ALIQUOT SEQUENCES WITH SMALL STARTING VALUES

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Abstract
We describe the results of the computation of aliquot sequences with small starting values. In particular, all sequences with starting values less than a million have been computed until either termination occurred (at 1 or a cycle), or an entry of 100 decimal digits was encountered. All dependencies were recorded, and numerous statistics, curiosities, and records are reported.

1. Introduction

Aliquot sequences arise from iterating the sum-of-proper-divisors function

$$s(n) = \sum_{\substack{d|n \\ d<n}} d,$$

assigning to an integer $n > 1$ the sum of its *aliquot* divisors (that is, excluding n itself). Iteration is denoted exponentially, so s^k is shorthand for applying $k \geq 1$ times the function s. We say that an aliquot sequence *terminates (at 1)* if $s^k(n) = 1$ for some k; this happens when and only when $s^{k-1}(n)$ is prime. It is possible that $s^{k+c}(n) = s^k(n)$, for some $c > 0$ and all $k \geq k_0$, that is, to hit an *aliquot cycle of length* c, where we take $c > 0$ and k_0 minimal. Case $c = 1$ occurs when n is a perfect number (like 6), and $c = 2$ when $n, m \neq n$ form a pair of *amicable numbers*: $s(n) = m$ and $s(m) = n$. See Section 6 for more cycles.

The main open problem regarding aliquot sequences is the conjecture attributed to Catalan [2] and Dickson [4].

Conjecture 1. All aliquot sequences remain bounded.

If true, it would imply that for every n after finitely many steps we either hit a prime number (and then terminate at 1) or we find an aliquot cycle. Elsewhere we comment upon some of the heuristics to support or refute this conjecture [1].

We will call an aliquot sequence *open* if it is not known to remain bounded. This notion depends on our state of knowledge. The point of view adopted in this paper is that we compute an aliquot sequence until either we find that it terminates or cycles, or we find that it reaches some given size. In particular, we pursued every sequence starting with at most 6 decimal digits to 100 decimal digits (if it did not terminate or cycle before).

The idea of computing aliquot sequences for small starting values n_0 is the obvious way to get a feeling for their behaviour, and hence has been attempted very often. The main problem with this approach is that for some n_0 the values of $s^k(n_0)$ grow rapidly with k. This causes difficulties because all known practical ways to compute $s(n)$ use the prime factorization of n in an essential way. Clearly, $s(n) = \sigma(n) - n$, where σ denotes the sum-of-*all*-divisors function, which has the advantage over s of being multiplicative, so it can be computed using the prime factorization of n:

$$\sigma(n) = \prod_{\substack{p^k \parallel n \\ p \text{ prime}}} (1 + p + \cdots + p^k),$$

where $p^k \parallel n$ indicates that p^k divides n but p^{k+1} does not.

Thus, it is no coincidence that similar computations have been performed over the past 40 years after new factorization algorithms were developed, and better hardware became much more widely available. There have been several initiatives following pioneering work of Wolfgang Creyaufmüller [3], and for ongoing progress one should consult webpages like [12] with contributions by many individuals.

Despite the extended experience and knowledge gained from computations such as reported here, it still seems unlikely that Conjecture (1.1) will be proved or disproved soon; certainly mere computation will not achieve this. Yet, valuable insight might be obtained.

digits	*terminating*	*cycle*	*open*
10	735,421	16,204	248,374
20	783,786	17,274	198,939
30	797,427	17,761	184,811
40	800,703	17,834	181,462
50	803,317	17,913	178,769
60	804,830	17,940	177,229
70	805,458	17,985	176,556
80	807,843	18,036	174,120
90	809,362	18,039	172,598
100	811,555	18,103	170,341

Table 1: *The number of aliquot sequences with starting values less than a million that terminate, cycle, or reach the indicated number of digits before doing so.*

The current paper grew out of intermittent attempts over 25 years to independently perform all necessary computations (at least twice), and, causing more headaches, to make sure that all confluences were faithfully recorded. The main findings are summarized in the table and charts given below.

The table above summarizes what happens if, for starting values up to 10^6, we pursue the aliquot sequences up to a size of d decimal digits, with d growing from 10 to 100. As more cycles and terminating sequences are found, the number of open sequences declines. In Section 5 a more detailed table is given for even starting values only.

We try to visualize the rate at which this process takes place in the pictures below: they plot the number of starting values (below 10^6) that terminate or cycle before the given number of digits (on the horizontal axis) is reached. Note that in both charts the absolute numbers are plotted vertically, but the scale differs markedly.

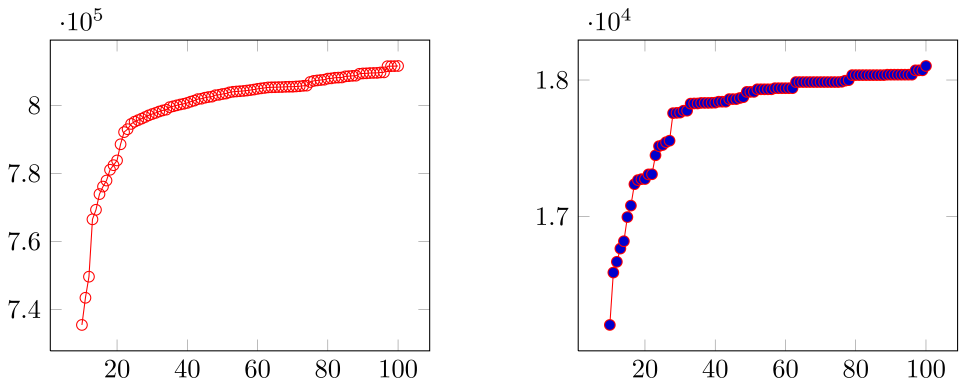

Figure 1: *The number of aliquot sequences with starting values less than a million that terminate (left) or cycle (right) as a function of the number of decimal digits reached.*

The next pair of pictures displays the effect of bounding the starting values. In the chart on the left, the bottom graph shows that almost no starting values less than 10^3 reach a size of 20 decimal digits, but for starting values up to 10^4 around 8% do, a percentage that grows to more than 13% for starting values up to 10^5 and 20% up to 10^6. The corresponding (growing) percentages for terminating sequences are displayed on the right. The corresponding percentages will almost, but not exactly, add up to 100%, as a small percentage (less than 2%) leads to aliquot cycles.

