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THE 4-COLOR CUBES PUZZLE

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Abstract

Starting with a palette of four colors, a 4-*color cube* is one where each face is colored with exactly one color and each color appears on some face–there are a total of sixty-eight distinct varieties of 4-color cubes. In the 4-Color Cube puzzle, one is given a set of 4-color cubes and tries to arrange a subset into a larger $n \times n \times n$ 4-color cube. To solve this puzzle, it is sufficient to fill in the large cube's n-*frame*, its corners and edges. For each n we determine a minimal value, $\text{fr}(n)$, so that given any arbitrary collection of $\text{fr}(n)$ 4-color cubes, there is always a subset which can be used to build an n-frame. In particular, we are able to show that for $n \geq 3$, $\text{fr}(n) = 12n - 16$, the smallest possible number. In addition, we describe a set of ten distinct 4-color cubes from which it is possible to build $2 \times 2 \times 2$ frames modeled on all sixty-eight color cube varieties and conclude that this is the smallest size of such a set.

1. Introduction

Given a palette of $k \leq 6$ colors, a k-*color cube* is one where each face is exactly one color and every color appears on some face. Using a quick counting argument one easily shows that when one has six colors to choose from, there are thirty different possible 6-color cube varieties. Among the earliest written references to this set is due to Percy MacMahon who, among other things, is known for writing one of the first books on enumerative combinatorics [15]. MacMahon also wrote about the collection of 6-color cubes in Section 34 of his *New Mathematical Pastimes* [14], from 1921. In it he stated that

> It is now some years since Colonel Julian R. Jocelyn communicated to the present writer the fact that he could select eight cubes and assemble

> them...so as to produce a cube of twice the linear dimensions which is a faithful copy in colours of any *given* member of the set of thirty cubes.

Whether or not Colonel Jocelyn provided the solution, it is likely that MacMahon already knew about it, since MacMahon published an article on this problem in 1893 [13]. The material in [14] contained questions not in the original paper, including whether it was possible to use twenty-seven distinct cubes in the collection to build a $3 \times 3 \times 3$ cube where each face was one color (it is); and whether the entire collection can similarly be assembled into a $2 \times 3 \times 5$ block with single-color faces (it can).

Interest in sets of face-colored cubes has continued long after their introduction by MacMahon. Martin Gardner wrote about them a number of times, including in “Thirty Color Cubes” (Chapter 6 of [8]), and “The 24 Color Squares and the 30 Color Cubes” (Chapter 16 of [9]). Another puzzle, known as “Eight Blocks to Madness,” was released in 1970 by Eric Cross [8]. In it, one is provided a particular collection of eight cubes; the goal is to arrange the cubes into a $2 \times 2 \times 2$ cube where each face is one color. Analysis of puzzles of this type are the subject of [11] as well as the more recent article [10]. A cube puzzle with four colors known as Instant Insanity™ was a top-seller in the late 1960’s, and this puzzle has a well-known graph theoretic solution (see [5], for example). Other analyses of related problems can be found in [2], [3], [4], and [7].

This paper focuses on another 4-color puzzle. It is motivated by a question posed in [4] that asks, if given an *arbitrary* collection of twenty-seven 6-color cubes, whether it is always possible to assemble a $3 \times 3 \times 3$ cube. The answer is affirmative, and the result holds for sets of cubes of size n^3 for an $n \times n \times n$ cube. We look at a related problem, where we start with a palette of four colors instead of six. We call such cubes 4-*color cubes.* In addition, we make the following assumption.

The Coloring Condition: For every cube we consider, all four colors in the palette appear on some face of the cube.

Under the coloring condition, a number of cubes in the original puzzle are superfluous. In an $n \times n \times n$ cube, the $(n-2) \times (n-2) \times (n-2)$ cube in the interior is hidden, so any collection of cubes can be used in its construction. And since the Coloring Condition ensures that all colors appear on every cube, the $(n-2) \times (n-2)$ cubes in the center of each face can be arbitrary as well. The only cubes whose composition really matter are the eight corner cubes and $12(n-2)$ cubes along the edges. We call this collection of $12n - 16$ cubes the *frame* of a cube. We can now give our formal statement of the Color Cubes puzzle.

The Color Cubes puzzle (Frame Version): What is the smallest size set of 4-color cubes so that, regardless of the set’s composition, one can always construct the frame of a $n \times n \times n$ 4-color cube where each face is a single color?

Versions of this puzzle, with palettes of two, three, and six colors, have been analyzed in [2] and [3]. It turns out that when $k = 6$ and $n \geq 4$, when one has a set of $12n - 16$ k-color cubes then one can always construct a frame. The same is true for $k = 2$ and $n \geq 3$. On the other hand, for the case $k = 3$, $16n - 17$ cubes are required to construct a frame, starting with $n = 4$. (Table 7 towards the end of the paper summarizes what is known.) One additional difficulty with the $k = 4$ case is the number of varieties of cube we need to consider in our sets. For $k = 2, 3$ and 6, the number of varieties are eight, thirty-two, and thirty. For $k = 4$, on the other hand, there are sixty-eight to consider. Nonetheless, in this paper we will prove:

Main Theorem: For $n > 2$, any collection of $12n - 16$ 4-colored cubes suffices to build a frame. When $n = 2$, any collection of eleven 4-color cubes always contains a subset of eight which can build a $2 \times 2 \times 2$ cube.

A naive approach to this problem might search through all possible sets of cubes. This would be a daunting task; for the case of a $3 \times 3 \times 3$ frame, there are $\binom{87}{20} \approx 2.38 \times 10^{19}$ sets to search through, a number that is within an order of magnitude of the number of configurations of a Rubik's cube. Clearly, brute force is out. There are a number of observations which make our proof possible. First, the order and number of colors on the faces of a particular cube are implicit in the number and type of corners and edges on each cube variety. Once we encoded corner and edge data from the set of sixty-eight 4-color cube varieties into a table (Table 8 in the Appendix–only non-zero entries are shown), we were able to discern a great deal of structure within the set. In addition, there is a partial order on cube varieties which allows us to restrict our attention to a smaller subset of only thirty-eight varieties. By grouping the varieties in subsets with similar characteristics, calculations became tractable, either by hand or via computer calculations in *Mathematica*. Even still, we were surprised by the number of arguments that seemed to just barely work out.

This paper is organized as follows. In Section 2 we introduce notation and cover major definitions, including the partial order. In Section 3 we determine conditions for a set of cubes to have a *corner solution*, a subset of eight cubes which solves the $2 \times 2 \times 2$ puzzle. In Sections 4 and 5 we tackle the general frame cases. We switch focus in Section 6, where we address a question posed in [10] and construct a smallest-sized example of a *universal* set of cubes, one that can be used to construct a corner solution modeled on any 4-color cube. In Section 7 we discuss the computational complexity of finding the solution to the 4-color cubes puzzle, and in Section 8 we discuss some further topics for study.

