#A69 INTEGERS 18 (2018)

TERNARY SMIRNOV WORDS AND GENERATING FUNCTIONS

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Received: 8/27/17, Accepted: 6/24/18, Published: 8/17/18

Abstract

We demonstrate that enumeration problems related to words with neighbouring letters being always different (Smirnov words) are most efficiently done using generating functions.

– Dedicted to Robert Tichy on the occasion of his 60th birthday.

1. Introduction

Smirnov words [1] are characterized by the property that the words $x_1 \dots x_n$ must satisfy the condition that $x_i \neq x_{i+1}$ for all $i = 1, \dots, n-1$. To obtain generating functions for them is a well known process [1]; for a recent application of the principle, the reader is referred to [2].

Koshy and Grimaldi [3] treated several enumerations around *ternary* (Smirnov) words (an alphabet with 3 letters is underlying) in an elementary fashion, which involves long computations. The aim of the present note is to show how Smirnov words and generating functions get such results in a painless way. Generalizations to more than 3 letters and additional parameters are immanent.

Many results may be expressed by Jacobsthal and Jacobsthal-Lucas numbers

$$J_n = \frac{2^n - (-1)^n}{3} \quad \text{and} \quad j_n = 2^n + (-1)^n.$$

This is not essential, and is always possible, since one can replace 2^n by $J_{n+1} + J_n$ and $(-1)^n$ by $J_{n+1} - 2J_n$.

[1]The author was supported by an incentive grant of the National Research Foundation of South Africa.

2. Ternary Smirnov Words Starting and Ending With the Letter 0

We assume that the 3 letters are $0, 1, 2$ and set up generating functions $f_i(z)$ via the system

$$\begin{aligned}
f_0(z) &= z + zf_1(z) + zf_2(z),\\
f_1(z) &= zf_0(z) + zf_2(z),\\
f_2(z) &= zf_0(z) + zf_1(z).
\end{aligned}$$

This system is inspired by random walks, and describes how a new letter is appended, which must be different from the previous one. The letter z marks the letters. The functions $f_i(z)$ count Smirnov words starting with 0 and ending with i. In [3] the main interest was in $f_0(z)$, but the other two are easy as well. The solutions are:

$$\begin{aligned}
f_0(z) &= \frac{1}{2} + \frac{1}{6}\frac{1}{1-2z} - \frac{2}{3}\frac{1}{1+z},\\
f_1(z) = f_2(z) &= -\frac{1}{2} + \frac{1}{6}\frac{1}{1-2z} + \frac{1}{3}\frac{1}{1+z},\\
f_0(z) + f_1(z) + f_2(z) &= -\frac{1}{2} + \frac{1}{2}\frac{1}{1-2z}.
\end{aligned}$$

Reading off the coefficient of z^n, we find for $n \ge 1$:

$$[z^n]f_0(z) = \frac{1}{6}2^n - \frac{2}{3}(-1)^n = 2J_{n-2}, \quad [z^n]f_1(z) = \frac{1}{6}2^n + \frac{1}{3}(-1)^n = J_{n-1}.$$

In total, we get of course 2^{n-1} Smirnov words of length n, starting with 0.

Corollary 1 of [3] gives J_{n-2} for the number of Smirnov words, starting with 01 and ending with 0. This follows without computation from the symmetry of such words having the second letter equal to 1 resp. 2.

3. Ternary Smirnov Words Starting and Ending With the Letter 0 – Counting the Letters 0

We can use essentially the same equations, but now using a second variable u, counting the numbers 0:

$$\begin{aligned}
f_0(z,u) &= zu + zf_1(z,u) + zf_2(z,u),\\
f_1(z,u) &= zuf_0(z,u) + zf_2(z,u),\\
f_2(z,u) &= zuf_0(z) + zf_1(z,u).
\end{aligned}$$