As a rather naive indication for the truth of the Catalan-Dickson conjecture, we have also calculated a first order linear approximation $(-0.019 \cdot x + 18.93)$ to the percentage of open sequences reaching to more than 20 digits, with starting values up to 10^6; this the line drawn on the top left.

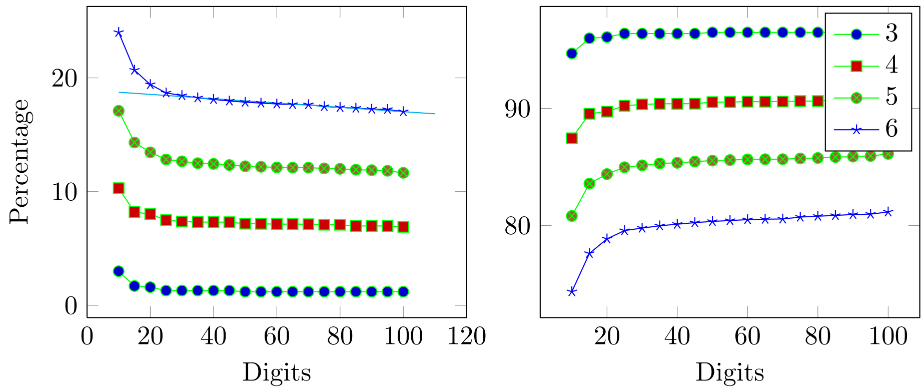

Figure 2: *The percentage of surviving (left) and terminating (right) aliquot sequences as a function of the number of decimal digits reached, for starting values with at most 3, 4, 5 or 6 decimal digits.*

There is no reason to believe (nor model to support) linear decay in the long run, but the line does reflect the downward tendency on the interval between 25 and 100 digits.

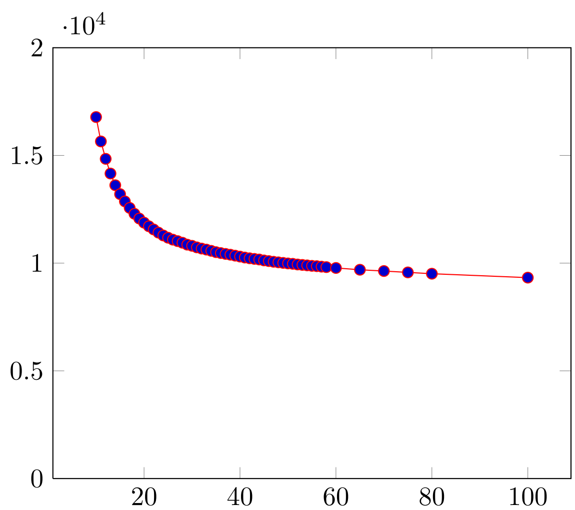

Figure 3: *Number of surviving main sequences at given number of digits*

Sometimes we find it useful identify aliquot sequences with the same tails; we say that they *merge* at some point. We call a sequence a *main* sequence if it has not merged with a sequence with a smaller starting value (yet). The final plot in this section shows data for the number of *different* small aliquot sequences in this sense: the number of aliquot sequences starting below 10^6 that exceed N digits for the first time at different values. At $N = 100$ there remain 9327 such main sequences.

2. Preliminaries

In this section we have collected some known results, with pointers to the existing literature, as well as some terminology (some standard, some ad hoc).

Arguments about random integers are not automatically applicable to heuristics for aliquot sequences due to the fact that certain factors tend to persist in consecutive values. The most obvious example of this phenomenon is parity preservation: $s(n)$ is odd for odd n unless n is an odd square, $s(n)$ is even for even n unless n is an even square or twice an even square. Guy and Selfridge introduced [8] the notion of a driver. A *driver* of an even integer n is a divisor $2^k m$ satisfying three properties: $2^k \parallel n$; the odd divisor m is also a divisor of $\sigma(2^k) = 2^{k+1} - 1$; and, conversely, 2^{k-1} divides $\sigma(m)$. As soon as n has an additional odd factor (coprime to m) besides the driver, the same driver will also divide $s(n)$. The even perfect numbers are drivers, and so are only five other integers (2, 24, 120, 672, 523,776). Not only do they tend to persist, but with the exception of 2, they also drive the sequence upward, as $s(n)/n$ is 1 for the perfect numbers, and $\frac{1}{2}$, $\frac{3}{2}$, $2, 2, 2$ for the other drivers.

More generally, it is possible to prove that arbitrarily long increasing aliquot sequences exist, a result attributed to H. W. Lenstra (see [7], [11], [5]).

Another heuristic reason to question the truth of the Catalan-Dickson conjecture was recently refuted in [1] (but see also [10]). We showed that, in the long run, the growth factor in an aliquot sequence with even starting value will be less than 1. Besides giving a probabilistic argument, this is not as persuasive as it may seem, since it assumes that entries of aliquot sequences behave randomly, which is not true, as we argued above. See also [6, 9].