2. Preliminaries

As we stated in the Introduction, a 4-color cube is a unit cube where each face is one color, and each of the four colors appears on some face. From this point in the paper forward we will mostly be working with 4-color cubes, so we will usually just refer to 4-color cubes as cubes. We identify cubes under the operation of rigid rotation, saying that they are of the same *variety*. Cubes can be distinguished by the way colors are assigned to their faces. We track this by noting the number of types of corners and edges on a cube. Edges can be single-color or two-color. Corners can be single-color, two-color, or three-color. In contrast to edges, the order of colors in corners makes a difference. We read colors going clockwise under the equivalence

$$\mathrm{XYZ} \sim \mathrm{YZX} \sim \mathrm{ZXY}.$$

However, XYZ $\not\sim$ XZY. Aided by a Polya counting argument, one finds that there are sixty-eight different 4-color cube varieties. Table 8 in the Appendix contains a description of all varieties as well as their corners and edges.

Given an arbitrary set of cubes, a *solution* to the $n \times n \times n$ Color Cubes puzzle is a collection of $12n - 16$ unit color cubes which can be placed in the corner and edge positions of an $n \times n \times n$ cube such that the colors on each face agree. This is the *frame* of a cube; an example can be seen in Figure 1. We say that a solution is *modeled* after the variety it resembles. The component unit cubes of a frame are either *corner cubes* or *edge cubes*. In particular, a *solution* to the Color Cubes puzzle is a collection of eight cubes that make up a *corner solution*, a $2 \times 2 \times 2$ cube modeled on some variety, as well as the $12(n-2)$ cubes that fill in the edges. We note that a corner solution determines the color and placement of edge cubes in a frame. In this paper we are interested in determining the size of the smallest set of cubes which guarantees that we can construct a frame regardless of the number and type of varieties in the set.

Definition 1. The frame number, $\mathrm{fr}(n)$, is the minimum size set of cubes which is guaranteed to construct an $n \times n \times n$ frame using four colors.

Although keeping track of sixty-eight different varieties within a given collection may seem like a hopeless task, fortunately it is not necessary to consider all of them. Given varieties v and w, we say that $v \prec w$, or v is smaller than w, if each type of corner of v is also a corner of w. This defines a partial order on the set of all 4-color cubes.

Definition 2. A variety v is minimal if it is smallest under the partial order $\prec$.

As an example, a cube of a single color (although not 4-color) is always minimal. In addition, we note that since the edges of a cube are implicit in its corners, if $v \prec w$, then all edges of v are also edges of w.

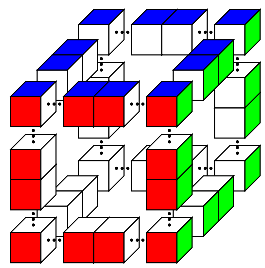

Figure 1: The Frame of an $n \times n \times n$ Cube

Restricting our analysis to minimal cube varieties changes a puzzle in the following way. Given a collection of cubes, replacing a cube of a non-minimal variety with one of a strictly smaller variety reduces the number of types of corners and edges available for constructing a solution, thereby decreasing the number of possible solutions. Conversely, if one can find a solution using cubes of minimal varieties, then that same solution is also constructible when one replaces any cube in the set with a cube of greater variety. Therefore, sets of minimal varieties are precisely the ones that are least likely to have a solution. If we can show that for a given frame every minimal cube collection of a certain size has a solution, then the frame is constructible from every possible cube set of that size.

Restricting our analysis to sets containing only minimal varieties is a useful reduction. Unfortunately, thirty-eight of the sixty-eight 4-color cube varieties are minimal, which means a direct computer search for a solution is still out of reach. We proceed by stratifying the collection of minimal cubes into sets based on structure observed in Table 8. We note that most minimal varieties have different mirror image, and that sixteen varieties have the four corners GBW, RGW, RBG, and RWB, whereas their sixteen mirror images have the four corners RBW, RGB, RWG, and GWB. We denote the first set of varieties by K_o ("original") and the set of their mirror images by K_m ("mirror"); any argument which leads to a conclusion about cubes of varieties from K_o has an analogous argument for K_m and vice versa. We call the four corners GBW, RGW, RBG, and RWB *o-universal* corners (or just *universal* when the context is clear). Their mirror images are the *m-universal* corners. We denote the union of K_o and K_m by K. There are also six minimal varieties which, being their own mirror image, are in neither K_o nor K_m. Because these varieties have two colors which alternate around the girth of the cube, we call these the *checkerboard* varieties, and denote their collection by K^c. From Table 8, we note that all checkerboard varieties have four o- and four m-universal corners. More specifically, each variety has two each of two types of o-universal corners as well as their mirror images. Given a checkerboard variety v, the unique checkerboard variety with the other four types universal corners is the variety's *complement*, denoted

v^c. In Table 8, varieties BW7 and GR7 are complements.

Since all varieties within K_o share four o-universal corners, the remaining four corners, which we call *characteristic* corners, are what distinguish varieties in K_o from one another. Using Table 8, we confirm that universal corners are always three-color, whereas characteristic corners are either single-color or two-color. Likewise, we call an edge a *universal edge* if it appears on all varieties in K; all four have two colors. This is in contrast to *characteristic edges* which are single-color and do not necessarily appear on all varieties in K. From Table 8, minimal varieties with a single-color corner have four characteristic corners and three characteristic edges. The other varieties in K have only two characteristic edges instead of three. Each checkerboard variety also contains five of the six types of universal edges but no characteristic edges.

Since characteristic corners contain at most two colors, every characteristic corner is its own mirror image. As a consequence, a variety and its mirror image both have the same characteristic corners. We say that the variety, X, and its mirror make up a *corner class*, [X]. In Table 1 we list the (K_o, K_m) mirror pairs which make up each corner class. We remark that edges of varieties in the same corner class appear with the same type and multiplicity.

(R1, R2)	(G1,G2)	(W1, W2)	(B2, B1)
(BG1, BG3)	(BG4, BG2)	(BR2, BR4)	(BR3, BR1)
(GR2, GR4)	(GR3, GR1)	(BW2, BW4)	(BW3, BW1)
(GW2, GW4)	(GW3, GW1)	(RW1, RW3)	(RW4, RW2)

Table 1: The 16 Corner Classes

3. The Corner Solution

From Figure 1, we see that a corner solution determines the number and type of edges that appear on a frame. Therefore, our first goal is to determine conditions for when we have a corner solution. Lemmas 1 and 2 provide sufficient conditions on certain restricted sets. Both lemmas can be proved by considering many cases and exhaustively checking each one by hand. However, the restricted sets are small enough that they can also be checked by an algorithm written in *Mathematica*—the code is available from the first author.

The *Mathematica* algorithm steps through subsets of cubes. For a fixed subset of cubes and a target cube whose corners we would like to match we construct a bipartite graph. One vertex set has a vertex for each cube in the subset; the other vertex set has a vertex for each corner of the target cube. We draw an edge from a cube in the first vertex set to a corner in the second if the corner is on the cube. A matching of size eight means that eight cubes can be arranged as a corner solution modeled on the target cube, although we sometimes only focus on characteristic corners.

