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ON SHORTENED RECURRENCE RELATIONS FOR GENOCCHI NUMBERS AND POLYNOMIALS

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Abstract
Motivated by the von Ettingshausen-Stern and Saalschütz-Gelfand formulas for Bernoulli numbers, in this paper we study shortened (or incomplete) recurrence relations for Genocchi numbers and polynomials, in which some of the preceding numbers and polynomials are completely excluded.

1. Introduction

Throughout this paper, we use the following symbolic notation for convenience. For any given $a, b \in \mathbb{C}$ (or $\mathbb{C}[x]$) and an infinite sequence $\{T_n\}_{n\geq 0}$ with $T_n \in \mathbb{C}$ (or $\mathbb{C}[x]$), we write for integers $m, k \geq 0$,

$$(aT_m + b)^k := \sum_{i=0}^{k} \binom{k}{i} a^i T_{m+i}\, b^{k-i}.$$

In other words, expand the left-hand side in full based on the binomial theorem and then replace $(T_m)^i$ by T_{m+i} for each $i = 0, 1, \ldots, k$. This is a modified version of Lucas' notation $T^m(aT + b)^k$ used in his paper [9].

This paper is related to the Genocchi numbers G_n and polynomials $G_n(x)$, $n = 0, 1, 2, \ldots$, that have been studied extensively in many areas of mathematics and physics such as number theory, combinatorial theory, differential topology, modular forms, p-adic analysis, quantum field theory, and others. They are formally defined by means of the generating functions

$$\mathcal{G}(t) : = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} \frac{G_n t^n}{n!} \quad (|t| < \pi);$$

$$\mathcal{G}(t; x) : = \frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{G_n(x) t^n}{n!} \quad (|t| < \pi).$$

The first few Genocchi numbers are: $0, 1, -1, 0, 1, 0, -3, 0, 17, 0, -155$, and so on (cf. the OEIS in [13]). Since $\mathcal{G}(-t) = -2t + \mathcal{G}(t)$, we have $(-1)^n G_n = G_n$ for all $n > 1$; and thus, $G_n = 0$ if $n > 1$ is odd.

As is easily seen from $\mathcal{G}(t; x) = \mathcal{G}(t)e^{xt}$, we have $G_n(0) = G_n$ and $G_n(x)$ is expressed in terms of Genocchi numbers, namely

$$G_n(x) = (G_0 + x)^n = \sum_{i=0}^{n} \binom{n}{i} G_i x^{n-i} \quad (n \geq 0). \tag{1}$$

Further, since $\mathcal{G}(t; x + y) = \mathcal{G}(t; x)e^{yt}$, we have $G_n(x + y) = (G_0(x) + y)^n$. Not surprisingly, this is actually equivalent to $\frac{d}{dx}G_n(x) = nG_{n-1}(x)$ (see, e.g., [3]).

Numerous number of recurrence relations for these numbers and polynomials have been developed over the years. Among them, the most basic ones are

$$\begin{aligned} &\text{(i)} \qquad G_0 = 0, \quad (G_0 + 1)^n + G_n = \begin{cases} 2 & (n = 1), \\ 0 & (n \geq 2); \end{cases} \\ &\text{(ii)} \qquad G_0(x) = 0, \quad (G_0(x) + 1)^n + G_n(x) = 2nx^{n-1} \quad (n \geq 1), \end{aligned} \tag{2}$$

which are deduced by observing the relations $\mathcal{G}(t)(e^t + 1) = 2t$ and $\mathcal{G}(t; x)(e^t + 1) = 2te^{xt}$, respectively. If we use the different relations $\mathcal{G}(t)(e^{-t} + 1) = 2te^{-t}$ and $\mathcal{G}(t; x)(e^{-t} + 1) = 2te^{(x-1)t}$, then the following recurrence relations are deduced:

$$\begin{aligned} &\text{(i)} \qquad (G_0 - 1)^n + G_n = (-1)^{n-1} 2n \quad (n \geq 0); \\ &\text{(ii)} \qquad (G_0(x) - 1)^n + G_n(x) = 2n(x - 1)^{n-1} \quad (n \geq 0). \end{aligned} \tag{3}$$

As is well-known, Genocchi numbers and polynomials are represented in terms of the Bernoulli numbers B_n and polynomials $B_n(x)$ defined by the generating functions

$$\mathcal{F}(t) := \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!} \quad (|t| < 2\pi);$$

$$\mathcal{F}(t; x) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x) t^n}{n!} \quad (|t| < 2\pi),$$

respectively. Indeed, by making use of the relations $\mathcal{G}(t) = 2\left(\mathcal{F}(t) - \mathcal{F}(2t)\right)$ and $\mathcal{G}(t; x) = 2\left(\mathcal{F}(t; x) - \mathcal{F}(2t; x/2)\right)$, it is shown that for all $n \geq 0$,

$$G_n = 2(1 - 2^n)B_n \quad \text{and} \quad G_n(x) = 2\left(B_n(x) - 2^n B_n(x/2)\right).$$

Therefore, by Fermat's little theorem and von Staudt-Clausen's theorem, we see that G_n is an integer; and thus from (1), $G_n(x)$ is a polynomial in $\mathbb{Z}[x]$.