Not only does parity tend to persist in aliquot sequences, the typical behavior of the two parity classes of aliquot sequences is very different. There is much stronger tendency for odd n to have $s(n) < n$. In all odd beginning segments, only four cases were encountered during our computations where four consecutive odd values were increasing:

$$\begin{array}{l} 38745, 41895, 47025, 49695 \\ 651105, 800415, 1019025, 1070127 \\ 658665, 792855, 819945, 902295 \\ 855855, 1240785, 1500975, 1574721. \end{array}$$

On the other hand, seeing the factorizations of these, as in the first quadruple

$$3^3 \cdot 5 \cdot 7 \cdot 41, \quad 3^2 \cdot 5 \cdot 7^2 \cdot 19, \quad 3^2 \cdot 5^2 \cdot 11 \cdot 19, \quad 3 \cdot 5 \cdot 3313,$$

it is not so difficult to generate longer (and larger!) examples, such as

$$25399054932615, 37496119518585, 48134213982855, 63887229572985,$$

$$72415060070535, 87397486554105, 101305981941255, 115587206570745,$$

$$133433753777415, 163310053403385, 174881380664583,$$

in the vein of the result of Lenstra, but such examples did not show up in our sequences.

We say that sequence s *merges* with sequence t (at value x) if s and t have x as first common value, t has a smaller starting value than s, and the common value occurs before s reaches its maximum. In this case t will be the *main* sequence (unless it merges with a 'smaller' sequence again). From x on, s and t will coincide of course.

We should point out again that the notion of being a main sequence is time dependent: sequences may merge beyond the point to which we have as yet computed them.

Since all of our sequences are finite (although, possibly, infinitely repeating at the end), we can speak of the *height* of a sequence: this is essentially the logarithm of its maximal value. Sometimes we measure this in number of decimal digits, sometimes in number of bits. The *volume* will be the sum of the number of digits of the entries of the sequence, without rounding first, so that $\mathrm{vol}(s) = \sum_{x \in s} \log_{10} x$.

3. Odd Cases

We first consider the 500,000 odd starting values, as they usually lead to termination quickly. In fact, 494,088 odd starting values terminated; 5119 sequences with odd starting values lead to a cycle (see next section) and 793 remain open, after merging with a sequence with an even starting value.

As we saw above, parity is not always maintained. Therefore, we need to distinguish in our 500,000 odd starting values between aliquot sequences consisting *only* of odd integers, and those containing even values as well.

For 440,239 odd starting values, an all-odd sequence ensues. The remaining 59,761 change over to even, after hitting an odd square. Of the 59,761 odd starting values that change over, 12,674 do so after hitting 3^2. Only the odd squares less than a million did occur.

Of the all-odd sequences, 208 end in an odd cycle.

Of the 59,761 odd-to-even starting values, 5119 lead to a cycle and 793 merge with an even sequence reaching 100 digits. Of the 54,057 terminating odd-to-even starting values, 17 take more than 1000 steps before terminating: 11 of them merge after a couple of steps with the 94-digit maximum sequence 16,302 of length 1602, and 6 of them merge after a couple of steps with the 76-digit maximum sequence 31,962 of length 1740.

parity	*terminating*	*cycle*	*open*
odd	494,088	5,119	793
all-odd	440,031	208	0
even	317,467	12,984	169,548
all	811,555	18,103	170,341

Table 2: *Numbers of terminating, cycling and open sequences at 100 digits starting below one million, by parity.*

4. Terminators

Of the 999,999 starting values, 811,555 terminated without reaching a 100-digit value.

4.1. All-odd Terminators

Among the 440,031 all-odd terminators, 78,497 terminate after 1 step. This reflects that there are 78,497 odd primes less than a million. The longest all-odd terminator has length 23:

$$966195, 856845, 807795, 643005, 606915, 445149, 214371, 95289, 37383,$$
$$15465, 9303, 4905, 3675, 3393, 2067, 957, 483, 285, 195, 141, 51, 21, 11, 1.$$

The further distribution of lengths is as follows (including the trivial sequence 1).

length	0	1	2	3	4	5	6	7
#	1	78,497	74,893	63,266	55,020	44,764	36,104	26,724
	8	9	10	11	12	13	14	15
	18,932	13,327	9,774	6,791	4,431	2,814	1,652	1,093
	16	17	18	19	20	21	22	23
	740	555	349	227	62	11	3	1

Table 3: *Length distribution for all-odd terminating sequences.*

The all-odd terminators never get very high. The maximum height is reached by the sequence starting with 855,855, which is merged by 886,545, as follows:

$$886545, 855855, 1240785, 1500975, 1574721, 777761, 1.$$

Eleven have volume exceeding 80; the maximum volume 88.8379 is reached by the 966,195 sequence, which was also the longest (see above).

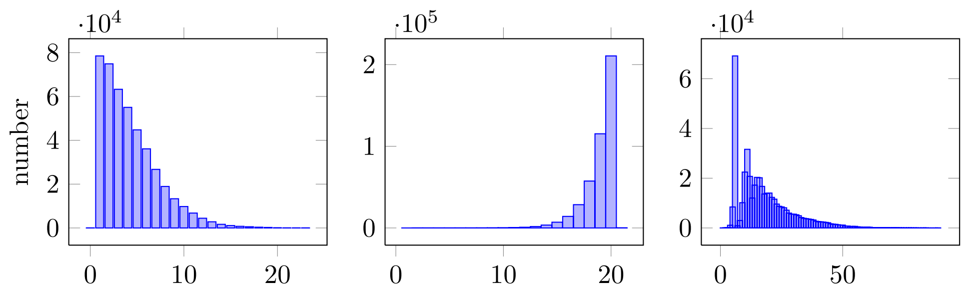

Figure 4: *Length, height (in bits) and volumes of all-odd terminators*

4.2. Odd-to-even Terminators

Next we look at the odd-to-evens terminating sequences; 1003 of them merge with an even sequence with smaller starting value. We consider the 53,054 sequences that do not merge.