Lemma 1. *Given seven cubes of varieties from K, we can always find a match for the four characteristic corners of some variety in K.*

Lemma 1 has an important consequence. Once there is a match for four characteristic corners for some corner class, four of any other cubes from K_o (resp. K_m) will fill in the o-universal (resp. m-universal) corners. We get:

Corollary 1. *Given eight cubes from K_o (resp. K_m), we can always build a corner solution modeled on some variety in K_o (resp. K_m).*

We recall a comment from Section 2, that by mirror symmetry a result that holds for K_o also holds for K_m. In the sequel, we will usually state results for K_o only.

Lemma 2. *Given i cubes of varieties from K^c, Table 2 shows the least number of corners, j, that can be matched with some variety in K^c. In particular, given 11 arbitrary checkerboard cubes, we are guaranteed a corner solution modeled on some checkerboard variety.* □

i	1	2	3	4	5	6	7	8	9	10	11
j	1	2	3	4	5	5	6	6	7	7	8

Table 2: Matching Numbers in K^c

These initial results show that given enough cubes from a particular subset, we can arrange eight of them into a corner solution modeled on a variety in the same subset. Moving forward, we want to show how the subsets interact. We will denote by $|S|_{K_o}$ the number of cubes in a collection S that are of varieties in K_o. We have analogous notation for K_m and K^c.

Lemma 3. *Given a set of cubes S with $|S|_{K^c} \geq 5$, we can match all m-universal corners of a variety in K^c.*

Proof. It is sufficient to consider $|S|_{K^c} = 5$. Let v be the variety of K^c with highest repetition number. Then:

$|v| \geq 4$: We use four v cubes for the universal K_m corners.

$|v| = 3$: From Table 3, any other non-complementary checkerboard variety shares two m-universal corners with v. So if we have a cube of a non-complementary variety we are done. Otherwise, we have two v^c cubes. In this case, by Table 3 we see that v and v^c share two m-universal corners with every other checkerboard variety. Since v and v^c have no corners in common, two copies of each can fill the m-universal corners of any other checkerboard variety.

$|v| = 2$: The set has either two non-complementary cubes to v, or two v^c cubes. This case is analogous to the case $|v| = 3$.

$|v| = 1$: We have one cube each of five varieties. By Table 3 one can match the m-universal corners of v using three other cubes that are not v^c.

□

	RBW	RGB	RWG	GWB
BW7	2			2
GR7		2	2	
BG7		2		2
RW7	2		2	
GW7			2	2
BR7	2	2		

Table 3: M-characteristic Corners for Checkerboard Varieties. (Only non-zero entries are shown.)

One interesting property of a checkerboard variety is that it is its own mirror image, so if a checkerboard variety is used in a corner position in a solution, then it can also be used for the mirror image of that corner. An implication to creating a frame modeled on a checkerboard variety is that we can always assume that we have matched at least as many m-universal corners as o-universal ones, since if this is not the case one can just take mirror images of everything.

Lemma 4. *Take a collection of eight cubes: four from K_o and the rest from K_m and/or K^c. Then we can build a corner solution modeled on a checkerboard variety unless there are four cubes from K^c, three of one variety of one of its complement.*

Proof. Using an argument similar to the one in Lemma 3, if there are $0 \leq k \leq 4$ cubes in K^c and they do not split as in the lemma, then we can match these with k distinct m-universal corners of some checkerboard variety v. We use the $4 - k$ cubes from K_m and four cubes from K_o to provide a matching for the remaining corners of v. □

Lemma 5. *Given a set S of eleven minimal cubes, if at least one cube is of a variety from each of the sets K_o, K_m, and K^c, then we are guaranteed a corner solution modeled on a checkerboard variety.*

Proof. We separate this into cases based on the number of cubes in K.

$|S|_K \leq 2$: In this case, we have nine cubes in K^c, which by Lemma 2 guarantees us a matching of seven corners of a checkerboard variety. Using one cube from K_o or K_m, we can match the eighth corner.

$3 \leq |S|_K \leq 6$: By taking mirror images, we can assume that this set contains at least one cube of a variety from K_o and two from K_m. We also have at least five cubes from K^c, so by Lemma 2 we can match at least five corners of some checkerboard variety.

We recall that checkerboard varieties have two each of two o- and m-universal corners as well as their mirror images. This means that if the matching consists of two o-universal and three m-universal corners from a checkerboard variety v, then there is also a matching that consists of three o-universal and two m-universal corners from the checkerboard variety v^c. Furthermore, if the matching with v consists of four o-universal and one m-universal corner, we can take one of the checkerboard cubes in an o-universal corner position and match it with its mirror, an m-universal corner.

In summary, we can assume that the checkerboard varieties provide a match for three o-universal and two m-universal corners of some checkerboard variety. The one K_o- and two K_m-variety cubes complete the corner solution.

$7 \leq |S|_K \leq 10$: Using the pigeonhole principle and mirror images we may assume that at least four of the cubes are from varieties in K_o. These cases are all covered by Lemma 4.

□

Theorem 1. *Given eleven arbitrary minimal cubes, we are guaranteed a corner solution.*

Proof. We have shown in Corollary 1 and Lemma 2 that any eight cubes from K_o (or K_m), or any eleven cubes from K^c guarantees a corner solution. Lemma 5 extends this result to eleven cubes where all subsets K_o, K_m, and K^c are represented. To prove the theorem, it is sufficient to show that eleven cubes from subsets K_o and K_m or from K_o and K^c always guarantee a solution as well. The former case is straightforward, since four each of cubes from subsets K_o and K_m guarantees a corner solution by Lemma 4. Otherwise there are at least eight cubes from one subset, and Corollary 1 applies. For the latter case, we proceed by the number of cubes in K_o.

$1 \leq |S|_{K_o} \leq 3$: If we have exactly eight cubes from K^c then by Lemma 2 we can match at least six corners of some checkerboard variety v using checkerboard cubes; otherwise, we can match seven. Using similar reasoning as in Lemma 5 we may assume that the four m-universal corner positions are filled. Then we can fill in the remaining two or fewer corners of a corner solution modeled on v using K_o cubes.

$4 \leq |S|_{K_o} \leq 6$: We always have at least four K_o cubes and at least five checkerboard cubes, so by Lemma 4, we can build a corner solution modeled on a checkerboard variety.

$|S|_{K_o} = 7$: By Lemma 1 we can fill in four characteristic corners of some variety in K, and by choosing the appropriate variety in the corner class we may assume that the variety is in K_o. Then we can use a checkerboard cube to fill in one of the o-universal corners, and the three remaining cubes from K_o to fill in the rest of them.

$|S|_{K_o} \geq 8$: These cases are handled by Corollary 1. □

Theorem 1 answers the first question in the Introduction, that any eleven cubes have a subset of eight which form a corner solution. In the next section, we will build off of this result to determine when a corner solution can be extended to a frame.

