SOME q-BINOMIAL IDENTITIES INVOLVING THE GENERALIZED q-FIBONACCI NUMBERS

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Abstract

In this paper, starting from the *shifting property* for the ordinary Fibonacci and q-Fibonacci numbers, we obtain some combinatorial identities involving the generalized Fibonacci and q-Fibonacci numbers of the first and second kind, and for the q-Fibonacci polynomials. In particular, we specialize these identities to the Fibonacci polynomials, the Pell polynomials, the Jacobstahl polynomials, the Chebyshev polynomials, the Fermat polynomials and the Morgan-Voyce polynomials.

1. Introduction

The *Fibonacci numbers* f_n, [4] [16, A000045], are defined by the recurrence $f_{n+2} = f_{n+1}+f_n$ with the initial values $f_0 = f_1 = 1$. These numbers have been generalized in several ways. For instance, Carlitz [2] defined the *q-Fibonacci numbers* $f_n(q)$ by employing a particular statistic on the set of Fibonacci strings (i.e. binary strings without two consecutive 1's), and proved that they satisfy the recurrence

$$f_{n+2}(q) = f_{n+1}(q) + q^{n+1} f_n(q)$$

with the initial values $f_0(q) = f_1(q) = 1$. These numbers have been further generalized as follows.

The *generalized Fibonacci numbers of the first kind* $f_n^{[m]}$, [8, 10], are defined by the recurrence

$$f_{n+m}^{[m]} = f_{n+m-1}^{[m]} + f_{n+m-2}^{[m]} + \cdots + f_n^{[m]}$$

with the initial values

$$f_0^{[m]} = 1\,, \quad f_1^{[m]} = 1\,, \quad f_2^{[m]} = 2\,, \quad \ldots\,, \quad f_{m-2}^{[m]} = 2^{m-3}\,, \quad f_{m-1}^{[m]} = 2^{m-2}\,.$$

The *generalized q-Fibonacci numbers of the first kind* $f_n^{[m]}(q)$, [10], are defined by the recurrence

$$f_{n+m}^{[m]}(q) = f_{n+m-1}^{[m]}(q) \\ + q^{n+m-1} f_{n+m-2}^{[m]}(q) + q^{n+m-2} f_{n+m-3}^{[m]}(q) + \cdots + q^{n+1} f_n^{[m]}(q) \tag{1}$$

with the initial values

$$f_{-m+1}^{[m]}(q) = f_{-m-2}^{[m]}(q) = \cdots = f_{-1}^{[m]}(q) = 0 \qquad \text{and} \qquad f_0^{[m]}(q) = 1\,.$$

The *generalized Fibonacci numbers of the second kind* $g_n^{[m]}$, [14], are defined by the recurrence

$$g_{n+m}^{[m]} = g_{n+m-1}^{[m]} + g_n^{[m]}$$

with the initial values $g_0^{[m]} = g_1^{[m]} = \cdots = g_{m-1}^{[m]} = 1$. The *generalized q-Fibonacci numbers of the second kind* $g_n^{[m]}(q)$, [14], are defined by the recurrence

$$g_{n+m}^{[m]}(q) = g_{n+m-1}^{[m]}(q) + q^{n+1} g_n^{[m]}(q) \tag{2}$$

with the initial values $g_0^{[m]}(q) = g_1^{[m]}(q) = \cdots = g_{m-1}^{[m]}(q) = 1$.

The *q-binomial coefficients*, or *Gaussian coefficients*, are defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

where $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ are the *q-factorial numbers* and $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ are the *q-numbers*, and satisfy the recurrence

$$\binom{n+1}{k+1}_q = \binom{n}{k}_q + q^{k+1} \binom{n}{k+1}_q\,. \tag{3}$$

For $q = 1$, we have the usual *binomial coefficients*.

In [12], we obtained the following identities (*shifting property*)

$$\sum_{k=0}^{n} \binom{n}{k} f_{k+1} = \sum_{k=0}^{n} \binom{n+1}{k+1} f_k \tag{4}$$

$$\sum_{k=0}^{n} \binom{n}{k}_q f_{k+1}(q) = \sum_{k=0}^{n} \binom{n+1}{k+1}_q f_k(q) \tag{5}$$

for the Fibonacci and the q-Fibonacci numbers. In [13], we obtained some generalizations of identity (4) for the bivariate Fibonacci polynomials. In this paper, we extend identities (4) and (5) to the generalized Fibonacci and q-Fibonacci numbers of the first and the second kind (Sections 2 and 3), and to the q-Fibonacci polynomials (Section 4). In particular, we specialize these identities to several classical polynomial sequences, such as the ones given by the Fibonacci polynomials, the Pell polynomials, the Jacobstahl polynomials, the Chebyshev polynomials, the Fermat polynomials and the Morgan-Voyce polynomials.

