ON THE SLACK EULER PAIR FOR VECTOR PARTITION

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Abstract

This work provides a slack version of Subbarao's generalization of Euler's partition theorem for vector partitions. We give two proofs; one uses multivariate generating functions, and the other is a direct bijection of Glaisher-type.

1. Introduction

The following result due to Euler [7] is arguably the first theorem in the theory of integer partitions.

Theorem 1 (Euler, 1748). *The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.*

There are numerous ways to generalize the above theorem. We mention four of them here to motivate our result. The interested readers are referred to [2, 5, 10, 11, 13] for other refinements and generalizations.

1. Glaisher's bijection. When odd parts are viewed as parts not divisible by 2, one wonders if something similar can be said for all moduli. In 1883, Glaisher [8] found a purely bijective proof of Euler's theorem and was able to extend it to cover all moduli.

   Theorem 2 (Glaisher, 1883). *For $t \geq 2, n \geq 1$, the number of partitions of n into parts not divisible by t equals the number of partitions of n into parts not repeated more than $t-1$ times.*

2. Cheema's vector version. When we consider a "k-partite" number (i.e., an ordered k-tuple of nonnegative integers not all zero), Euler's theorem naturally extends to vector partitions. By *vector partition* we mean a representation of vector $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ as a sum of k-partite numbers: $\mathbf{n} = \boldsymbol{\xi}^{(1)} + \boldsymbol{\xi}^{(2)} + \cdots$ subject to the lexicographic ordering $\boldsymbol{\xi}^{(i)} \geq \boldsymbol{\xi}^{(i+1)}$ of the parts:

$$\boldsymbol{\xi}^{(i)} = (\xi_1^{(i)}, \ldots, \xi_k^{(i)}) > (\xi_1^{(i+1)}, \ldots, \xi_k^{(i+1)}) = \boldsymbol{\xi}^{(i+1)}$$

provided $\xi_j^{(i)} > \xi_j^{(i+1)}$, where j is the least inter such that $\xi_j^{(i)} \neq \xi_j^{(i+1)}$. Cheema [6] observed the following extension of Euler's theorem.

Theorem 3 (Cheema, 1964)**.** *For a fixed k and every $\mathbf{n}$, the number of vector partitions of $\mathbf{n}$ into distinct k-partite numbers equals the number of vector partitions of $\mathbf{n}$ in which each part $(\xi_1^{(i)}, \xi_2^{(i)}, \ldots, \xi_k^{(i)})$ has at least one odd component.*

3. Andrews' Euler pair. There is yet another direction to generalize Euler's theorem. The original idea was due to Schur, but Andrews [1] was the first to present it in the following full generality.

Theorem 4 (Andrews, 1969)**.** *If S_1 and S_2 are any two sets of positive integers such that $2S_1 \subseteq S_1$ and $S_2 = S_1 - 2S_1$, then the number of partitions of n into distinct parts taken from S_1 equals the number of partitions of n into parts taken from S_2.*

4. Subbarao's unification. Shortly after, Subbarao [12] was able to establish a common generalization of 1, 2 and 3 (see also Theorem 12.2 in [3]).

Theorem 5 (Subbarao, 1971)**.** *Let $t \geq 2$ and $k \geq 1$ be integers. Let S_1 and S_2 be sets of positive integers. The number of vector partitions of $\mathbf{n}$ into parts $\boldsymbol{\xi}^{(i)} = (\xi_1^{(i)}, \ldots, \xi_k^{(i)})$ in which $\xi_j^{(i)} \in S_1$ for all i and $1 \leq j \leq k$, and where no part repeats more than $t-1$ times, always equals the number of vector partitions of $\mathbf{n}$ into parts $\boldsymbol{\xi}^{(i)}$ with $\xi_j^{(i)} \in S_1$, $1 \leq j \leq k$, and some $\xi_{j_0}^{(i)} \in S_2$ for all i, given that*

$$tS_1 \subseteq S_1 \quad \text{and} \quad S_2 = S_1 - tS_1. \tag{1}$$

Those pairs (S_1, S_2) satisfying condition (1) are the so-called "Euler pairs of order t" [12].