This paper is considerably motivated by the following two classical formulas for Bernoulli numbers, which are so-called *shortened* (or *incomplete*) recurrence relations. For brevity, denoting $\widetilde{B}_n := (n+1)B_n$ for $n \geq 0$ and $B_n^* := B_n/n$ for $n \geq 1$, they are stated as follows:

- Von Ettingshausen-Stern's formula (cf. [7, 14]):

$$(\widetilde{B}_m + 1)^{m+1} = 0 \quad (m \geq 0); \tag{4}$$

- Saalschütz-Gelfand's formula (cf. [11, 8]):

$$(-1)^k (B_{k+1}^* + 1)^m + (-1)^m (B_{m+1}^* + 1)^k = -\frac{k!m!}{(k+m+1)!} \quad (m, k \geq 0). \tag{5}$$

The biggest feature of these formulas is that some of the preceding Bernoulli numbers are completely missing. So that it does not require the knowledge of all the preceding numbers up to B_{n-1} in order to compute B_n for any specified $n > 1$. Both formulas can be deduced by making use of the polynomial identity

$$(B_m(x) + y)^k = (B_k(x+y) - y)^m \quad (m, k \geq 1). \tag{6}$$

An elementary proof of this identity can be found, e.g., in [1, 2].

It is the main purpose of this paper to study shortened recurrence relations for Genocchi numbers and polynomials, referring to (4) and (5) in the Bernoulli number case. In Section 2, as preliminary, we present some elementary lemmas related to Genocchi polynomials which will be needed in the forthcoming discussion. In Section 3 we first prove a Genocchi polynomial analogue of (6) such that

$$\left(G_{m+1}^*(x) + y\right)^k = \left(G_{k+1}^*(x+y) - y\right)^m \quad (m, k \geq 0), \tag{7}$$

where $G_n^*(x) := G_n(x)/n$ for $n \geq 1$. Subsequently, by applying this identity we derive various kinds of shortened recurrence relations for Genocchi numbers and polynomials. We conclude this paper, in Section 4, with some additional remarks on shortened recurrence relations for Euler and tangent numbers and polynomials, which are closely related to the results obtained in Section 3.

2. Some Lemmas

Here, and in what follows, we use the following notation for simplicity:

$$\begin{aligned}
&G_n^*(x) := \frac{G_n(x)}{n}, \quad G_n^* := G_n^*(0) = \frac{G_n}{n} \quad \text{for } n \geq 1; \\
&\widetilde{G}_n(x) := (n+1)G_n(x), \quad \widetilde{G}_n := \widetilde{G}_n(0) = (n+1)G_n \quad \text{for } n \geq 0.
\end{aligned}$$

At the beginning of this section, we wish to present the following elementary lemma telling us that Genocchi polynomials form an Appell sequence, as well as Bernoulli polynomials.

Lemma 2.1. *For an integer $n \geq 0$ we have*

$$\text{(i)} \quad \frac{d}{dx} G_n(x) = \widetilde{G}_{n-1}(x); \quad \text{(ii)} \quad \int_y^x G_n(t)dt = G^*_{n+1}(x) - G^*_{n+1}(y).$$

Proof. The proof is quite easy. By direct calculations, we get from (1),

$$\begin{aligned} \frac{d}{dx} G_n(x) &= \sum_{i=0}^{n-1} \binom{n}{i} (n-i) G_i x^{n-1-i} = n \sum_{i=0}^{n-1} \binom{n-1}{i} G_i x^{n-1-i} \\ &= nG_{n-1}(x) = \widetilde{G}_{n-1}(x), \end{aligned}$$

as desired in (i). On the other hand, noting that $\frac{n+1}{n+1-i}\binom{n}{i} = \binom{n+1}{i}$, we have

$$\begin{aligned} \int_y^x G_n(t)dt &= \sum_{i=0}^{n} \binom{n}{i} \frac{G_i}{n+1-i} \left[t^{n+1-i}\right]_y^x \\ &= \frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} G_i \cdot (x^{n+1-i} - y^{n+1-i}) \\ &= \frac{G_{n+1}(x) - G_{n+1}(y)}{n+1} = G^*_{n+1}(x) - G^*_{n+1}(y), \end{aligned}$$

and hence, (ii) follows. □

As already mentioned in Section 1, the above (i) is equivalent to $G_n(x+y) = (G_0(x)+y)^n$ for $n \geq 0$.

For integers $n, k \geq 0$, let $P_k^{(n)}(x)$ be the polynomial in $\mathbb{Z}[x]$ defined by

$$P_k^{(n)}(x) := \sum_{j=0}^{k} (-1)^j (x+k-j)^n.$$

In particular, we have $P_{2k}^{(0)}(x) = 1$ and $P_{2k+1}^{(0)}(x) = 0$ for all $k \geq 0$.