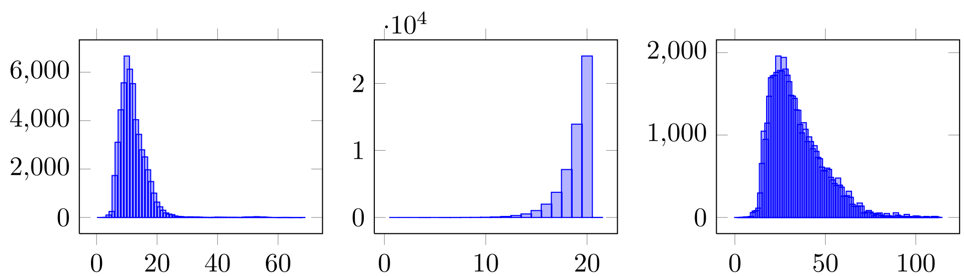

Figure 5: *Lengths, heights (in bits), and volumes of odd-to-even terminating sequences.*

There is one sequence in this category that is simultaneously longest, most voluminous. It is the sequence

- 855,441 of length 68, height 5.932 and volume 267.309, which is in full (note that $229{,}441 = 479^2$):

$$855441, 451359, 229441, 480, 1032, 1608, 2472, 3768, 5712, 12144, 23568, 37440, 101244, 180996, 241356, 321836, 251044, 188290, 168830, 135082, 88478, 59698, 34622, 24754, 12380, 13660, 15068, 11308, 10364, 7780, 8600, 11860, 13088, 12742, 7274, 3640, 6440, 10840, 13640, 20920, 26240, 38020, 41864, 36646, 19298, 9652, 8268, 12900, 25292, 18976, 18446, 10498, 5882, 3514, 2534, 1834, 1334, 826, 614, 310, 266, 214, 110, 106, 56, 64, 63, 41, 1.$$

Again, the sequences in this category do not reach high. The maximum height is 6.17 for the value 1,480,761 (requiring 21 bits) reached by the sequence for 945,945.

4.3. Even Terminators

In this subsection we consider all even terminating starting values, where we include the mergers and also the odd-to-even sequences (considered separately above); there are 371,543 of them.

Below, the distribution of the lengths of these is depicted. The result looks much nicer if we only count main (non-merging) terminators.

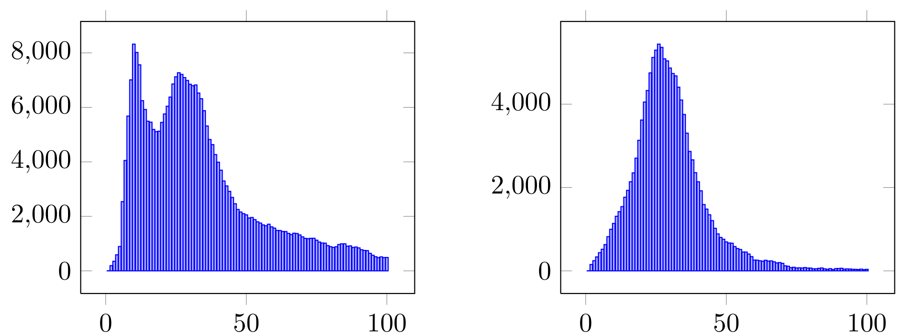

Figure 6: *Number of terminating sequences (left; cut off at length 100) and of terminating main sequences (right) of given length.*

In fact, the thin tail of this distribution extends all the way to 6585, with 476 starting values here having length at least 1000 and three of the 136,318 even starting values have a terminating sequence extending to over 5000 terms:

- 414,288 of length 6585, height 91.2754 and volume 325676.634,
- 565,680 of length 5309, height 98.6734 and volume 259264.265,
- 696,780 of length 5294, height 97.3217 and volume 239530.611.

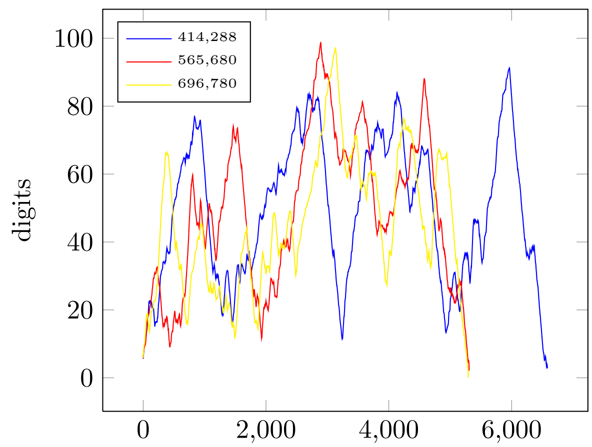

Figure 7: *Size 'profile', as function of index, for three long terminating sequences.*

The tail of the heights is in fact long and thin, reaching up to 333 bits. Indeed, several of these sequences reach up to 98 or even 99 digits before terminating. Record holders are

- 261,306 of length 2173, height 98.8504 and volume 86,295.954,
- 108,072 of length 1503, height 98.7872 and volume 77,131.106,

profiled in Figure 9.

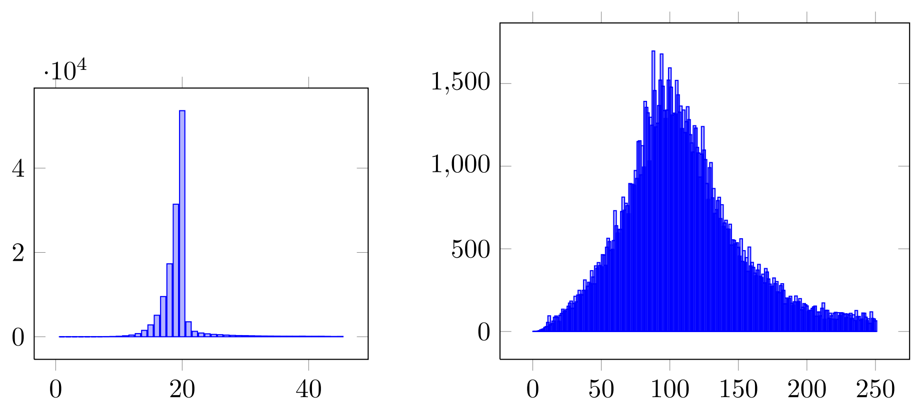

Figure 8: *The distribution of heights (left; in bits) and volumes (right) of the terminating sequences.*