4. Constructing The 3-Frame

In this section we show that, given twenty arbitrary minimal four-colored color cubes, we can complete the frame of a $3 \times 3 \times 3$ cube modeled on one of the minimal varieties. A number of our initial results focus on having sufficient cubes to fill in the characteristic corners and characteristic edges of a $3 \times 3 \times 3$ frame. We note that all varieties in K_o, resp. K_m, have the o-universal, resp. m-universal, corners, and each checkerboard variety possesses two types of m-universal corners and two types of o-universal corners. Furthermore, as previously mentioned, varieties from K have all universal edges and each checkerboard variety has five out of the six universal edges, with each individual variety missing a unique edge. Thus, if we can fill in the characteristic positions of a frame, it is usually a simple matter to fill the universal positions of the frame given a sufficient number of cubes.

We start with a lemma about very specific collections of varieties. Fixing two colors, we look at the two varieties that have single-color corners of those colors and the two varieties that have both single-color edges of those colors. Table 4 shows the case of green and red.

	RRR	RRG	RRW	RRB	GGG	GGR	GGW	GGB	RR	GG
R1	1	1	1	1					3	
G1					1	1	1	1		3
GR2		1	1			1		1	1	1
GR3		1		1		1	1		1	1

Table 4: Characteristic Corners and Edges for Some Red and Green Varieties

Lemma 6. *Fix two colors. If for both colors there is one cube with a single-color corner and two cubes with a single-color edge, then those six cubes can fill the characteristic corners and edges of the frame of a $3 \times 3 \times 3$ cube modeled on some variety in K.*

Proof. Since cubes of the same corner class have the same characteristic corners and edges, we will work in K_o. Further, using a color permutation we may assume the set of six cubes consists of varieties R1, G1, and two each with red-red and green-green edges.

From Table 8, we see that every cube with a characteristic edge has at least two out of three two-color corners (all three if the variety has a single-color corner). In particular, every cube with a red-red edge has either a RRG or RRW corner, and every cube with a green-green edge has either a GGB or GGR corner. Since R1 and G1 have all four characteristic corners, we can match R1, G1, and one each of the cubes with single-color edges to the four characteristic corners of GR2. The remaining two cubes with single-color edges fill in the characteristic edges. (Note: by an analogous argument, we could have matched to a GR3 variety.) □

We come to the first main result of the section, which determines the number of cubes from K required to build a $3 \times 3 \times 3$ frame modeled on some variety in K. This is a significant enough reduction that a computer search can be implemented as part of the proof.

Theorem 2. *Given 16 cubes of varieties in K, we can always fill the characteristic corners and characteristic edges of the frame of a $3 \times 3 \times 3$ cube modeled on a variety in K.*

Proof. Given the set of sixteen cubes, we know from Lemma 1 that we can take any subset of seven cubes and fill in the characteristic corners of the frame of a $3 \times 3 \times 3$ cube modeled on a variety in K. If we can use the remaining cubes to fill in the characteristic edges of this frame we are done, so we assume that this is not the case. We will consider two situations, based on whether there is a single-color corner in the corner solution or not.

In the first situation, let us say that the corner solution has a two-color red and green corner. From Table 8 these varieties have two characteristic edges, so we may assume that the single-color red edge in the frame cannot be filled. It is possible that there are no cubes with a red characteristic edge, or perhaps that of the twelve cubes remaining, there is precisely one with both a characteristic red and green edge (varieties in the corner classes of [GR2] or [GR3]). In any case, this means that there are at least twelve cubes of some combination of the nine corner classes without red-red edges: [B1], [W1], [G1], [BG1], [BG4], [GW2], [GW3], [BW2], and [BW3]. There are $\binom{19}{8} = 75,582$ cases to consider, and we wish to show for each case that we can match the characteristic corners and edges for some variety in K. In fact, since no cube we consider has a red characteristic edge, the only varieties we need to consider are the K_o-representatives from the nine corner classes.

We check each case using Mathematica code which is similar to the one described at the beginning of Section 3. Here, we use David Bevan's Multisets package [1] to enumerate each possible option and convert it to a bipartite graph for each of the varieties we would like to match. We then look for matchings of size six for varieties with two-color corners and matchings of size seven for varieties with single-color corners. Our code produces a match for every case.

We next address the situation where the corner solution has a single-color corner, say red (variety R1). Since R1 has three characteristic red edges, if we cannot complete the characteristic positions in a frame then there are at most two other cubes in the collection with a red characteristic edge, leaving ten without. When we rerun the Mathematica code for ten cube collections, we find that there are characteristic matchings as claimed except for three cases: when all ten cubes are of varieties B1, G1, and W1, where one appears six times and the other two appear twice. In this case we can appeal to Lemma 6 using red and the color associated to the variety which appears six times. □

Lemma 7. *Take an arbitrary set of twenty minimal* 4*-color cubes, at most five of which are varieties in* K^c*. Assume that we can use cubes in the set to fill in the characteristic corners and edges of the frame of a* $3\times3\times3$ *cube modeled on a variety in* K*. Then we can complete the frame of a cube modeled on a variety in* K*.*

Proof. By assumption, the four characteristic corners and two or three characteristic edges of some frame have already been filled, leaving the four universal corners and nine or ten universal edges to be filled. By Table 8, varieties in K^c have five out of six universal edges, so we can fit as many of these varieties as we have into universal edge positions, leaving eight or nine cubes left to be matched. These remaining cubes are of varieties from K_o or K_m. They all have all six universal edges, so they can be used to fill any remaining open edge positions. And since there are at least eight cubes remaining, there are at least four of one type to fill in the universal corners–this will determine whether the final frame models a variety from K_o or from K_m. ☐

This next result says that checkerboard corner solutions are very flexible. We state this result for an $n \times n \times n$ frame, since we will also use it in Section 5.

Lemma 8. *Take an arbitrary collection of* $12n - 16$ *minimal cubes. If one can construct a checkerboard corner solution from this set, one can also construct the frame of a (possibly different)* $n \times n \times n$ *cube modeled on some checkerboard variety.*

Proof. Let v be the modeled variety whose corner solution we can construct, and assume that we have filled in as many edge positions in the frame as we can. We have reproduced part of Table 8 showing corner and edge data for checkerboard varieties in Table 5.

	RBW	RGB	RWG	GWB	GBW	RGW	RBG	RWB	RG	WB	RW	GB	RB	GW
BW7	2			2	2			2		4	2	2	2	2
GR7		2	2			2	2		4		2	2	2	2
BG7		2		2	2		2		2	2		4	2	2
RW7	2		2			2		2	2	2	4		2	2
GW7			2	2	2	2			2	2	2	2		4
BR7	2	2					2	2	2	2	2	2	4	

Table 5: Characteristic Corners and Edges for Checkerboard Varieties

Table 5 shows that all edges of a checkerboard variety are universal. In addition, each checkerboard variety lacks a unique universal edge, and no two lack the same edge. Therefore, if there are two different unfilled edge positions in the frame, one can always match a checkerboard cube with at least one of those positions. We conclude that if we cannot complete a frame for a corner solution modeled on v, then we are missing at most one type of universal edge, which we denote by e. Let v_1 be the unique checkerboard variety that lacks e. We will determine lower bounds on the number of copies of variety v_1 in the original collection of cubes.