Lemma 2.2. *With the above notation, we have*

$$2(n+1)P_k^{(n)}(x) = (G_0(x)+k+1)^{n+1} + (-1)^k G_{n+1}(x), \tag{8}$$

or equivalently,

$$2P_k^{(n)}(x) = G^*_{n+1}(x+k+1) + (-1)^k G^*_{n+1}(x). \tag{9}$$

Proof. Consider the functional identity

$$2t \sum_{j=0}^{k} (-1)^j e^{(x+k-j)t} = 2t \frac{e^{xt}(e^{(k+1)t} - (-1)^{k+1})}{e^t + 1} = \mathcal{G}(t;x)(e^{(k+1)t} + (-1)^k).$$

Differentiating the both ends of this identity $(n+1)$ times with respect to t based on Leibniz's rule and then putting $t=0$, it follows that

$$2(n+1)\sum_{j=0}^{k}(-1)^j(x+k-j)^n = \sum_{i=0}^{n+1}\binom{n+1}{i}G_i(x)(k+1)^{n+1-i} + (-1)^k G_{n+1}(x),$$

which shows that (8) follows. The second identity (9) is an easy translation of (8) based on the fact that $(G_0(x)+y)^{n+1}/(n+1) = G^*_{n+1}(x+y)$. □

Lemma 2.3. *For an integer $n \geq 0$ and an odd prime p with $p \nmid n+1$ we have*

$$P^{(n)}_{p-1}(x) \equiv G^*_{n+1}(x) \pmod{p}. \tag{10}$$

Proof. Take $k = p-1$ in (8). Noting that $G_n(x) \in \mathbb{Z}[x]$ for all $n \geq 0$, we obtain

$$2(n+1)P^{(n)}_{p-1}(x) = (G_0(x)+p)^{n+1} + (-1)^{p-1}G_{n+1}(x) \equiv 2G_{n+1}(x) \pmod{p},$$

which leads to (10) when dividing by $2(n+1)$. □

3. Main Results

In this section we derive, as main results, various types of shortened recurrence relations for Genocchi numbers and polynomials.

In what follows, we assume that l and m are arbitrary non-negative integers, unless otherwise noted. At first, by making use of Lemma 2.3, we give a complete proof of (7) that has many applications and uses.

Theorem 3.1. *We have*

$$(G^*_{m+1}(x)+y)^l = (G^*_{l+1}(x+y)-y)^m. \tag{11}$$

Proof. For an odd prime p with $p > m+l+1$, consider the following function of t having two indeterminate parameters x and y:

$$T^{(m)}_{p-1}(t;x,y) := e^{yt}\sum_{j=0}^{p-1}(-1)^j(x+p-1-j)^m e^{(x+p-1-j)t}.$$

Differentiate this function l times with respect to t based on Leibniz's rule and set $t=0$. Then, since $p \nmid m+1+r$ for $r = 0,1,\ldots,l$, we get from (10) in Lemma 2.3,

$$\begin{aligned}\frac{d^l}{dt^l}T^{(m)}_{p-1}(t;x,y)\Big|_{t=0} &= \sum_{r=0}^{l}\binom{l}{r}y^{l-r}\sum_{j=0}^{p-1}(-1)^j(x+p-1-j)^{m+r}\\ &= \sum_{r=0}^{l}\binom{l}{r}y^{l-r}P^{(m+r)}_{p-1}(x) \equiv \sum_{r=0}^{l}\binom{l}{r}y^{l-r}G^*_{m+1+r}(x)\\ &\equiv (G^*_{m+1}(x)+y)^l \pmod{p}.\end{aligned} \tag{12}$$

On the other hand, by using the binomial theorem, rewrite $T_{p-1}^{(m)}(t;x,y)$ as

$$\begin{aligned} T_{p-1}^{(m)}(t;x,y) &= \sum_{j=0}^{p-1}(-1)^j\left\{((x+y)+p-1-j)-y\right\}^m e^{((x+y)+p-1-j)t} \\ &= \sum_{r=0}^{m}\binom{m}{r}(-y)^{m-r}\sum_{j=0}^{p-1}(-1)^j((x+y)+p-1-j)^r e^{((x+y)+p-1-j)t}. \end{aligned}$$

Based upon this expression, since $p \nmid l+1+r$ for $r=0,1,\ldots,m$, we have

$$\begin{aligned} \frac{d^l}{dt^l}T_{p-1}^{(m)}(t;x,y)\Big|_{t=0} &= \sum_{r=0}^{m}\binom{m}{r}(-y)^{m-r}\sum_{j=0}^{p-1}(-1)^j\{(x+y)+p-1-j\}^{l+r} \\ &= \sum_{r=0}^{m}\binom{m}{r}(-y)^{m-r}P_{p-1}^{(l+r)}(x+y) \equiv \sum_{r=0}^{m}\binom{m}{r}(-y)^{m-r}G_{l+1+r}^*(x+y) \\ &\equiv (G_{l+1}^*(x+y)-y)^m \pmod p. \end{aligned} \tag{13}$$

So that, equating (12) and (13), it is shown that

$$(G_{m+1}^*(x)+y)^l \equiv (G_{l+1}^*(x+y)-y)^m \pmod p.$$

This congruence is valid for infinitely many odd primes $p > m+l+1$ and both sides do not depend on p, which imply that (11) must hold unconditionally. □

Applying (11), we can deduce the following shortened recurrence relations for $G_n^*(x)$ $(n \geq 1)$, in which the first $\min\{l,m\}$ polynomials are completely missing.