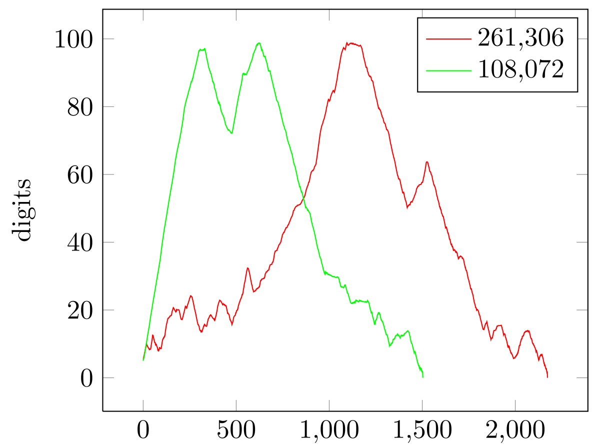

Figure 9: *Size 'profile', as function of index, for two high terminating sequences.*

The three most voluminous are in fact also the longest three we saw before! There is only one more of volume exceeding 200,000:

- 320,664 of length 4293, height 97.7939 and volume 205,004.62.

4.4. Penultimate Primes

To conclude this section, we consider the penultimate prime values for all terminating sequences together. It turns out that the most popular values are 43 and 59, with 11 different primes being hit more than 10,000 times:

p:	43	59	41	7	601	37	3	11
#:	77,947	53,159	50,903	42,293	26,726	24,946	21,934	17,193
	73	31	19					
	13,570	12,160	10,495					

Table 4: *Number of occurrences of most popular penultimate primes.*

In all, 78,572 different primes appear, among them, of course, the 78,498 primes below 10^6 (of which 56,513 *only* appear with the prime as starting value).

Of the 74 primes larger than 10^6, the largest is 4,737,865,361 (appearing only for 891,210), and the second largest is 870,451,093, which appears 216 times, for three different main sequences: 54,880 (with 203 mergers), 397,416 (with 9 mergers), 780,456 (with 1 merger).

Only one prime less than 10,000 appears just once as a penultimate value, namely 9173 for the sequence $11 \cdot 9161$, 9173, 1. A similar phenomenon occurs for two larger solitary penultimate primes in our range: $83 \cdot 9923, 10007, 1$ and $47 \cdot 12743, 12791, 1$.

5. Open

The following table breaks up the range of starting values into ten sub-intervals from $k \cdot 10^5$ to $(k+1) \cdot 10^5 - 1$, for $k = 0, 1, \ldots, 9$, and for those the number of even starting values reaching d decimal digits is given. Note that (as 0 is not included as starting value) the first column concerns 49,999 starting values, and the other columns 50,000. The final column is the sum of the first 10 columns, and counts how many of the 499,999 even starting values less than 10^6 reach d decimal digits (that is, a value of at least 10^{d-1}).

d	$n < 10^5$	$2 \cdot 10^5$	$3 \cdot 10^5$	$4 \cdot 10^5$	$5 \cdot 10^5$	$6 \cdot 10^5$	$7 \cdot 10^5$	$8 \cdot 10^5$	$9 \cdot 10^5$	10^6	total
10	17517	22117	23804	24377	25293	25982	26333	26843	27220	27705	247,191
15	14432	18578	20157	20644	21350	21924	22169	22579	23042	23351	208,226
20	13715	17671	19274	19658	20280	20862	21137	21442	21857	22176	198,072
25	12780	16642	18184	18491	19143	19663	19827	20256	20628	20829	186,443
30	12588	16407	17913	18233	18923	19439	19584	19984	20362	20533	183,966
35	12425	16236	17692	18033	18706	19235	19322	19778	20180	20295	181,902
40	12404	16149	17548	17880	18543	19088	19198	19643	20034	20161	180,648
45	12260	16033	17397	17716	18379	18971	19034	19475	19828	20007	179,100
50	12159	15953	17317	17625	18261	18821	18913	19349	19703	19858	177,959
55	12107	15890	17244	17534	18167	18741	18804	19269	19605	19757	177,118
60	12047	15842	17176	17464	18075	18668	18737	19186	19550	19674	176,419
65	12029	15796	17124	17417	18021	18605	18668	19125	19495	19606	175,886
70	12025	15793	17123	17407	18012	18586	18648	19105	19471	19576	175,746
75	11984	15678	16983	17276	17874	18426	18485	18943	19281	19418	174,348
80	11925	15602	16901	17137	17758	18328	18363	18825	19172	19305	173,316
85	11856	15569	16850	17078	17698	18246	18288	18748	19098	19223	172,654
90	11807	15519	16780	16986	17610	18140	18168	18658	18996	19130	171,794
95	11753	15469	16747	16958	17593	18114	18147	18644	18983	19107	171,515
100	11574	15280	16535	16785	17390	17901	17934	18468	18786	18895	169,548

Table 5: *Decay of number of surviving even sequences at d digits by sub-interval*

At 100 digits, there are still 9327 different open main sequences, all with even starting values; 160,221 even (and 793 odd) starting values merge somewhere with these.