We note that we can place v_1 into any of the positions it matches with the frame modeled on v, and thereby swap v_1 with the cube originally matched into that position. Therefore, if we cannot complete edges of type e in the frame, this implies that the cubes in those positions in the frame are also of variety v_1. This is in addition to at least one more copy of v_1, since we cannot fill in all edges of type e. From Table 5 we see that the variety v_1 shares four corners and ten edges with v when $v_1 \neq v^c$, v_1's complement. On the other hand, when $v_1 = v^c$, the varieties share no corners and eight edges. So when $v_1 = v^c$, there are at least $8(n-2)+1$ copies of v_1, otherwise, there are at least $10(n-2)+5$ copies.

When $v_1 \neq v^c$ and $n \geq 3$

$$10(n-2) + 5 \geq 7(n-2) + 8.$$

That is, there are always enough cubes of variety v_1 to fill all characteristic corners and seven edges of a frame modeled on variety v_1. Since all varieties lack at most one edge, one can fill at least five edges of a frame modeled on v_1 with cubes of other varieties. The collection of cubes of variety v_1 complete the frame.

When $v_1 = v^c$, as any collection of cubes can be matched with at least eight edges of any variety, as long as we have enough copies of v_1 to fill in four edges and eight corners, we can complete a frame modeled on v_1. In other words, we need

$$8(n-2)+1 \geq 4(n-2)+8.$$

This occurs when $n \geq 4$.

The last case is when $v_1 = v^c$ and $n = 3$. We break the collection down as follows: let S be the set of eight cubes used for the corner solution of v; let T be the collection of at least nine copies of $v_1 = v^c$; and let U be the remaining three or fewer cubes. We note that there is no copy of v_1 in S since all the cubes in S have at least one corner in the complementary corner set to v_1. We build a new checkerboard corner solution by taking four copies of v_1 from T and four cubes from S, specifically two that were used for two identical corners in the original corner solution of v and two that were used for the identical corners' mirror images. By Table 5 these eight cubes form a corner solution of a unique checkerboard variety different from v and v_1. We start filling in the frame using cubes from U. Since T contains cubes which are not of variety v_1, the sets S and T can fill in the up to four edge positions that the other subset cannot. Together, they finish the frame. □

Theorem 3. *Given an arbitrary collection of twenty minimal cubes, we can construct the frame of a $3 \times 3 \times 3$ cube modeled on a minimal variety.*

Proof. By Lemma 2, if there are eleven cubes of varieties in K^c, then we have a corner solution modeled on a checkerboard variety, which can be completed to a frame by Lemma 8. Therefore, we may assume that at least ten cubes are in K_o and K_m, and using mirror images if necessary, that there are at least as many cubes from varieties from K_o as from K_m. Next, Lemma 4 implies that when there are at least four other cubes from K_m and K^c (with one exception) then we can build a corner solution modeled on a checkerboard variety, which again can be completed by Lemma 8. Consequently, we only need to consider the eleven cases listed in Table 6. (The * indicates that the four cubes of varieties in K^c are split so that three are of a single variety and one is of that variety's complement, as in Lemma 4.)

For these cases, we have at least sixteen cubes from K, so Theorem 2 implies that we can always fill the characteristic positions of a frame modeled on some variety in K. Then since fewer than five cubes are from K^c, Lemma 7 implies that we can complete the frame. □

K_o	20	19	19	18	18	18	17	17	17	17	16
K_m	0	1	0	2	1	0	3	2	1	0	0
K^c	0	0	1	0	1	2	0	1	2	3	4*

Table 6: Cases with No Checkerboard Solutions

5. Constructing The General n-Frame

The argument for constructing an $n \times n \times n$ frame with $n > 3$ is less involved than the $n = 3$ case. Roughly speaking, we look at which single-color edge occurs most in a collection and try to model a variety that uses it.

Definition 3. For a fixed color, a bucket is the collection of the seven corner classes with a single-color edge of that color.

As an example, the red bucket consists of the corner classes [R1], [GR2], [GR3], [RW1], [RW4], [BR2], and [BR3]. From Table 8, these are the varieties that have characteristic red-red edges, and at least two corners where red occurs twice. Notice also that most cube varieties are in two buckets, except for the corner classes of {[B2], [G1], [R1], [W1]}, which are in only one. Given a set of s cubes, S, let t be the total number of cubes in S which are in {[B2], [G1], [R1], [W1]}. Then the set S contributes

$$2(s - t) + t = 2s - t \tag{1}$$

to the four buckets, so some bucket contains at least $\lceil \frac{2s-t}{4} \rceil$ cubes.

Theorem 4. *When $n \geq 4$, given $12(n - 2) + 8$ minimal cubes, we are guaranteed a frame solution for an $n \times n \times n$ cube.*

Proof. We have a couple of previous results that allow us to restrict the cube sets we consider. By Lemma 8, we know that whenever we can construct a corner solution modeled on a checkerboard variety, the frame can be completed, and Lemma 4 provides conditions when this happens. Therefore, we may assume that we have no more than four cubes from K_m and/or K^c in the collection, and at least $12(n-2)+4 = 12n - 20$ from K_o. Furthermore, if two varieties with single-color corners, say G1 and R1, occur n times each, then we can fill in the characteristic positions of a variety with edges of those colors, like GR2 and GR3: two copies of G1 and R1 each share two disjoint characteristic corners with GR2 and GR3, then the remaining $n - 2$ copies of G1 and R1 fill in the single-color (characteristic) edges. Place K_o varieties into o-universal corners positions, then fill the universal edge positions with the remaining cubes, starting with those from K^c. We conclude that if there are more than $4(n - 1) + 1$ cubes of varieties in {B2, G1, R1, W1}, then we can complete the frame.

When these two situations do not happen, we know from Equation 1 that some bucket must contain at least

$$\left\lceil \frac{2\ (12n-20)-(4n-3)}{4} \right\rceil = 5n-9$$

cubes of K_o varieties. For the bucket with the most cubes (say red), we consider cases arranged by the multiplicity of the cube with a single-colored corner (variety R1). We need to show that we can fill the characteristic corners and edges of some variety. Because all the cubes are in the red bucket, and $5n-9>4+3(n-2)$ for $n>3$, once the characteristic corners of a frame are filled in it is always possible to fill in the remaining characteristic red edge(s). Four other cubes of varieties from K_o fill the o-universal corners, and any cube from the collection can be used to fill in universal edge positions in the frame.