Theorem 3.2. *For an arbitrary integer $q \geq 1$ we have*

$$\begin{aligned} (G_{m+1}^*(x)+q)^l &+ (-1)^{q-1}(G_{l+1}^*(x)-q)^m \\ &= 2\sum_{k=1}^{q}(-1)^{k-1}(x+q-k)^l(x-k)^m. \end{aligned} \tag{14}$$

In particular,

$$(G_{m+1}^*(x)+1)^l + (G_{l+1}^*(x)-1)^m = 2x^l(x-1)^m. \tag{15}$$

Proof. Set $y=q$ in (11), namely

$$(G_{m+1}^*(x)+q)^l = (G_{l+1}^*(x+q)-q)^m. \tag{16}$$

Using (9) with $k = q - 1$, the right-hand side of this identity can be written as

$$\begin{aligned}
(G^*_{l+1}(x+q) - q)^m &= \sum_{i=0}^{m} \binom{m}{i} (-q)^{m-i} G^*_{l+1+i}(x+q) \\
&= \sum_{i=0}^{m} \binom{m}{i} (-q)^{m-i} \big\{ (-1)^q G^*_{l+1+i}(x) + 2P^{(l+i)}_{q-1}(x) \big\} \\
&= (-1)^q \sum_{i=0}^{m} \binom{m}{i} (-q)^{m-i} G^*_{l+1+i}(x) \\
&\quad + 2 \sum_{i=0}^{m} \binom{m}{i} (-q)^{m-i} \sum_{j=0}^{q-1} (-1)^j (x+q-1-j)^{l+i} \\
&= (-1)^q (G^*_{l+1}(x) - q)^m + 2 \sum_{j=0}^{q-1} (-1)^j (x+q-1-j)^l \\
&\qquad \times \sum_{i=0}^{m} \binom{m}{i} (-q)^{m-i} (x+q-1-j)^i \\
&= (-1)^q (G^*_{l+1}(x) - q)^m + 2 \sum_{j=0}^{q-1} (-1)^j (x+q-1-j)^l (x-1-j)^m \\
&= (-1)^q (G^*_{l+1}(x) - q)^m + 2 \sum_{k=1}^{q} (-1)^{k-1} (x+q-k)^l (x-k)^m.
\end{aligned}$$

Substituting this into (16), we get (14). For (15), take $q = 1$ in (14). □

Corollary 3.3. *For an arbitrary integer $q \geq 1$ we have*

$$\begin{aligned}
(-1)^{m+q} (G^*_{m+1} + q)^l + (-1)^l (G^*_{l+1} + q)^m \\
= 2 \sum_{k=1}^{q} (-1)^{q+k-1} (q-k)^l k^m + \varepsilon_{l,m}(q),
\end{aligned} \tag{17}$$

where $\varepsilon_{0,m}(q) := 2q^m$ and $\varepsilon_{l,m}(q) := 0$ for $l \geq 1$. In particular,

$$(-1)^m (G^*_{m+1} + 1)^l = (-1)^l (G^*_{l+1} + 1)^m. \tag{18}$$

Proof. Put $x = 0$ in (14), namely

$$(G^*_{m+1} + q)^l + (-1)^{q-1} (G^*_{l+1} - q)^m = (-1)^m 2 \sum_{k=1}^{q} (-1)^{k-1} (q-k)^l k^m. \tag{19}$$

Since $G^*_1 = (-1)^1 G^*_1 + 2$ and $G^*_n = (-1)^n G^*_n$ for all $n > 1$, the second term on the left-hand side of (19) is expressed as, without the sign $(-1)^{q-1}$,

$$\begin{aligned}(G^*_{l+1}-q)^m &= \sum_{i=0}^{m}\binom{m}{i}(-q)^{m-i}G^*_{l+1+i}\\ &= \sum_{i=0}^{m}\binom{m}{i}(-q)^{m-i}(-1)^{l+1+i}G^*_{l+1+i}+(-1)^m\varepsilon_{l,m}(q)\\ &= (-1)^{l+m+1}\sum_{i=0}^{m}\binom{m}{i}q^{m-i}G^*_{l+1+i}+(-1)^m\varepsilon_{l,m}(q)\\ &= (-1)^{l+m+1}(G^*_{l+1}+q)^m+(-1)^m\varepsilon_{l,m}(q).\end{aligned}$$

Substitute this into (19) and then multiply by $(-1)^{m+q}$ to obtain (17). The second identity (18) is nothing but the special case of (17) with $q=1$. □