In an effort to generalize Subbarao's theorem, we are led to the following slack version of Euler pair.

Definition 6. Two sets of nonnegative integers S_1 and S_2 are called a *slack Euler pair of order t* if they satisfy

$$tS_1 \cap S_2 = \emptyset \quad \text{and} \quad S_1 \subseteq tS_1 \cup S_2. \tag{2}$$

Clearly when the second condition becomes tight, i.e., $S_1 = tS_1 \cup S_2$, and with 0 excluded from S_1, S_2, then (2) reduces to (1) and we get back to the original Euler pair of order t. For a positive number n, denote $[n] = \{1, 2, \ldots, n\}$. We need one more definition to make a neat theorem.

Definition 7. Let S be any set of positive integers. We define the *set of k-partite numbers with respect to S* as

$$V(S) := \{\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_k) : \xi_j \in S, 1 \le j \le k\}.$$

Now we can state the main result of this note, which is the following generalization of Subbarao's theorem.

Theorem 8. *Fix $k \ge 1, t \ge 2$, a k-partite number $\mathbf{n}$, and take any slack Euler pair of order t, say (S_1, S_2). Take $T = tS_1 \cup S_2$. Denote by $\mathcal{D}_t(\mathbf{n})$ the set of vector partitions $\mathbf{n} = \boldsymbol{\xi}^{(1)} + \boldsymbol{\xi}^{(2)} + \cdots$, such that for each i, $\boldsymbol{\xi}^{(i)} \in V(T)$, and whenever $\boldsymbol{\xi}^{(i)} \in V(S_1)$, then $\boldsymbol{\xi}^{(i)}$ can be repeated at most $t-1$ times. And denote by $\mathcal{O}_t(\mathbf{n})$ the set of vector partitions $\mathbf{n} = \boldsymbol{\eta}^{(1)} + \boldsymbol{\eta}^{(2)} + \cdots$, such that for each i, $\boldsymbol{\eta}^{(i)} \in V(T) - V(tS_1)$. Then we have*

$$|\mathcal{D}_t(\mathbf{n})| = |\mathcal{O}_t(\mathbf{n})|.$$

We note that in two recent papers, Andrews [4] presented a Glaisher-type proof of a finite version for Euler's theorem, and Nyirenda [9] derived a Glaisher-type proof of a finite version for Glaisher's theorem. Their work has motivated us to look for a similar finite version for Subbarao's theorem. Interestingly, all these finite versions emerge as special cases of the above theorem by taking $S_1 = [N]$ and $T = [tN]$ for some given positive integer N.

In the following two sections, we supply two different proofs of Theorem 8. We end our work with a concrete example to illustrate our bijection.

2. The Proof Using Generating Functions

We first give a proof using multivariate generating functions. Given $\mathcal{O}_t(\mathbf{n})$ and $\mathcal{D}_t(\mathbf{n})$ as defined in Theorem 8, denote by $f(x_1, x_2, \cdots, x_k)$ (resp. $g(x_1, x_2, \cdots, x_k)$) the generating function of $\mathcal{O}_t(\mathbf{n})$ (resp. $\mathcal{D}_t(\mathbf{n})$).

1st proof of Theorem 8.

