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ROBIN'S INEQUALITY FOR NEW FAMILIES OF INTEGERS

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Abstract

Robin's criterion states that the Riemann Hypothesis is true if and only if Robin's inequality $\sigma(n) := \sum_{d|n} d < e^\gamma n \log\log n$ is satisfied for each $n > 5040$, where γ denotes the Euler-Mascheroni constant. We show that if a positive integer n satisfies either $\nu_2(n) \le 19$, $\nu_3(n) \le 12$, $\nu_5(n) \le 7$, $\nu_7(n) \le 6$ or $\nu_{11}(n) \le 5$ then Robin's inequality is satisfied, where $\nu_p(n)$ is the p-adic order of n. In the end we show that $\sigma(n)/n < 1.0000005645\, e^\gamma \log\log n$ holds unconditionally for $n > 5040$.

1. Introduction

Let n be an integer satisfying $\sigma(n) := \sum_{d|n} d < e^\gamma n \log\log n$, where γ denotes the Euler-Mascheroni constant. This inequality is called *Robin's inequality*. Robin [8] proved that the Riemann Hypothesis (RH) is true if and only if his inequality holds for every integer $n > 5040$. So far Robin's inequality has been proven unconditionally for families of integers that are

- odd and greater than 9 [5]
- square-free and greater than 30 [5]
- a sum of two squares and greater than 720 [2]
- not divisible by the fifth power of a prime [5]
- not divisible by the seventh power of a prime [9]
- not divisible by the eleventh power of a prime [4].

Here, we extend Robin's inequality. We first provide a modified algorithm of the one obtained by Akbary et al. [1] to establish the exceptions to the inequality $n/\varphi(n) < (1771561/1771560)e^\gamma \log\log n$, where φ stands for Euler's totient function. With this we then show that if n has a 2-adic order smaller or equal to 19 or satisfies either $\nu_3(n) \le 12$, $\nu_5(n) \le 7$, $\nu_7(n) \le 6$ or $\nu_{11}(n) \le 5$ then Robin's inequality holds.

Then we find that $\sigma(n)/n < 1.0000005645\ e^\gamma \log\log n$ holds unconditionally for all $n > 5040$.

2. Theorems

We first want to show the case where we know that the 2-adic order of n is lower or equal to 19.

Theorem 1. *Robin's inequality holds for $n > 5040$ when $\nu_2(n) \leq 19$.*

We then go on to partially prove a result of Choie et. al [5].

Theorem 2. *Consider those integers n which satisfy $\nu_3(n) \leq 12$, $\nu_5(n) \leq 7$, $\nu_7(n) \leq 6$ or $\nu_{11}(n) \leq 5$. Then, Robin's inequality holds for all such integers $n > 5040$.*

An improved unconditional upper bound of $\sigma(n)/n$ is provided by the following.

Theorem 3. *The inequality*

$$\sigma(n)/n < 1.0000005645\ e^\gamma \log\log n \tag{1}$$

holds for all $n > 5040$.

3. Proofs

Lemma 1. *Let $\prod_{i=1}^{r} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < ... < q_r$ with positive exponents $a_1 < ... < a_r$. Then*

$$\frac{\sigma(n)}{n} = \frac{n}{\varphi(n)} \prod_{i=1}^{r} \left(1 - \frac{1}{q_i^{a_i+1}}\right). \tag{2}$$

Proof. This is Lemma 2 in [6]. □

We now take a look at a way to establish a new upper bound for $n/\varphi(n)$. First we provide an algorithm which is derived from Akbary et al. [1]. They developed an algorithm that calculates the exceptions to the following inequality where $0 < \epsilon < 1$ and $\omega(n)$ is the number of distinct prime divisors of n:

$$f(n) := \prod_{\substack{p \leq p_{\omega(n)} \\ p\ prime}} \frac{p}{p-1} < e^\gamma (1+\epsilon) \log\log n. \tag{3}$$

For an integer n and an integer $\beta \geq \omega(n) \geq 2$ they showed that if

$$n > n_\beta := \exp\left(\exp\left(\frac{1}{(1+\epsilon)e^\gamma} \prod_{p \leq p_\beta} \frac{p}{p-1}\right)\right) \tag{4}$$

then inequality (3) is satisfied. According to Lemma 3.4 in [1], we only need to find the first β for a given ϵ for which $\prod_{p \leq p_\beta} p < n_\beta$ does not hold in order to get to the largest possible exception of (3). We call this largest possible exception of inequality (3) $n_{\beta_{max}}$. We can now describe the modified algorithm which is proven to be correct by Lemma 3.4 in [1].

Algorithm 1 Largest possible exception to $f(n) < e^\gamma(1+\epsilon)\log\log n$

Require: $0 < \epsilon < 1$
Ensure: Largest possible exception to the inequality.
  while $\prod_{p \leq p_\beta} p < n_\beta$ **do**
    $\beta \rightarrow \beta + 1$
  end while
  $\beta_{max} \rightarrow \beta$
  $n_{\beta_{max}} \rightarrow n_\beta$

We can now go on to find an upper bound for $n/\varphi(n)$.

Lemma 2. *The inequality*

$$\frac{n}{\varphi(n)} < \frac{1771561}{1771560} e^\gamma \log\log n \tag{5}$$

is satisfied for all $n > c_0 := e^{e^{23.762143}}$.