Theorem 3.4. *For an arbitrary integer $r \geq 1$ we have*

$$\begin{aligned}&\sum_{i=0}^{l+r}\binom{l+r}{i}\binom{m+r+i}{r-1}G_{m+1+i}(x)\\ &\qquad+\sum_{j=0}^{m+r}(-1)^{m+r-j}\binom{m+r}{j}\binom{l+r+j}{r-1}G_{l+1+j}(x) \qquad (20)\\ &= 2r\sum_{k=0}^{r}\binom{l+r}{k}\binom{m+r}{r-k}x^{l+r-k}(x-1)^{m+k}.\end{aligned}$$

Proof. Consider (15) with l and m replaced by $l+r$ and $m+r$, respectively. That is,

$$\begin{aligned}&(G^*_{m+r+1}(x)+1)^{l+r}+(G^*_{l+r+1}(x)-1)^{m+r}\\ &=\sum_{i=0}^{l+r}\binom{l+r}{i}G^*_{m+r+1+i}(x)+\sum_{j=0}^{m+r}(-1)^{m+r-j}\binom{m+r}{j}G^*_{l+r+1+j}(x) \qquad (21)\\ &=2x^{l+r}(x-1)^{m+r}.\end{aligned}$$

Using (i) in Lemma 2.1 repeatedly, we have, for an integer a with $1 \leq a \leq n$,

$$\frac{d^a}{dx^a}G^*_n(x)=\frac{1}{n}\frac{d^a}{dx^a}G_n(x)=(n-1)_{a-1}G_{n-a}(x),$$

where $(z)_k := z(z-1)\cdots(z-k+1)$ is the falling factorial function of z. Based on this fact, by differentiating both sides of (21) r times with respect to x, we get

$$\begin{aligned}\sum_{i=0}^{l+r}\binom{l+r}{i}(m+r+i)_{r-1}G_{m+1+i}(x)&\\ +\sum_{j=0}^{m+r}(-1)^{m+r-j}\binom{m+r}{j}(l+r+j)_{r-1}G_{l+1+j}(x)&\\ =2\sum_{k=0}^{r}\binom{r}{k}(l+r)_k(m+r)_{r-k}x^{l+r-k}(x-1)^{m+k},&\end{aligned}$$

which leads to (20) after dividing by $(r-1)!$. □

Corollary 3.5. *For an integer $r \geq 1$ we have*

$$\sum_{i=0}^{l+r}\binom{l+r}{i}\binom{m+r+i}{r-1}G_{m+1+i} = (-1)^{m+l+r}\sum_{j=0}^{m+r}\binom{m+r}{j}\binom{l+r+j}{r-1}G_{l+1+j}. \tag{22}$$

In particular, if $r \geq 1$ is odd, then

$$\sum_{i=0}^{m+r}\binom{m+r}{i}\binom{m+r+i}{r-1}G_{m+1+i} = 0. \tag{23}$$

Proof. Putting $x = 0$ in (20), we obtain immediately

$$\sum_{i=0}^{l+r}\binom{l+r}{i}\binom{m+r+i}{r-1}G_{m+1+i} + (-1)^{m+r}\sum_{j=0}^{m+r}(-1)^j\binom{m+r}{j}\binom{l+r+j}{r-1}G_{l+1+j} = (-1)^{m+r}\delta_l(r), \tag{24}$$

where $\delta_0(r) := 2r$ and $\delta_l(r) := 0$ for $l \geq 1$. Noting that $G_1 = (-1)^1 G_1 + 2$ and $G_n = (-1)^n G_n$ for all $n > 1$, we can rewrite the second sum on the left-hand side of (24) as follows: without the sign $(-1)^{m+r}$,

$$\sum_{j=0}^{m+r}(-1)^j\binom{m+r}{j}\binom{l+r+j}{r-1}G_{l+1+j} = (-1)^{l+1}\sum_{j=0}^{m+r}\binom{m+r}{j}\binom{l+r+j}{r-1}G_{l+1+j} + \delta_l(r). \tag{25}$$

Here we used the fact that $l+1+j=1$ if and only if $l=j=0$; therefore, if $l=0$, then the term corresponding to $j=0$ in the sum on the left-hand side of (25) becomes

$$\binom{m+r}{0}\binom{r}{r-1}G_1 = rG_1 = r(-G_1+2) = -rG_1 + \delta_0(r).$$

To obtain (22) we have only to substitute (25) into (24). Further, if r is odd, then (23) is deduced from (22) by taking $l=m$ and dividing by 2. □

Theorem 3.6. *We have*

$$(G_m(x)+1)^l + (G_l(x)-1)^m = 2\{(l+m)x-l\}\, x^{l-1}(x-1)^{m-1}. \tag{26}$$

In particular,

$$\begin{aligned} \sum_{i=0}^{m}\binom{2m}{2i}G_{2(m+i)}(x) &= 2m(2x-1)\left(x(x-1)\right)^{2m-1}; \\ \sum_{i=0}^{m}\binom{2m+1}{2i+1}G_{2(m+i+1)}(x) &= (2m+1)(2x-1)\left(x(x-1)\right)^{2m}. \end{aligned} \tag{27}$$

