Since $x + y < (38 \cdot 1) + 26 \leq 38m + 26$ for all $x, y \in S_7$, we have $x \in T_k$ and either $y \in T_k$ or $y \in \{4, 12, 16\}$. The first case is ruled out by considering residue classes modulo 38, while the second case is eliminated because none of the numbers $38m + 10, 38m + 14, 38m + 22$ belong to T_k. Hence the claim that $38m + 26$ cannot be expressed as the sum of two elements of $S_k = S_7 \cup T_k$ is verified.

  We now claim that if $t^\star \neq s^\star$, then at least one of $38m + 17, 38m + 19, 38m + 20, 38m + 25$ cannot be written as the sum of two elements of $S_k = S_{k-1} \cup \{t^\star\}$. Let, if possible, $t^\star + a = 38m + 17$, $t^\star + b = 38m + 19$, $t^\star + c = 38m + 20$ and $t^\star + d = 38m + 25$, where $a, b, c, d \in S_k$.

  We claim that $t^\star \neq a, b, c, d$. Note that $t^\star \neq a, b, d$ (since $t^\star + a = 38m + 17$, $t^\star + b = 38m + 19$ and $t^\star + d = 38m + 25$ are all odd), so $a, b, d \in S_{k-1}$. If $t^\star = c$, then $t^\star = 19m + 10$, and $a = 19m + 7$, $b = 19m + 9$, $d = 19m + 15$ are integers in S_{k-1}. This is impossible, since the largest pair of integers differing by 2 in S_{k-1} is $9, 11$. This proves our claim that $t^\star \neq a, b, c, d$.

  Thus, $a, b = a + 2, c = a + 3$ and $d = a + 8 \in S_{k-1}$; this is easily seen to be impossible. This proves our claim that at least one of $38m + 17, 38m + 19, 38m + 20, 38m + 25$ cannot be written as the sum of two elements of $S_k = S_{k-1} \cup \{t^\star\}$ when $t^\star \neq s^\star$.

  This completes the proof of Case 6, and of the theorem.

$\square$

Concluding Remarks. The results of Theorem 1 were guided by an extensive computer program that we ran for all values of $k \leq 200$. The extremal sets exhibited a recurring pattern after $k = 7$, in blocks of length 6. In spite of running a computer program to list all extremal sets and determine the corresponding $n(3, S_k)$ for all values of $k \leq 300$, we were unable to find a pattern similar to the one we found for the case $h = 2$. We did not look to compute the cases corresponding to higher values of h.

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