5.1. Odd Opens

Only 793 odd starting values lead to open sequences. All of them do so after merging with an open sequence with a smaller even starting value. In the table we list the number of consecutive odd steps in these cases, before the first even number appears.

odd length :	1	2	3	4	5	6	7	8	9	10
number :	111	208	179	112	72	58	22	13	12	6

Table 6: *Distribution of number of initial odd values of open sequences.*

Among these cases are 111 squares of odd integers less than 1000, which immediately have an even successor. Of these, 57 occur more often; the smallest, 55^2 occurs a total of 233 times. The following is a table listing all squares that occur more than ten times among these open sequences. These 793 odd starting values merge

square of :	55	85	115	121	125	129	205	235	243	265
times :	233	51	37	25	24	127	15	16	19	12
merge with :	1074	1134	2982	1464	3906	5400	3876	3270	1134	18528

Table 7: *Number of times most popular squares appear in odd sequences.*

with 80 different open sequences. Some of these are more 'popular' than others; we list the ones occurring more than 12 times: For 1074, all of the 233 merge

open starting value :	1074	1134	1464	2982	3270	3876	3906	5400	7044
number of times :	233	70	25	37	16	25	24	127	13

Table 8: *Number of times most popular open even sequences appear as merger for odd sequences.*

at $s^2(1074) = 1098$ after $55^2 = 3025$. For 5400, all of the 127 merge through $16{,}641 = 129^2 \to 7968 \to 13{,}200 = s(5400)$.

Comparing these tables, it will be clear that sometimes more than one square must give entry to the same open sequence. Indeed, the following is a list of starting values for opens for which several squares give entry from an odd starting sequence (with the total number of times):

276 :	$\{473^2, 793^2, 493^2\}$	(6)
564 :	$\{563^2, 625^2\}$	(2)
660 :	$\{957^2, 551^2, 659^2, 827^2, 999^2\}$	(6)
1134 :	$\{243^2, 85^2\}$	(70)
1632 :	$\{803^2, 925^2, 289^2\}$	(6)
1734 :	$\{391^2, 897^2, 799^2, 855^2\}$	(6)
3432 :	$\{451^2, 225^2, 365^2, 535^2\}$	(12)
3876 :	$\{869^2, 205^2, 447^2, 459^2, 899^2\}$	(25)
4800 :	$\{335^2, 533^2, 371^2\}$	(11)
5208 :	$\{295^2, 975^2\}$	(6)
6552 :	$\{417^2, 441^2\}$	(5)
7044 :	$\{595^2, 873^2, 495^2, 879^2, 411^2, 843^2\}$	(13)
17,352 :	$\{979^2, 943^2\}$	(2)
27,816 :	$\{831^2, 939^2\}$	(2)

Table 9: *Open even sequences with multiple squares as entry from odd sequence.*

5.2. Even Mergers

For the 160,221 even starting values merging with an open sequence, the histogram shows how many among the 9327 open sequences have k mergers; 22 have more than 1000 merging sequences, the record holders being 660 (with 7090 mergers), 3876 (with 4307 mergers) and 7044 (with 3093 mergers).

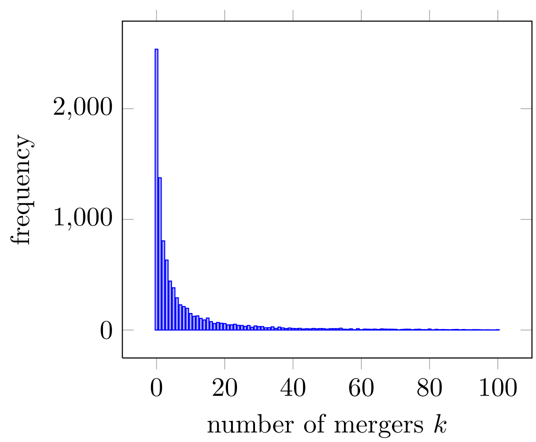

Figure 10: *The number of times an even open sequence has k mergers.*

5.3. Big Opens and Mergers

It does happen that an aliquot sequence reaches almost 100 digits, then decreases before merging with an as yet open sequence. There are 15 starting values (the least being 472,836) that lead to a common 99 digit maximum before merging with the 32,064 open sequence (472,836:2284=32,064:173=1,358,054). Similarly, the 679,554 sequence merges (like 2 others) with the 31,240 open sequence after reaching a 99 digit local maximum (679,554:2672=31,240:35=50,871,436).

There are examples that are even longer (but not higher) before merging:

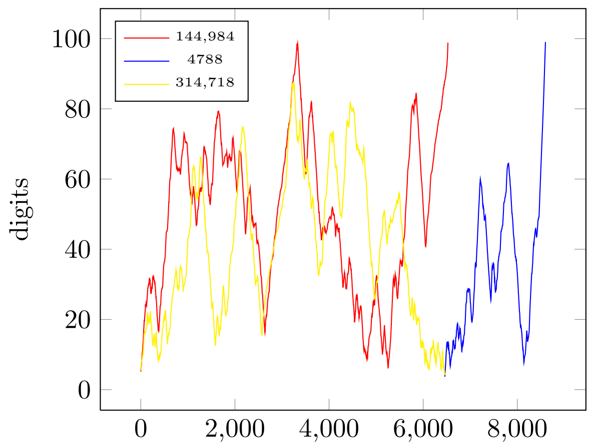

Figure 11: *Examples of long open sequences, one with long pre-merger.*

the 461,214 sequence merges with the open 4788 sequence after 6467 steps (after reaching a 88 digit local maximum). To complicate the situation, it first merges with the 314,718 sequence (461,214:5=314,718:4=1,372,410) which in turn merges with the 4788 sequence (on its way picking up 14 more sequences that have the same local maximum). The longest of these, the 461,214 and 580,110 sequences, reach 100 digits (with 4788) after 8599 steps. The next longest pre-merger example is a group of 4 sequences merging with the open 1920 sequence after 4656 steps and a 76 digit maximum.

The total length of the 461,214 sequence (which merges with 4788) is the largest for any open sequence (8599). Ignoring similar mergers with 4788, next in length is the 7127 step long sequences for 389,508 and 641,956, merging with 34,908 (like a few others that are slightly shorter), and then mergers 910,420 and 638,352 with 556,276 of length 6715 and 6713. Several mergers with 144,984 and 1920 extend also beyond a length of 6500.