Proof. If $l=m=0$, then (26) is trivial, because both sides vanish. If one of m and l equals 0, then (26) coincides with (2) (ii) or (3) (ii) in Section 1. When $l,m\geq 1$, take $r=1$ in (20), and replace l by $l-1$ and m by $m-1$. Then we have

$$\begin{aligned} \sum_{i=9}^{l}\binom{l}{i}G_{m+i}(x) + \sum_{j=9}^{m}(-1)^{m-j}\binom{m}{j}G_{l+j}(x) \\ = 2\left(mx^l(x-1)^{m-1} + lx^{l-1}(x-1)^m\right), \end{aligned}$$

which implies (26). Next, by taking $l=m$ in (26), we have

$$\sum_{i=0}^{m}(1+(-1)^{m+i})\binom{m}{i}G_{m+i}(x) = 2m(2x-1)\left(x(x-1)\right)^{m-1}.$$

In order to deduce the identities in (27), taking into account of the parity of m, pick up only the i's such that $m+i$ is even in the sum on the left-hand side. □

Corollary 3.7. *We have*

$$(G_m+1)^l + (-1)^{l+m}(G_l+1)^m = 0. \tag{28}$$

In particular,

$$(G_m+1)^m = 0. \tag{29}$$

Proof. Putting $x=0$ in (26), we have

$$(G_m+1)^l+(G_l-1)^m=\lambda_{l,m}, \tag{30}$$

where

$$\lambda_{l,m}:=\begin{cases}(-1)^{m-1}2m & (l=0,\, m\geq 0);\\ (-1)^m 2 & (l=1,\, m\geq 0);\\ 0 & (l>1,\, m\geq 0).\end{cases}$$

As mentioned repeatedly, using the fact that $G_1=1=(-1)^1G_1+2$ and $G_n=(-1)^nG_n$ unless $n=1$, the second term on the left-hand side of (30) becomes

$$(G_l-1)^m=\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}G_{l+j}=(-1)^{m+l}(G_l+1)^m+\lambda_{l,m}.$$

Substituting this into (30), we obtain (28). For (29), take $l=m$ in (28) and divide by 2. Here note that (29) is equivalent to the special case $r=1$ of (23). □

Incidentally, it should be mentioned that the above (29) is known as Seidel's identity (cf. Nielsen [10]). Actually, Seidel [12] has observed this identity together with a certain Pascal-type triangle that provides us with a combinatorial interpretation of Genocchi and tangent numbers (for details, see also [4, 5, 6]). As is easily seen, if $m>1$, then (29) is equivalent to $(G_m-1)^m=0$.

Putting $x=0$ in (27), we get immediately the following identities.

Corollary 3.8. *For an integer $m>0$ we have*

$$\sum_{i=0}^{m}\binom{2m}{2i}G_{2(m+i)}=0 \quad \textit{and} \quad \sum_{i=0}^{m}\binom{2m+1}{2i+1}G_{2(m+i+1)}=0.$$

The following are the shortened recurrence relations for $\widetilde{G}_n(x)$ and $\widetilde{G}_n$ obtained by observing a special case of Theorem 3.4.

Corollary 3.9. *We have*

$$\begin{aligned}(\widetilde{G}_m(x)+1)^{l+1}&+(\widetilde{G}_l(x)-1)^{m+1}\\ &=4\sum_{k=0}^{2}\binom{l+1}{k}\binom{m+1}{2-k}x^{l+1-k}(x-1)^{m-1+k};\end{aligned} \tag{31}$$

$$(\widetilde{G}_m+1)^{l+1}+(-1)^{l+m+1}(\widetilde{G}_l+1)^{m+1}=0. \tag{32}$$

Proof. Taking $r=2$ in (20), and replacing l by $l-1$ and m by $m-1$, we get

$$\sum_{i=0}^{l+1}\binom{l+1}{i}(m+1+i)G_{m+i}(x)+\sum_{j=0}^{m+1}\binom{m+1}{j}(-1)^{m+1-j}(l+1+j)G_{l+j}(x)$$
$$=4\sum_{k=0}^{2}\binom{l+1}{k}\binom{m+1}{2-k}x^{l+1-k}(x-1)^{m-1+k},$$

which is just the same as (31). For (32), putting $x = 0$ in (31), we have

$$(\widetilde{G}_m + 1)^{l+1} + (\widetilde{G}_l - 1)^{m+1} = \mu_{l,m}, \tag{33}$$

where

$$\mu_{l,m} := \begin{cases} (-1)^m 4(m+1) & (l = 0,\, m \geq 0); \\ (-1)^{m-1} 4 & (l = 1,\, m \geq 0); \\ 0 & (l > 1,\, m \geq 0). \end{cases}$$