$|R1| \geq 2$: By Table 8, any two cubes in the red bucket provide two characteristic corners of a corner solution modeled on R1, and two copies of R1 supply the other two.

$|R1| = 1$: We start by trying to build a frame modeled on R1. The copy of R1 is used for the single-color corner, and any two other cubes can be used to match two other two-color characteristic corners. If another cube can be found for the last characteristic corners then it is straightforward to finish the frame. The only way this cannot happen is if all remaining $5n-10$ cubes are of two varieties with the same two-color red corners, say BR2 and GR3 (see Table 8). Assume that variety GR3 occurs with greatest multiplicity of at least

$$\left\lceil \frac{5n-10}{2} \right\rceil = 2n-5+\left\lceil \frac{n}{2} \right\rceil,$$

which is always greater than n. We will complete a frame modeled on GR3. Up to two copies of BR2 can be used to fill in the characteristic corners that BR2 shares with GR3. Then two copies of GR3 complete the two characteristic corners, and $n-2$ copies of GR3 complete the green characteristic edge. If variety BR2 occurs with multiplicity 0 or 1, then there are enough copies of GR3 to fill the four characteristics corners and green characteristic edge.

$|R1| = 0$: This means there are at most six varieties represented in the red bucket. We separate the varieties into pairs based on the colors of their characteristic edge colors: {BR2, BR3}, {GR2, GR3}, and {RW1, RW4}. We note from Table 8 that any cube in the red bucket can be used in some corner position with two red faces on any frame modeled after a red bucket cube. Therefore, to complete the characteristic positions of a frame, we need to verify that we can fill in the other three positions as well as the non-red characteristic edge. This requires $n+1$ cubes. Now for $n \geq 5$, $\lceil \frac{5n-9}{3} \rceil$ is always at least as big as $n+1$, so some pair of varieties will occur at least $n+1$ times, say {BR2, BR3}. If BR2 occurs the most, use a cube from a different pair to fill in either corner RRG or RRB. Use up to two copies of BR3 to fill in at least one other characteristic corner, two copies of BR2 to finish filling in the last two characteristic corners, and remaining copies of BR2 and BR3 to fill the blue characteristic edge.

We treat the case $n = 4$ separately. If there are at least five cubes in some pair then the approach in the last paragraph works. We may therefore assume that the number of cubes in the pairs is four, four and three, specifically four cubes each in $\{$BR2, BR3$\}$ and $\{$GR2, GR3$\}$, and three in $\{$RW1, RW4$\}$. We assume that variety BR2 occurs at least as much as BR3. If we can use cubes from GR2 and GR3 to fill in corners RRG and RRB in a frame modeled on BR2, then two copies of BR2 complete the corners and the two remaining cubes in BR2, BR3 fill in the two blue characteristic edge positions in the frame. From Table 8, the only way this cannot happen is if there are four copies of GR2 and three copies of RW4. In this case, we build a frame modeled after GR2. We use two copies of RW4 to fill in corners RRG and RRW, two copies of GR2 to complete the characteristic corners, and the last two copies of GR2 to fill in the two green characteristic edge positions. □

We now have that the minimum number of cubes suffices for the construction of a frame when $n \geq 3$, so we have proved our main theorem.

Theorem 5. *The frame numbers for the* 4*-Color Cubes puzzle are* $\mathrm{fr}(2) = 11$ *and* $\mathrm{fr}(n) = 12n - 16$ *for* $n \geq 3$*.*

6. Universal Sets

In the context of the 6-Color Cubes puzzle, Haraguchi asks in [10] if there is a minimal set of cubes which can be used to construct all possible corner solutions. He answers his question in the affirmative, providing a collection of twelve cubes with this "universal" property. We answer the analogous question in our situation.

A *universal set* of four-color cubes is one that can be used to construct corner solutions modeled on any variety of four-colored color cubes. An interesting question is to determine universal sets of minimum size.

Lemma 9. *The minimum possible size of a universal set of* 4*-color cubes is ten.*

Proof. The smallest possible number of cubes that can be contained in a universal set of 4-color cubes is ten. There are precisely four corner classes with a single-color corner, so we need at least those four varieties in the collection. There are also twelve characteristic corners which have two colors, three each of which are used to model R1, W1, G1, and B1. By Table 8, a variety can fill in characteristic corners on at most two out of the four varieties with single-color corners. Therefore, we need at least another six cubes in the minimal collection, for a total of ten cubes. □

This bound is realizable: there is a set of ten cubes that is universal, consisting of the varieties

R1, W1, B1, G2, BR8, BG8, BW8, GR8, GW8, and RW8.

This set has the desirable property of being invariant under permutations of the color palette. Therefore, instead of needing to check all sixty-eight color cube varieties, it is sufficient to show that one can construct a corner solution modeled on one representative of each color permutation equivalence class of cubes. One possibility for the representatives consists of the thirteen cubes in Figures 2 and 3 at the end of this paper. Note that our universal set contains non-minimal varieties. This makes sense; if minimal cubes are in some sense most restrictive, then non-minimal cubes, having a larger collection of corners, should be more flexible.

7. Complexity

By Theorem 5, we know that a collection of eleven cubes always contains a subset of eight which can be used to build a corner solution, and that for $n \geq 3$, as long as there are enough cubes to build a frame, then it is always possible to do so. In this short section we address the complexity of finding an explicit solution, showing that this is a polynomial time problem.

Start with an arbitrary collection C of $12n - 16$ cubes. For a given variety v of 4-color cubes we build a bipartite graph G_v with two sets of $12n - 16$ vertices. The first set has one vertex for each cube in C. The other set has a vertex for each corner and edge position in an n-frame modeled on v. We connect two vertices with an edge when the cube from C can be used in the appropriate position for a solution modeled on v. Therefore, G_v is a graph with $O(n)$ vertices and $O(n^2)$ edges. Clearly, there is a solution to the Color Cubes Puzzle when G_v has a maximum matching of size $12n - 16$. The complexity of matching algorithms has been studied extensively, and it is known, for example, that finding a maximum matching in bipartite graphs under our assumptions is solvable in polynomial time [6, Section 27.3]. One may have to look through all sixty-eight varieties to find the solution, but this only adds a constant to the calculation. And Theorem 5 guarantees that for $n > 2$, some solution will be found.

8. Further Investigations

The results in this paper represent our first look at some of the problems that originate in color cubes, and we end with a number of open questions for the interested reader. One obvious generalization is to vary the number of colors on each cube.

In fact, this paper is one in a series that considers this option; results in this vein have been completed for two and three colors [2] and six colors [3]. We summarize in Table 7 what is known about the number of cubes, $\mathrm{fr}(n)$, required to solve the Color Cubes puzzle for different size color palettes. The case with five colors seems to be very difficult and, to our knowledge, has not been analyzed. One can also consider the Color Cubes puzzle with more than six colors. In this case, a natural Coloring Condition would have no color appear on any cube more than once.