The sequences for 144,984 (length 6527) and 556,276 (of length 6510) are record-length non-merging open sequences, followed by 842,592 of length 6455, which has

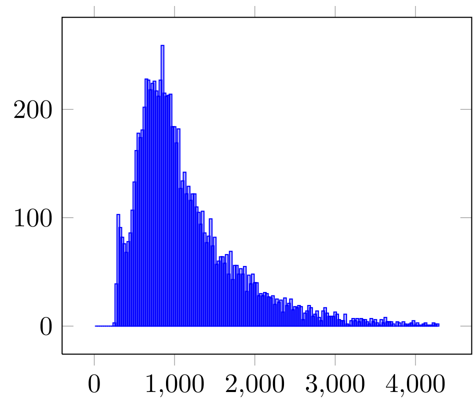

Figure 12: *Distribution of lengths of main open sequences*

no mergers at all.

The 638,352 and 910,420 mergers are the most voluminous ones (with a volume of just over 365,000).

The fastest growing open sequence is 993,834, reaching 100 digits after only 245 steps; it has no mergers. The sequence starting with 267,240 takes 248 steps to reach 100 digits, but two of its mergers (588,120 and 693,960) take one step fewer. With the 235,320 sequence (after 249 steps) and its merger 503,400 (248 steps) these are the only examples hitting the 100 digit ceiling in fewer than 250 steps.

6. Cycles

6.1. Odd Cyclers

Of the 208 all-odd cyclers, only 2 have length 8 (and none are longer):

$$854217, 701883, 547365, 533211, 279333, 134535, 80745, 67095, 71145, 67095, \ldots$$
$$894735, 687105, 503955, 392205, 292659, 97557, 36843, 12285, 14595, 12285, \ldots$$

Presented below are all lengths and their frequencies:

length	1	2	3	4	5	6	7	8
number	15	24	55	50	40	18	4	2

Table 10: *Lengths of all-odd sequences ending in a cycle, with frequencies.*

They all end in one of the eight odd amicable pairs listed in the table below.

Of the odd starting values, 5026 lead immediately to an aliquot cycle, and 93 do so after merging with a smaller sequence. The table shows which cycles are hit, and how often by both main and merging sequences.

[6]	:	4774	42
[496]	:	1	0
[220, 284]	:	8	0
[1184, 1210]	:	2	1
[2620, 2924]	:	1	0
[5020, 5564]	:	26	0
[6232, 6368]	:	17	0
[12285, 14595]	:	104	2
[67095, 71145]	:	45	2
[69615, 87633]	:	36	3
[79750, 88730]	:	0	39
[100485, 124155]	:	3	1
[122265, 139815]	:	2	1
[522405, 525915]	:	5	1
[802725, 863835]	:	1	1
[947835, 1125765]	:	1	0
	:		
16 cycles	:	5026	93

Table 11: *Aliquot cycles, and for how many starting values they are reached by main and merging sequences.*

Only one of the main sequences leading to a cycle has length larger than 10:

$$783225, 643798, \ldots, 14206, 7106, 5854, 2930, 2362, 1184, 1210,$$

of length 48. But note that 783,225 $= 885^2$ and from there on the sequence is even. The first entry is the maximum.

Of the 93 merging cyclers, on the other hand, 40 have length greater than 11, but 39 of these have the same 14 digit maximum 56,365,247,896,588, ending in [79750, 88730], as mergers of the main sequence of length 95 starting at 50,106. The other one (the sequence starting with 949^2) has length 575 and hits maximum

129,948,923,412,692,571,824,805,719,693,528,658,164,860,246,112 (48 digits),

almost halfway, having merged with the open sequence 15,316 after six steps, ending in $[1210, 1184]$.

Of the 59,761 odd starting values hitting a square, 4911 end in a cycle (of which 4812 are going through $25 \to 6$). Interestingly, the 3 sequences hitting $573^2 =$ 328,329, like 681831, 328329, 148420, 172628, 133132, 103244, 81220, 96188, 74332, 55756, 44036, 34504, 33896, 33304, 32216, 28204, 25724, 20476, 15364, 12860, 14188, 10648, 11312, 13984, 16256, 16384, 16383, 6145, 1235, 445, 95, 25, and 6 hit $16384 = 2^{14}$, and then six odd numbers again, finishing with 5^2 and then the perfect number 6. The 39 odd starters hitting 285^2 merge with the 50,106 sequence obtaining a 14-digit maximum before ending after around 100 steps in the $[\,79750, 88730\,]$ amicable pair.

6.2. All Cyclers

In all, 18,103 starting values lead to a cycle. Among these are 5119 odd starting values, 93 mergers. Of the 12,984 even ones, 6954 are mergers.

Fifty-six different cycles occur; four of these are the perfect numbers 6, 28, 496, 8128. Two are cycles of length four:

$$[\,1264460, 1547860, 1727636, 1305184\,], \text{and}[\,2115324, 3317740, 3649556, 2797612\,],$$

one is a cycle of length five: $[\,12496, 14288, 15472, 14536, 14264\,]$, and one of length 28:

$$C_{28} = [\,14316, 19116, 31704, 47616, 83328, 177792, 295488, 629072, 589786, 294896, 358336, 418904, 366556, 274924, 275444, 243760, 376736, 381028, 285778, 152990, 122410, 97946, 48976, 45946, 22976, 22744, 19916, 17716\,].$$

The remaining 48 are amicable pairs.

Below, the distribution of the lengths of all of these is depicted. Not shown is the long tail, with 123 sequences even having length exceeding 1000, of which 9 are main sequences.