We find
$$f_0(z,u)=\frac{zu(1-z)}{1-z-2z^2u}=\frac{zu}{1-\frac{2z^2}{1-z}u}.$$
Therefore (for $n\ge 2$ and $k\ge 2$)
$$[u^k]f_0(z,u)=2^{k-1}\frac{z^{2k-1}}{(1-z)^{k-1}},$$
and
$$[z^nu^k]f_0(z,u)=[z^n]2^{k-1}\frac{z^{2k-1}}{(1-z)^{k-1}}=[z^{n+1-2k}]\frac{2^{k-1}}{(1-z)^{k-1}}=2^{k-1}\binom{n-1-k}{k-2}.$$
The total number of letters 0 in Smirnov words of length n is given via $\frac{\partial}{\partial u}f_0(z,1)$:
$$\begin{aligned}&[z^n]\bigg(\frac{1}{18}\frac{1}{(1-2z)^2}+\frac{5}{54}\frac{1}{1-2z}-\frac{4}{9}\frac{1}{(1+z)^2}+\frac{8}{27}\frac{1}{1+z}\bigg)\\&=\frac{1}{18}(n+1)2^n+\frac{5}{54}2^n-\frac{4}{9}(n+1)(-1)^n+\frac{8}{27}(-1)^n\\&=\frac{1}{18}n2^n+\frac{4}{27}2^n-\frac{4}{9}n(-1)^n-\frac{4}{27}(-1)^n.\end{aligned}$$
Analogous computations are as follows:
$$f_1(z,u)=\frac{z^2u^2}{1-z-2z^2u}=\frac{z^2u^2}{1-z}\frac{1}{1-\frac{2z^2}{1-z}u},$$
and
$$[u^k]f_1(z,u)=\frac{z^2}{1-z}[u^{k-2}]\frac{1}{1-\frac{2z^2}{1-z}u}=2^{k-2}\frac{z^{2k-2}}{(1-z)^{k-1}},$$
and
$$[z^nu^k]f_1(z,u)=2^{k-2}[z^{n+2-2k}]\frac{1}{(1-z)^{k-1}}=2^{k-2}\binom{n-k}{k-2}.$$
Furthermore,
$$\frac{\partial}{\partial u}f_1(z,1)=-\frac{1}{2}+\frac{1}{18}\frac{1}{(1-2z)^2}+\frac{4}{27}\frac{1}{1-2z}+\frac{2}{9}\frac{1}{(1+z)^2}+\frac{2}{27}\frac{1}{1+z},$$
and the coefficient of z^n is
$$\frac{1}{18}n2^n+\frac{11}{54}2^n+\frac{2}{9}n(-1)^n+\frac{8}{27}(-1)^n.$$

4. Ternary Smirnov Words Starting and Ending With the Letter 0 – Counting the Letters 1

In a very similar way we can count the total number of occurrences of the letter 1. A second variable u is counting the numbers 1:

$$
\begin{aligned}
f_0(z,u) &= z + zuf_1(z,u) + zf_2(z,u),\\
f_1(z,u) &= zf_0(z,u) + zf_2(z,u),\\
f_2(z,u) &= zf_0(z) + zuf_1(z,u).
\end{aligned}
$$

We find, for instance,

$$\frac{\partial}{\partial u} f_0(z,1) = \frac{1}{18}\frac{1}{(1-2z)^2} - \frac{7}{54}\frac{1}{1-2z} - \frac{1}{9}\frac{1}{(1+z)^2} + \frac{5}{27}\frac{1}{1+z}$$

and the coefficient of z^n is

$$\frac{1}{18}n2^n - \frac{2}{27}2^n - \frac{1}{9}n(-1)^n + \frac{2}{27}(-1)^n.$$

Other results are similar.

5. Words Interpreted as Decimal (and Other) Numbers

Let $q \geq 3$ be a base, and define $\mathsf{value}(x_1 \dots x_n) = x_1 q^{n-1} + x_2 q^{n-2} + \cdots + x_n$. As in [3], we consider S_n, the sum over the values of all Smirnov words of length n, rendered by zeros. The recursion $\mathsf{value}(x_1 \dots x_n) = q\mathsf{value}(x_1 \dots x_{n-1}) + x_n$ is obvious.

We can set up the following equations:

$$
\begin{aligned}
f_0(z,u) &= z + zf_1(z,u^q) + zf_2(z,u^q),\\
f_1(z,u) &= zuf_0(z,u^q) + zuf_2(z,u^q),\\
f_2(z,u) &= zu^2 f_0(z,u^q) + zu^2 f_1(z,u^q).
\end{aligned}
$$

Then the coefficient of $z^n u^k$ in $f_i(z,u)$ is the number of Smirnov words ending in i, having length n and $\mathsf{value} = k$. We are only aiming at the total value, so in other words we should differentiate $f_i(z,u)$ and evaluate it at $u = 1$. The evaluations of $f_i(z) := f_i(z,1)$ have been given before, so, with $g_i(z) = \frac{\partial}{\partial u} f_i(z,1)$, we get

$$
\begin{aligned}
g_0(z) &= z + zg_1(z) + zg_2(z),\\
g_1(z) &= qzg_0(z) + qzg_2(z,u) + zf_0(z) + zf_2(z),\\
g_2(z) &= qzg_0(z) + qzg_1(z) + 2zf_0(z) + 2f_1(z).
\end{aligned}
$$