$$
\begin{aligned}
f(x_1, x_2, \cdots, x_k) &= \prod_{(i_1,i_2,\cdots,i_k)\in V(T)-V(tS_1)} (1 - x_1^{i_1}, x_2^{i_2}, \cdots, x_k^{i_k})^{-1} \\
&= \frac{\prod_{(i_1,i_2,\cdots,i_k)\in V(tS_1)}(1 - x_1^{i_1}x_2^{i_2}\cdots\ x_k^{i_k})}{\prod_{(i_1,i_2,\cdots,i_k)\in V(T)}(1 - x_1^{i_1}x_2^{i_2}\cdots\ x_k^{i_k})} \\
&= \frac{\prod_{(i_1,i_2,\cdots,i_k)\in V(S_1)}(1 - x_1^{ti_1}x_2^{ti_2}\cdots\ x_k^{ti_k})}{\prod_{(i_1,i_2,\cdots,i_k)\in V(T)}(1 - x_1^{i_1}x_2^{i_2}\cdots\ x_k^{i_k})} \\
&= \frac{\prod_{(i_1,i_2,\cdots,i_k)\in V(S_1)}(1 + x_1^{i_1}\cdots\ x_k^{i_k} + \cdots + x_1^{(t-1)i_1}\cdots\ x_k^{(t-1)i_k})}{\prod_{(i_1,i_2,\cdots,i_k)\in V(T)-V(S_1)}(1 - x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k})} \\
&= g(x_1, x_2, \cdots, x_k).
\end{aligned}
$$

□

3. The Bijective Proof of Glaisher's Type

The Glaisher-type bijection, utilized by Andrews [4] and Nyirenda [9], can be applied to prove our theorem as well, with appropriate modifications.

2nd proof of Theorem 8. We first construct a map ϕ from $\mathcal{O}_t(\mathbf{n})$ to $\mathcal{D}_t(\mathbf{n})$. Given a vector partition of $\mathbf{n} = \boldsymbol{\eta}^{(1)} + \boldsymbol{\eta}^{(2)} + \cdots$, where $\boldsymbol{\eta}^{(i)} \in V(T) - V(tS_1)$, we first compute the smallest nonnegative integer a_i for each $\boldsymbol{\eta}^{(i)}$, such that $t^{a_i}\boldsymbol{\eta}^{(i)} \in V(T) - V(S_1)$. Note that $t^b\boldsymbol{\eta}^{(i)} \in V(S_1)$ for each $b = 0, 1, \ldots, a_i - 1$. In the case that such an a_i does not exist (which can never happen if S_1 is a finite set, the situation with Andrews' and Nyirenda's finite versions), simply take $a_i = \infty$, and consequently $t^b\boldsymbol{\eta}^{(i)} \in V(S_1)$ for all $b \geq 0$. Note that the following statements hold.

1) If $a_i = 0$, then $\boldsymbol{\eta}^{(i)}$ is fixed under ϕ.

2) If $a_i > 0$ (or $a_i = \infty$), then $\boldsymbol{\eta}^{(i)} \in V(S_1)$ and we may need to transform this vector.

Let s_i be the number of times the vector $\boldsymbol{\eta}^{(i)}$ appears in the original vector partition, then we can uniquely write

$$s_i = b_i t^{a_i} + h_i, \quad \text{where } 0 \leq h_i \leq t^{a_i} - 1. \tag{3}$$

Then, write the t-ary expansion of h_i:

$$h_i = \sum_{l=0}^{r} m_i(l)t^l, \quad \text{where } 0 \leq m_i(l) \leq t - 1.$$

Hence $s_i = b_i t^{a_i} + \sum_{l=0}^{r} m_i(l) t^l$. Now we let $s_i \cdot \boldsymbol{\eta}^{(i)}$ denote s_i occurrences of $\boldsymbol{\eta}^{(i)}$, and define

$$\phi(s_i \cdot \boldsymbol{\eta}^{(i)}) := b_i \cdot t^{a_i} \boldsymbol{\eta}^{(i)} + \sum_{l=0}^{r} m_i(l) \cdot t^l \boldsymbol{\eta}^{(i)}.$$

Note that in either case, the map ϕ preserves the total weight of all the vectors, and therefore when we sum up all the "vector parts" obtained in 1) and 2), this gives rise to another vector partition of $\mathbf{n}$. And since $t^{a_i}\boldsymbol{\eta}^{(i)} \in V(T) - V(S_1), t^l \boldsymbol{\eta}^{(i)} \in V(S_1)$ with $0 \leq m_i(l) \leq t-1$, we see that this new vector partition is indeed in $\mathcal{D}_t(\mathbf{n})$.