Since $\widetilde{G}_1 = (-1)^1 \widetilde{G}_1 + 4$ and $\widetilde{G}_n = (-1)^n \widetilde{G}_n$ if $n = 0$ or $n > 1$, we may rewrite the second term on the left-hand side of (33) as follows:

$$\begin{aligned} (\widetilde{G}_l - 1)^{m+1} &= \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^{m+1-j} \widetilde{G}_{l+j} \\ &= \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^{m+1-j} (-1)^{l+j} \widetilde{G}_{l+j} + \mu_{l,m} \\ &= (-1)^{l+m+1} (\widetilde{G}_l + 1)^{m+1} + \mu_{l,m}. \end{aligned}$$

Substitute this identity into (33) in order to deduce (32). □

We note here that the right-hand side of (31) is equal to

$$2N_{l,m}(x) x^{l-1} (x-1)^{m-1},$$

where $N_{l,m}(x)$ is the quadratic polynomial in x such that

$$\begin{aligned} N_{l,m}(x) : &= (l^2 + m^2 + 2lm + 3l + 3m + 2)x^2 \\ &\quad - 2(l^2 + lm + 2l + m + 1)x + l^2 + l. \end{aligned}$$

Further note that the identity (32) is interesting only in the case when $l \neq m$, because the left-hand side vanishes if $l = m$.

4. Additional Remarks

In this last section, as some additional remarks, we deal with shortened recurrence relations for Euler and tangent numbers and polynomials. As will be seen below, they can be deduced automatically from the recurrence relations for Genocchi numbers and polynomials obtained in Section 4, and vice versa.

(I) Let E_n and $E_n(x)$, $n = 0, 1, 2, \ldots$, be the Euler numbers and polynomials defined by the generating functions

$$\mathcal{E}(t) := \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n t^n}{n!} \quad (|t| < \pi/2);$$
$$\mathcal{E}(t;x) := \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n(x) t^n}{n!} \quad (|t| < \pi),$$

respectively. From the relation $\mathcal{E}(2t;x) = \mathcal{E}(t)e^{(2x-1)t}$, we see that $E_n(x)$ can be expressed in terms of Euler numbers as follows:

$$E_n(x) = \frac{1}{2^n}\,(E_0 + 2x - 1)^n = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(2x-1)^i E_{n-i} \ \ (n \geq 0);$$

and thus we have $E_n(1/2) = E_n/2^n$. Further, it is easily shown that

$$\frac{d}{dx}E_n(x) = nE_{n-1}(x) \ \text{ and } \ \int_y^x E_{n-1}(t)dt = \frac{E_n(x) - E_n(y)}{n} \ \text{ for } n \geq 1.$$

On the other hand, since $\mathcal{G}(t;x) = \mathcal{E}(t;x)t$, we have, for all $n \geq 0$,

$$G_{n+1}(x) = (n+1)E_n(x); \ \text{ or equivalently, } \ G^*_{n+1}(x) = E_n(x).$$

Therefore, we see that every recurrence relation for Genocchi polynomials can be expressed by means of Euler polynomials, and vice versa.

For example, the identities (15) and (26) are transformed into

$$(E_m(x) + 1)^l + (E_l(x) - 1)^m = 2x^l(x-1)^m; \tag{34}$$
$$(\widetilde{E}_m(x) + 1)^{l+1} + (\widetilde{E}_l(x) - 1)^{m+1} = 2\{(l+m+2)x - l - 1\}x^l(x-1)^m, \tag{35}$$

respectively, where $\widetilde{E}_n(x) := \frac{d}{dx}E_{n+1}(x) = (n+1)E_n(x)$ for $n \geq 0$. Further, putting $x = 1/2$ in these identities and noting that $E_n = (-1)^n E_n$ for all $n \geq 0$, we can get the following shortened recurrence relations for Euler numbers:

$$(-1)^m(E_m + 2)^l + (-1)^l(E_l + 2)^m = 2;$$
$$(-1)^m(\widetilde{E}_m + 2)^{l+1} - (-1)^l(\widetilde{E}_l + 2)^{m+1} = 2(m - l),$$

where $\widetilde{E}_n := (n+1)E_n$. For other examples, see [2, Section 3].

It may be worth mentioning that above type recurrences for Euler polynomials can be deduced independently of Genocchi polynomials. Indeed, letting

$$Q_k^{(n)}(x) := \sum_{j=0}^{k}(-1)^j(x+j)^n \ \ (n, k \geq 0),$$

we can prove that $Q^{(n)}_{p-1}(x) \equiv E_n(x) \pmod{p}$, where p is an odd prime. Based on this congruence, the following Euler polynomial analogue of (11) is deduced by almost the same arguments as in the proof of Theorem 3.1:

$$(E_m(x)+y)^l = (E_l(x+y)-y)^m \quad (l, m \geq 0). \tag{36}$$

Applying (36), we are able to obtain various shortened recurrence relations for Euler polynomials including (34) and (35) as the special cases (cf. [1, (3.1) (ii)]).