This system is readily solved, with result

$$g_0(z) = \frac{1}{2(q-1)(1+2q)(1-2qz)} - \frac{1}{(2+q)(q-1)(1+qz)} - \frac{q}{2(2+q)(q-1)(1-2z)} + \frac{q}{(q-1)(1+2q)(1+z)}.$$

Reading off the coefficient of z^n leads to the numbers S_n:

$$S_n = \frac{1}{2(q-1)(1+2q)}2^n q^n - \frac{1}{(2+q)(q-1)}(-1)^n q^n - \frac{q}{2(2+q)(q-1)}2^n + \frac{q}{(q-1)(1+2q)}(-1)^n.$$

Of course, the coefficients of $g_1(z)$ and $g_2(z)$ could also be determined, as well as higher moments, by more differentiations. We are not doing this here; we just want to indicate how things could be done efficiently.

The values for $q = 3$ are derived in [3] using long computations.

6. Inversions

An inversion (in a Smirnov word as always in this paper) is a pair $x_i > x_j$ and $1 \le i < j \le n$. To count the total number of inversions, we proceed as follows. We consider a *marked* letter i, so that the word can be written as $x = wiy$, and we count the inversions with first letter i. They are given by the number of symbols $< i$ in y. So the generating functions decompose:

$$\frac{1}{z}f_0(z)f_0(z) + \frac{1}{z}f_1(z)g_1(z,u) + \frac{1}{z}f_2(z)h_2(z,u).$$

The first factor refers to the words wi (ending at i), the second factor to the reversed word $y^R i$, where in $g_1(z,u)$ the second variable is used to count the letters 0, and in $h_2(z,u)$ the second variable is used to count the letters 0 or 1. Since the designated letter i is part of both factors, we must divide by z.

Now we have again

$$\begin{aligned} f_0(z) &= z + zf_1(z) + zf_2(z), \\ f_1(z) &= zf_0(z) + zf_2(z), \\ f_2(z) &= zf_0(z) + zf_1(z) \end{aligned}$$

and also

$$\begin{aligned} g_0(z,u) &= zu + zug_1(z,u) + zug_2(z,u), \\ g_1(z,u) &= zg_0(z,u) + zg_2(z,u), \\ g_2(z,u) &= zg_0(z,u) + zg_1(z,u) \end{aligned}$$

and

$$\begin{aligned}
h_0(z,u) &= zu + zuh_1(z,u) + zuh_2(z,u),\\
h_1(z,u) &= zuh_0(z,u) + zuh_2(z,u),\\
h_2(z,u) &= zh_0(z,u) + zh_1(z,u).
\end{aligned}$$

These systems are readily solved, and we only give the final answer:

$$\begin{aligned}
\frac{\partial}{\partial u}\Big(\frac{1}{z}f_0(z)f_0(z) + \frac{1}{z}f_1(z)g_1(z,u) + \frac{1}{z}f_2(z)h_2(z,u)\Big)\Big|_{u=1}&\\
= \frac{1}{18}\frac{1}{(1-2z)^3} - \frac{5}{54}\frac{1}{(1-2z)^2} + \frac{1}{9}\frac{1}{(1+z)^3} + \frac{4}{27}\frac{1}{(1+z)^2}&.
\end{aligned}$$

The coefficient of z^n in this is routinely found to be

$$\Big(\frac{n^2}{36} - \frac{n}{108} - \frac{1}{27}\Big)2^n - \Big(\frac{n^2}{18} + \frac{n}{54} - \frac{1}{27}\Big)(-1)^n.$$

This formula was derived in [3] with significantly more effort.

7. Conclusion

The methods described here allow for all kinds of more refined results; we don't have space here to present more calculations.

The restriction to ternary words and first/last letter being 0 is superficial and could easily be replaced by larger alphabets and different conditions on the letters.

Calculations were done by Maple; they are easy to reproduce by the reader using any computer algebra system.

References

[1] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.

[2] U. Freiberg, C. Heuberger, and H. Prodinger, Application of Smirnov words to waiting time distributions of runs, *Electron. J. Combin.* **24**(3) (2017), P3.55, 12 pages.

[3] T. Koshy and R. Grimaldi, Ternary words and Jacobsthal numbers, *Fibonacci Quart.* **55** (2017), 129–136.