Colors	2	3	4	5	6
$\mathrm{fr}(2)$	11	23	11	unknown	24
$\mathrm{fr}(3)$	20	35	20	unknown	24
$\mathrm{fr}(n)$, $n \geq 4$	$12n-16$	$16n-17$	$12n-16$	unknown	$12n-16$

Table 7: Frame number for k-color cubes, $2 \leq k \leq 6$

The solutions presented in this paper involve only the exterior faces of the frame, but none of the faces that are hidden. The requirement that faces that touch also match is the *Domino Condition*. MacMahon's original statement of the eight cubes puzzle in [14] was actually under the assumption of the Domino Condition, and this adds a significant layer of complexity to our problem. For example, we note that eight cubes of a single 4-color variety will never result in a $2 \times 2 \times 2$ solution satisfying the Domino Condition.

Problem 1. Determine conditions on a set a cubes so that there is a corner solution that also satisfies the Domino Condition.

J. Conway constructed an elegant way to build corner solutions satisfying the Domino Condition for the 6-color case [12]. Is there an analogous solution for four colors? There are related and more difficult versions of this problem for n-frames in general.

Another set of open questions deals with changing the coloring assumption for frames. In the proofs in this paper, all constructed solutions were modeled on 4-color cubes. However, one should be able to construct cubes with fewer colors. For example, given eight copies of R1, a cube with a single-color red corner, one can construct an all-red corner solution.

Problem 2. Determine conditions under which a set of 4-color cubes contains a subset of eight which forms a corner solution modeled on some 3-color cube variety.

We expect that there is a minimal number so that a set of 4-color cubes of that size or larger always has such a subset. What is that number?

Finally, there are some questions dealing with universal sets.

Problem 3. Investigate which sets of minimal size can used to construct n-frames for all sixty-eight 4-color cube varieties. How does this number change if we only use minimal varieties in the universal set?

In [8], Gardner writes that MacMahon's color cubes "...have become a classic of recreational geometry" and that "it is a chore to make a set, but the effort brings rich rewards." We believe the results in this paper with a related set of cubes show that Gardner's words remain timely, and that there is still a rich collection of open questions to pursue.

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References

[1] D. Bevan, Wolfram library archive: Multisets package, `http://library.wolfram.com/infocenter/MathSource/8115/` (2011), online, accessed June 13, 2018.

[2] E. Berkove, D. Cervantes-Nava, D. Condon, A. Eickemeyer, R. Katz, and M. Schulman, The color cubes puzzle with 2 and 3 colors, in *The Mathematics of Various Entertaining Subjects, Volume 2*, Princeton University Press, Princeton, 2017.

[3] E. Berkove, D. Condon, D. Cervantes-Nava, and R. Katz, Automorphisms of S_6 and the color cubes puzzle, *Australas. J. Combin.* **68** (2017), 71-93.

[4] E. Berkove, J. Van Sickle, B. Hummon, and J. Kogut. An analysis of the (colored cubes)^3 puzzle, *Discrete Math.* **308** (7) (2008), 1033-1045.

[5] G. Chartrand, *Graphs as Mathematical Models*, Prindle, Weber & Schmidt, Inc., Boston, 1977.

[6] T. Cormen, C. Leiserson, and R. Rivest, *Introduction to Algorithms*, The MIT Electrical Engineering and Computer Science Series, MIT Press, Cambridge, MA; McGraw-Hill Book Co., New York, 1990.

[7] E. Demaine, M. Demaine, S. Eisenstat, T. Morgan, and R. Uehara, Variations on instant insanity, in *Space-efficient Data Structures, Streams, and Algorithms*, volume 8066 of *Lecture Notes in Comput. Sci.*, Springer, Heidelberg, 2013.

[8] M. Gardner, *Fractal Music, Hypercards and More...*, W. H. Freeman and Company, New York, 1992.

[9] M. Gardner, *Sphere Packing, Lewis Carroll, and Reversi*, volume 3 of *New Martin Gardner Mathematical Library*, Cambridge University Press, Cambridge; Mathematical Association of America, Washington, DC, 2009. Martin Gardner's new mathematical diversions, Reprinted from *Scientific American* **1959** (chapters 2, 14) **1960** (chapters 1, 3–10, 12, 13) and **1961** (chapters 11, 15–20).

[10] K. Haraguchi, On a generalization of "Eight Blocks to Madness" puzzle, *Discrete Math.* **339** (4) (2016), 1400-1409.

[11] S. Kahan, "Eight blocks to madness"—a logical solution, *Math. Mag.* **45** (1972), 57-65.

[12] J. Köller. Macmahon's coloured cubes, `http://www.mathematische-basteleien.de/macmahon.htm`, online, accessed June 13, 2018.

[13] P. MacMahon, On the thirty cubes that can be constructed with six differently coloured squares, *Proc. London Math. Soc.* **S1-24** (1893), 145-155.

[14] P. MacMahon, *Combinatory Analysis*, Cambridge University Press, London, 1915.

[15] P. MacMahon, *New Mathematical Pastimes*, Cambridge University Press, London, 1921.

9. Appendix

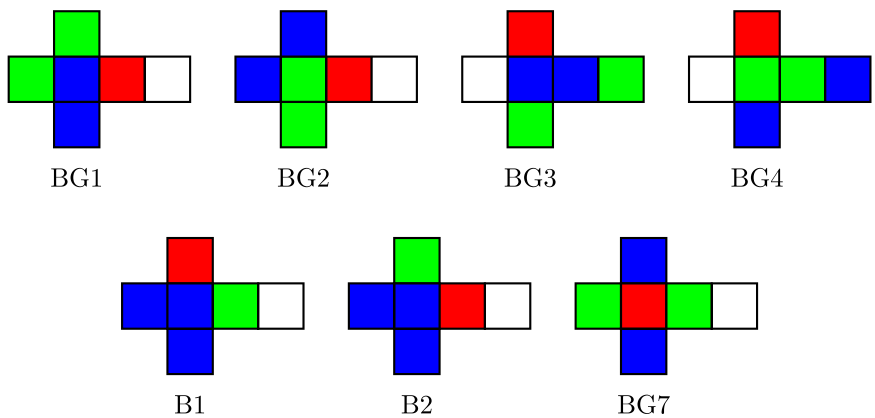

Figure 2: Representatives of the Seven Minimal Varieties

The nomenclature for cube varieties is determined by the color(s) that appear the most. Varieties X# have color X appearing three times. There are five equivalence classes under color permutation of varieties of this type. Varieties XY#, on the other hand, have colors X and Y both occurring twice. These varieties fall into eight equivalence classes. To go from X# to Y#, one swaps colors X and Y. To go from XY# to ZW#, if colors in exactly one position are different (XY and XZ, for example) colors Y and Z are exchanged. Otherwise, both colors are exchanged (Z for X and W for Y).