It is clear how to construct the inverse of ϕ. Namely, we take any vector partition $\mathbf{n} = \boldsymbol{\xi}^{(1)} + \boldsymbol{\xi}^{(2)} + \cdots$ in $\mathcal{D}_t(\mathbf{n})$, and transform each $\boldsymbol{\xi}^{(i)}$ as

$$\boldsymbol{\xi}^{(i)} \rightarrow t^{c_i} \cdot \frac{1}{t^{c_i}} \boldsymbol{\xi}^{(i)},$$

where $c_i \geq 0$ is the smallest integer such that $\frac{1}{t^{c_i}} \boldsymbol{\xi}^{(i)} \notin V(tS_1)$. □

We include the following example to illustrate our bijection.

Example 9. For $t = k = 3$,

$$\begin{aligned} S_1 &= \{3^m, m \geq 1\} \cup \{2, 5, 6, 8, 15\}, \\ 3S_1 &= \{3^m, m \geq 2\} \cup \{6, 15, 18, 24, 45\}, \\ S_2 &= \{0, 1, 2, 3, 5, 8, 10, 21, 33\}, \\ T &= \{3^m, m \geq 0\} \cup \{0, 2, 5, 6, 8, 10, 15, 18, 21, 24, 33, 45\}. \end{aligned}$$

For convenience, we denote r occurrences of $\boldsymbol{\eta}^{(i)}$ as $(\eta_1^{(i)}, \cdots, \eta_k^{(i)})^r$.

$$\mathbf{n} = (8,1,2)^4 + (6,0,6) + (3,9,9)^{11} + (3,2,5)^{50}, \quad \mathbf{n} = (221, 203, 363).$$

Denote this vector partition as $\boldsymbol{\eta}$. It is easy to check that all vector parts of $\boldsymbol{\eta}$ are in $V(T) - V(3S_1)$, so $\boldsymbol{\eta} \in \mathcal{O}_3(\mathbf{n})$. To compute $\phi(\boldsymbol{\eta})$, first we examine the a_i for each $\boldsymbol{\eta}_i$.

$(8,1,2) : (8,1,2) \in V(T) - V(S_1)$, so $a_i = 0$;
$(6,0,6) : (6,0,6) \in V(T) - V(S_1)$, so $a_i = 0$;
$(3,9,9) : t^m(3,9,9) = (3t^m, 9t^m, 9t^m) \in V(S_1)$, for all $m \geq 0$, so $a_i = \infty$;
$(3,2,5) : (3,2,5) \in V(S_1)$, $(9,6,15) \in V(S_1)$, $(27,18,45) \in V(T) - V(S_1)$, so $a_i = 2$.

Therefore $(8,1,2)$ and $(6,0,6)$ are fixed by ϕ, while for $(3,9,9)$, we compute $11 = 3^2 + 2 \cdot 3^0$, hence

$$(3,9,9)^{11} \to (27,81,81) + (3,9,9)^2.$$

Lastly, for $(3,2,5)$, we compute $50 = 5 \cdot 3^2 + 3^1 + 2 \cdot 3^0$, and hence

$$(3,2,5)^{50} \to (27,18,45)^5 + (9,6,15) + (3,2,5)^2.$$

To summarize, we have the correspondence:

$$\begin{aligned}\boldsymbol{\eta} &= (8,1,2)^4 + (6,0,6) + (3,9,9)^{11} + (3,2,5)^{50} \mapsto \\ \phi(\boldsymbol{\eta}) &= (27,81,81) + (27,18,45)^5 + (9,6,15) + (8,1,2)^4 \\ &\qquad + (6,0,6) + (3,9,9)^2 + (3,2,5)^2.\end{aligned}$$

We conclude by remarking that in Glaisher's original map, all $a_i = \infty$, while for Andrews' and Nyirenda's finite versions, all $a_i < \infty$, so our map weaves together these two extremes.

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