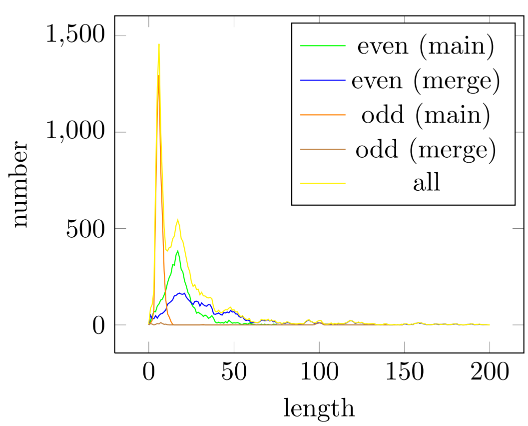

Figure 13: *Distribution of the lengths of sequences ending in cycles (cut off at 200).*

The longest main sequences ending in a cycle are 133,596 and 105,384,

- 133,596 of length 3961, height 98.614 and volume 217,737.45,
- 105,384 of length 2847, height 95.155 and volume 121,142.480.

They are profiled below; both end in amicable pair [1184, 1210].

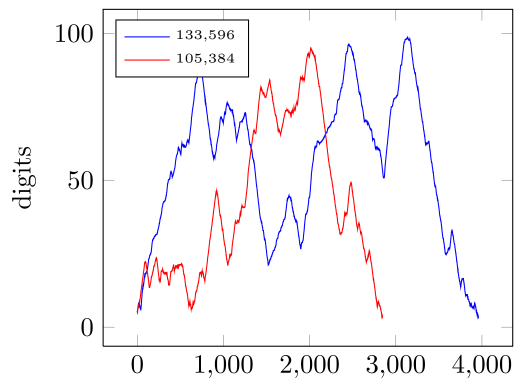

Figure 14: *Size 'profiles' of the two longest main sequences ending in cycles.*

The final tables list all cycles that occur, with their popularity.

cycle	:	#	(#main)	even	entry
	:				
[6]	:	5395	(5132)	579	5395
[28]	:	1	(1)	1	1
[496]	:	13	(11)	12	13
[8128]	:	1408	(460)	1408	1408
[1264460, 1547860,	:				
1727636, 1305184]	:	13	(2)	13	13|0|0|0
[2115324, 3317740,	:	1	(1)	1	1|0|0|0
3649556, 2797612]	:	1	(1)	1	1|0|0|0
[12496, 14288, 15472,	:				
14536, 14264]	:	150	(109)	150	72|2|1|74|1
C_{28}	:	741	(131)	741	8|1|3|3|1|6|1|2
	:				33|1|5|1|2|19|15
	:				1|157|1|1|1|3|5
	:				1|35|1|49|269|123
total	:	18,103	(11,056)	12,984	

Table 12: *Aliquot cycles, how many (main) sequences lead to them, and at which point of the cycle.*

The second and third columns list the number of starting values ending in the cycle listed in the first column and (in parentheses) the number of *main* sequences among these. The fourth column lists the number of even starting values among those of the second column. In the fifth column, it is shown how often each entry of the cycle is first hit by some sequence.

Thus, for example, the values 1|10 in the row for the amicable pair $[220, 284]$ reflect that, besides the starting values 220 and 284 themselves, only 9 other starting up to 10^6 lead to this cycle (7 of them with odd starting value according to the fourth column), and all of those will hit 284 first.

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[12] See `http://www.rechenkraft.net/aliquot/AllSeq.html`

[220, 284]	:	11	(10)	3	1|10
[1184, 1210]	:	7564	(3841)	7561	3599|3965
[2620, 2924]	:	1153	(533)	1152	9|1144
[5020, 5564]	:	50	(44)	24	1|49
[6232, 6368]	:	27	(26)	10	26|1
[10744, 10856]	:	249	(125)	249	1|248
[12285, 14595]	:	106	(104)	0	56|50
[17296, 18416]	:	202	(100)	202	200|2
[63020, 76084]	:	9	(2)	9	1|8
[66928, 66992]	:	6	(5)	6	5|1
[67095, 71145]	:	47	(45)	0	43|4
[69615, 87633]	:	39	(36)	0	21|18
[79750, 88730]	:	342	(102)	303	306|36
[100485, 124155]	:	4	(3)	0	2|2
[122265, 139815]	:	3	(2)	0	2|1
[122368, 123152]	:	3	(2)	3	2|1
[141664, 153176]	:	10	(6)	10	1|9
[142310, 168730]	:	5	(4)	5	1|4
[171856, 176336]	:	23	(17)	23	8|15
[176272, 180848]	:	17	(7)	17	16|1
[185368, 203432]	:	106	(56)	106	102|4
[196724, 202444]	:	25	(19)	25	6|19
[280540, 365084]	:	121	(41)	121	120|1
[308620, 389924]	:	6	(5)	6	5|1
[319550, 430402]	:	17	(8)	17	15|2
[356408, 399592]	:	2	(1)	2	1|1
[437456, 455344]	:	12	(6)	12	2|10
[469028, 486178]	:	34	(10)	34	30|4
[503056, 514736]	:	9	(5)	9	8|1
[522405, 525915]	:	6	(5)	0	4|2
[600392, 669688]	:	3	(2)	3	1|2
[609928, 686072]	:	3	(1)	3	2|1
[624184, 691256]	:	5	(1)	5	3|2
[635624, 712216]	:	39	(10)	39	31|8
[643336, 652664]	:	2	(1)	2	1|1
[667964, 783556]	:	7	(5)	7	4|3
[726104, 796696]	:	4	(3)	4	3|1
[802725, 863835]	:	2	(1)	0	1|1
[879712, 901424]	:	35	(4)	35	15|20
[898216, 980984]	:	9	(1)	9	8|1
[947835, 1125765]	:	1	(1)	0	1|0
[998104, 1043096]	:	2	(1)	2	2|0
[1077890, 1099390]	:	19	(1)	19	19|0
[2723792, 2874064]	:	13	(3)	13	9|4
[4238984, 4314616]	:	16	(1)	16	0|16
[4532710, 6135962]	:	6	(1)	6	6|0
[5459176, 5495264]	:	6	(1)	6	6|0
[438452624, 445419376]	:	1	(1)	1	1|0