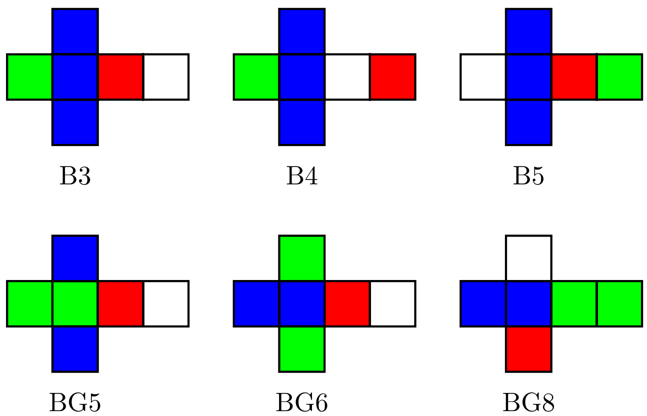

Figure 3: Representatives of the Six Non-Minimal Varieties

		Characteristic Corners																*m*-Universal Corners				*o*-Universal Corners				Characteristic Edges				Universal Edges					
	Variety	RRR	RRG	RRW	RRB	GGG	GGR	GGB	GGW	WWW	WWR	WWG	WWB	BBB	BBR	BBG	BBW	RBW	RGB	RWG	GWB	GBW	RGW	RBG	RWB	RR	GG	WW	BB	RG	WB	RW	GB	RB	GW
K_o	R1	1	1	1	1																	1	1	1	1	3				2	1	2	1	2	1
	G1					1	1	1	1													1	1	1	1		3			2	1	1	2	1	2
	W1									1	1	1	1									1	1	1	1			3		1	2	2	1	1	2
	B2													1	1	1	1					1	1	1	1				3	1	2	1	2	2	1
	BG1							1	1						1	1						1	1	1	1		1		1	1	1	1	3	2	2
	BG4						1	1								1	1					1	1	1	1		1		1	2	2	1	3	1	1
	BR2		1		1										1		1					1	1	1	1	1			1	2	2	1	1	3	1
	BR3			1	1										1	1						1	1	1	1	1			1	1	1	2	2	3	1
	GR2		1	1			1	1														1	1	1	1	1	1			3	1	2	2	1	1
	GR3		1		1		1		1													1	1	1	1	1	1			3	1	1	1	2	2
	BW2										1		1			1	1					1	1	1	1			1	1	1	3	2	2	1	1
	BW3											1	1		1		1					1	1	1	1			1	1	1	3	1	1	2	2
	GW2						1		1			1	1									1	1	1	1		1	1		2	2	1	1	1	3
	GW3							1	1		1	1										1	1	1	1		1	1		1	1	2	2	1	3
	RW1			1	1						1	1										1	1	1	1	1		1		1	1	3	1	2	2
	RW4		1	1							1		1									1	1	1	1	1		1		2	2	3	1	1	1
K_m	RW2		1	1							1		1					1	1	1	1					1		1		2	2	3	1	1	1
	RW3			1	1						1	1						1	1	1	1					1		1		1	1	3	1	2	2
	GW1							1	1		1	1						1	1	1	1						1	1		1	1	2	2	1	3
	GW4						1		1			1	1					1	1	1	1						1	1		2	2	1	1	1	3
	BW1											1	1		1		1	1	1	1	1							1	1	1	3	1	1	2	2
	BW4										1		1			1	1	1	1	1	1							1	1	1	3	2	2	1	1
	GR1		1		1		1		1									1	1	1	1					1	1			3	1	1	1	2	2
	GR4		1	1			1	1										1	1	1	1					1	1			3	1	2	2	1	1
	BR1			1	1										1	1		1	1	1	1					1			1	1	1	2	2	3	1
	BR4		1		1										1		1	1	1	1	1					1			1	2	2	1	1	3	1
	BG2						1	1								1	1	1	1	1	1						1		1	2	2	1	3	1	1
	BG3							1	1						1	1		1	1	1	1						1		1	1	1	1	3	2	2
	B1													1	1	1	1	1	1	1	1								3	1	2	1	2	2	1
	W2									1	1	1	1					1	1	1	1							3		1	2	2	1	1	2
	G2					1	1	1	1									1	1	1	1						3			2	1	1	2	1	2
	R2	1	1	1	1													1	1	1	1					3				2	1	2	1	2	1
K^c	BW7																	2			2	2			2						4	2	2	2	2
	GR7																		2	2			2	2						4		2	2	2	2
	BG7																		2		2	2		2						2	2		4	2	2
	RW7																	2		2			2		2					2	2	4		2	2
	GW7																			2	2	2	2							2	2	2	2		4
	BR7																	2	2					2	2					2	2	2	2	4	
Non-Minimal Cubes	GR6		2																1	1	1	1	1	1		1				4	1	1	2	1	2
	RW6			2														1		1	1	1	1		1	1				1	2	4	1	1	2
	BR5				2													1	1		1	1		1	1	1				1	2	1	2	4	1
	GR5						2											1	1	1			1	1	1		1			4	1	2	1	2	1
	BG5							2										1	1		1	1		1	1		1			1	2	1	4	2	1
	GW5								2									1		1	1	1	1		1		1			1	2	2	1	1	4
	RW5										2							1	1	1			1	1	1			1		2	1	4	1	2	1
	GW6											2							1	1	1	1	1	1				1		2	1	1	2	1	4
	BW5												2					1	1		1	1		1	1			1		1	4	1	2	2	1
	BR6														2			1	1	1			1	1	1				1	2	1	2	1	4	1
	BG6															2			1	1	1	1	1	1					1	2	1	1	4	1	2
	BW6																2	1		1	1	1	1		1				1	1	4	2	1	1	2
	R4		2	2														1	1					1	1	2				3	1	3	1	2	
	R5			2	2														1	1			1	1		2				2		3	1	3	1
	R3		2		2													1		1			1		1	2				3	1	2		3	1
	G3						2	2												1	1	1	1				2			3	1	1	3		2
	G4							2	2										1	1			1	1			2			2		1	3	1	3
	G5						2		2										1		1	1		1			2			3	1		2	1	3
	W3										2	2						1			1	1			1			2			2	3	1	1	3
	W4											2	2					1		1			1		1			2		1	3	2		1	3
	W5										2		2							1	1	1	1					2		1	3	3	1		2
	B3														2	2		1			1	1			1				2		2	1	3	3	1
	B4															2	2	1	1					1	1				2	1	3	1	3	2	
	B5														2		2		1		1	1		1					2	1	3		2	3	1
	BW8										1	1			1	1		1			1	1			1			1	1		2	2	2	2	2
	GR8			1	1			1	1										1	1			1	1		1	1			2		2	2	2	2
	BG8						1		1						1		1		1		1	1		1			1		1	2	2		2	2	2
	RW8		1		1							1	1					1		1			1		1	1		1		2	2	2		2	2
	GW8						1	1			1		1							1	1	1	1				1	1		2	2	2	2		2
	BR8		1	1												1	1	1	1					1	1	1			1	2	2	2	2	2	

Table 8: Table of Corners and Edges for all sixty-eight 4-color Cubes