(II) The tangent numbers T_n and polynomials $T_n(x)$, $n = 0, 1, 2, \ldots$, are defined by the generating functions

$$\mathcal{T}(t) := \frac{2}{e^{2t}+1} = \sum_{n=0}^{\infty} \frac{T_n t^n}{n!} \quad (|t| < \pi/2);$$
$$\mathcal{T}(t;x) := \frac{2e^{xt}}{e^{2t}+1} = \sum_{n=0}^{\infty} \frac{T_n(x) t^n}{n!} \quad (|t| < \pi/2),$$

respectively. It is easily shown that $T_{2n} = 0$ and $(-1)^n T_{2n-1} > 0$ for all $n \geq 1$. Further, from the relation $\mathcal{T}(t;x) = \mathcal{T}(t)e^{xt}$, we see that $T_n(0) = T_n$ and $T_n(x)$ is expressed in terms of tangent numbers as follows:

$$T_n(x) = (T_0 + x)^n = \sum_{i=0}^{n} \binom{n}{i} T_i x^{n-i}.$$

A number of recurrence relations for these numbers and polynomials are known. Among them, the most basic ones are

$$T_0 = 1, \quad (T_0+2)^n + T_n = 0 \quad (n \geq 1);$$
$$T_0(x) = 1, \quad (T_0(x)+2)^n + T_n(x) = 2x^{n-1} \quad (n \geq 1).$$

They can be easily deduced by observing the functional relations $\mathcal{T}(t)(e^{2t}+1) = 2$ and $\mathcal{T}(t,x)(e^{2t}+1) = 2e^{xt}$, respectively.

We do not mention in details, but as well as (11) and (36), it follows that

$$(T_m(x)+y)^l = (T_l(x+y)-y)^m \quad (l, m \geq 0).$$

By making use of this identity, we are able to deduce various kinds of shortened recurrence relations for tangent polynomials.

As will be seen below, all the tangent numbers (resp. polynomials) can be expressed in terms of Genocchi numbers (resp. polynomials). Indeed, by observing the relation $e^{2t}\mathcal{G}(2t) = 2te^{2t}\mathcal{T}(t) = 2t\,(2-\mathcal{T}(t))$, we have

$$2^n(G_0+1)^n = \begin{cases} G_0 = 0, & \text{if } n = 0; \\ 2(2-T_0) = 2, & \text{if } n = 1; \\ -2nT_{n-1}, & \text{otherwise.} \end{cases}$$

Hence, it follows from (2) (i) that for $n \geq 0$,

$$G_{n+1} = -(G_0+1)^{n+1} = \frac{n+1}{2^n}T_n; \text{ or equivalently, } 2^n G^*_{n+1} = T_n.$$

Furthermore, from the relation $\mathcal{G}(t;x/2) = t\mathcal{T}(t/2;x)$, we have

$$G_{n+1}(x/2) = \frac{n+1}{2^n}T_n(x) \quad (n \geq 0).$$

Therefore, based on this mutual relation, we can transform all the recurrences for Genocchi numbers and polynomials into those for tangent ones.

For example, the identity (15) in Theorem 3.2 leads to

$$\begin{aligned}
&(G^*_{m+1}(x/2)+1)^l + (G^*_{l+1}(x/2)-1)^m \\
&= \sum_{i=0}^{l}\binom{l}{i}\frac{G_{m+1+i}(x/2)}{m+1+i} + \sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}\frac{G_{l+1+j}(x/2)}{l+1+j} \\
&= \frac{1}{2^m}\sum_{i=0}^{l}\binom{l}{i}\frac{T_{m+i}(x)}{2^i} + \frac{1}{2^l}\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}\frac{T_{l+j}(x)}{2^j} \\
&= \frac{1}{2^m}\left(\frac{1}{2}T_m(x)+1\right)^l + \frac{1}{2^l}\left(\frac{1}{2}T_l(x)-1\right)^m \\
&= \frac{1}{2^{l+m}}\left\{(T_m(x)+2)^l + (T_l(x)-2)^m)\right\} \\
&= 2\,(x/2)^l\,(x/2-1)^m = \frac{1}{2^{l+m-1}}x^l(x-2)^m.
\end{aligned}$$

So that, multiplying by 2^{l+m}, we obtain

$$(T_m(x)+2)^l + (T_l(x)-2)^m = 2x^l(x-2)^m \quad (l,m \geq 0), \tag{37}$$

which is a typical shortened recurrence relation for tangent polynomials. Needless to say, (15) is equivalent to (37).

On the other hand, since $\mathcal{E}(2t;x) = \mathcal{T}(t;x)$ and $\mathcal{T}(t;x) = \mathcal{E}(2t;x)e^{-xt}$, we can give the mutual relations between Euler and tangent polynomials such that

$$2^n E_n(x) = (T_0(x)+x)^n \quad \text{and} \quad T_n(x) = (2E_0(x)-x)^n \quad (n \geq 0).$$

Putting here $x=0$ and $x=1/2$, we have

$$T_n = 2^n E_n(0) = 2^n G^*_{n+1} \quad \text{and} \quad 2^n T_n(1/2) = (2E_0-1)^n.$$

As a conclusion of this section, we may assert that it suffices to discuss only recurrences either for Genocchi, Euler or tangent numbers and polynomials.

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