Proof. On noting that

$$\frac{n}{\varphi(n)} \leq \prod_{p \leq p_\beta} \frac{p}{p-1} < e^\gamma(1+\epsilon)\log\log n, \tag{6}$$

we run the algorithm from Lemma 3 with $\epsilon = 1/1771560$ such that the RHS of (6) matches the RHS of (5). The result of the algorithm, namely β_{max} and $n_{\beta_{max}}$, is

$$\beta_{max} = 919356257 \qquad n_{\beta_{max}} < e^{e^{23.762143}}$$

We note that $n_{\beta_{max}}$ cannot be exactly numerically calculated to integer precision, which is mainly due to the sheer size of the number. Fortunately, this is not necessary, since we can bound $n_{\beta_{max}}$ from above in our numerical calculation and still maintain the correctness of the algorithm. This is why we limit the numerical computation of the exponent of $n_{\beta_{max}}$ to 200 digits and then use the exponent 23.762143. Since this calculated bound is important throughout our proofs we set $c_0 := e^{e^{23.762143}}$.

The algorithm guarantees that all exceptions to inequality (3) are below c_0, which allows us to conclude that for all $n > c_0$ the inequality (5) holds. □

Lemma 3. *Robin's inequality is true for all* $5040 < n \leq 10^{10^{10}}$.

Proof. Robin showed in [8], Prop.1, p.192 that if Robin's inequality holds for consecutive colossally abundant numbers n_1 and n_2 then it also holds for all $n \in [n_1, n_2]$. By definition an integer n is colossally abundant if there exists a positive ϵ for which $\sigma(n)/n^{1+\epsilon} \geq \sigma(k)/k^{1+\epsilon}$ for all $k > 1$. Briggs [3] showed that Robin's inequality holds for all colossally abundant numbers between 5040 and $10^{10^{10}}$. We may therefore conclude that Robin's inequality is also satisfied for all integers $5040 < n < 10^{10^{10}}$. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. We now let n have a 2-adic order of $\nu_2(n) \leq 19$. From Lemma 1 we note that

$$\frac{\sigma(n)}{n} = \frac{n}{\varphi(n)} \prod_{i=1}^{r} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \leq \frac{n}{\varphi(n)} \left(1 - \frac{1}{2^{\nu_2(n)+1}}\right). \tag{7}$$

We only need to look at the case where $\nu_2(n) = 19$ since the weaker cases follow because

$$\left(1 - \frac{1}{2^{1+1}}\right) < \left(1 - \frac{1}{2^{1+2}}\right) < \dots < \left(1 - \frac{1}{2^{1+19}}\right).$$

With Lemma 2 we have for $n > c_0$

$$\begin{aligned} \frac{\sigma(n)}{n} &\overset{\nu_2(n)=19}{\leq} \frac{n}{\varphi(n)} \left(1 - \frac{1}{2^{1+19}}\right) = \frac{1048575}{1048576} \frac{n}{\varphi(n)} \\ &< \frac{1048575}{1048576} \frac{1771561}{1771560} e^{\gamma} \log\log n < e^{\gamma} \log\log n. \end{aligned} \tag{8}$$

In light of Lemma 3 and the fact that $c_0 < 10^{10^{10}}$ we then conclude that Robin's inequality is true for those $n > 5040$ for which $\nu_2(n) \leq 19$. □

Our proof of Theorem 2 is now done with other p-adic orders used to partially prove Theorem 6 of [5].

Proof of Theorem 2. We now consider n with an 11-adic order satisfying $\nu_{11}(n) \leq 5$. The cases for the 3-adic, 5-adic or 7-adic order follow directly since

$$\left(1 - \frac{1}{5^{1+7}}\right) < \left(1 - \frac{1}{7^{1+6}}\right) < \left(1 - \frac{1}{3^{1+12}}\right) < \left(1 - \frac{1}{11^{1+5}}\right).$$

With Lemma 1 and 2 we then have for $n > c_0$

$$\begin{aligned} \frac{\sigma(n)}{n} &\stackrel{\nu_{11}(n)=5}{\leq} \frac{n}{\varphi(n)}\left(1 - \frac{1}{11^{1+5}}\right) = \frac{1771560}{1771561}\frac{n}{\varphi(n)} \\ &< \frac{1771560}{1771561}\frac{1771561}{1771560}e^{\gamma}\log\log n = e^{\gamma}\log\log n. \end{aligned} \tag{9}$$

By invoking Lemma 3 and noting that $c_0 < 10^{10^{10}}$ we then conclude that Robin's inequality is true for those integers $n > 5040$ for which $\nu_3(n) \leq 12$, $\nu_5(n) \leq 7$, $\nu_7(n) \leq 6$ or $\nu_{11}(n) \leq 5$. □

With these results, we can now also improve the unconditional bound for $\sigma(n)/n$ from Akbary et al. [1].

Proof of Theorem 3. First, note that $1771561/1771560 = 1.000000\overline{564474248684775}$. Then similar to Theorem 1, it follows from Lemma 2 that for $n > c_0$,

$$\frac{\sigma(n)}{n} \leq \frac{n}{\varphi(n)} < \frac{1771561}{1771560}e^{\gamma}\log\log n < 1.0000005645\ e^{\gamma}\log\log n \tag{10}$$

On invoking Lemma 3 we then find that the above inequality holds unconditionally for $n > 5040$. □

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