2. Generalized Shifting Property

We start by generalizing identities (4) and (5) to the generalized Fibonacci and q-Fibonacci numbers of the first kind. Specifically, the following theorem.

Theorem 1. *For every* $m, n \in \mathbb{N}$, $m \geq 2$, *we have the q-binomial identity*

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k}_q f_k^{[m]}(q) \cdots f_{k+m-3}^{[m]}(q) A_k^{[m]}(q) \\ = \sum_{k=0}^{n} \binom{n+1}{k+1}_q f_k^{[m]}(q) f_{k+1}^{[m]}(q) \cdots f_{k+m-2}^{[m]}(q)\,,\end{aligned} \tag{6}$$

where

$$A_k^{[m]}(q) = f_{k+m-1}^{[m]}(q) - q^{k+m-2} f_{k+m-3}^{[m]}(q) - \cdots - q^{k+1} f_k^{[m]}(q)\,.$$

In particular, for $q = 1$, *we have the binomial identity*

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} f_k^{[m]} \cdots f_{k+m-3}^{[m]} \left(f_{k+m-1}^{[m]} - f_{k+m-3}^{[m]} - \cdots - f_k^{[m]} \right) \\ = \sum_{k=0}^{n} \binom{n+1}{k+1} f_k^{[m]} f_{k+1}^{[m]} \cdots f_{k+m-2}^{[m]}\,.\end{aligned} \tag{7}$$

Proof. From recurrence (1), by replacing n by k, we have

$$f_{k+m}^{[m]}(q) - q^{k+m-1} f_{k+m-2}^{[m]}(q) - \cdots - q^{k+2} f_{k+1}^{[m]}(q) = f_{k+m-1}^{[m]}(q) + q^{k+1} f_k^{[m]}(q)$$

from which we obtain

$$\begin{aligned}&f_{k+1}^{[m]}(q) \cdots f_{k+m-2}^{[m]}(q) (f_{k+m}^{[m]}(q) - q^{k+m-1} f_{k+m-2}^{[m]}(q) - \cdots - q^{k+2} f_{k+1}^{[m]}(q)) \\ &\qquad = f_{k+1}^{[m]}(q) \cdots f_{k+m-2}^{[m]}(q) f_{k+m-1}^{[m]}(q) + q^{k+1} f_k^{[m]}(q) f_{k+1}^{[m]}(q) \cdots f_{k+m-2}^{[m]}(q)\,,\end{aligned}$$

that is,

$$\begin{aligned}&f_{k+1}^{[m]}(q) \cdots f_{k+m-2}^{[m]}(q) A_{k+1}^{[m]}(q) \\ &\qquad = f_{k+1}^{[m]}(q) \cdots f_{k+m-2}^{[m]}(q) f_{k+m-1}^{[m]}(q) + q^{k+1} f_k^{[m]}(q) f_{k+1}^{[m]}(q) \cdots f_{k+m-2}^{[m]}(q)\,.\end{aligned}$$

Then, we have

$$\begin{aligned}&\sum_{k=0}^{n-1} \binom{n}{k+1}_q f_{k+1}^{[m]}(q) \cdots f_{k+m-2}^{[m]}(q) A_{k+1}^{[m]}(q) \\ &\quad = \sum_{k=0}^{n-1} \binom{n}{k+1}_q f_{k+1}^{[m]}(q) \cdots f_{k+m-1}^{[m]}(q) + \sum_{k=0}^{n-1} \binom{n}{k+1}_q q^{k+1} f_k^{[m]}(q) \cdots f_{k+m-2}^{[m]}(q)\,.\end{aligned}$$

Replacing k by $k-1$ in the first two sums, we obtain

$$\begin{aligned}\sum_{k=1}^{n}&\binom{n}{k}_q f_k^{[m]}(q)\cdots f_{k+m-3}^{[m]}(q)A_k^{[m]}(q)\\&=\sum_{k=1}^{n}\binom{n}{k}_q f_k^{[m]}(q)\cdots f_{k+m-2}^{[m]}(q)+\sum_{k=0}^{n-1}\binom{n}{k+1}_q q^{k+1}f_k^{[m]}(q)\cdots f_{k+m-2}^{[m]}(q)\,,\end{aligned}$$

and then, by recurrence (3), we have

$$\begin{aligned}\sum_{k=0}^{n}&\binom{n}{k}_q f_k^{[m]}(q)\cdots f_{k+m-3}^{[m]}(q)A_k^{[m]}(q)\\&-f_0^{[m]}(q)\cdots f_{m-3}^{[m]}(q)(f_{m-1}^{[m]}(q)-q^{m-2}f_{m-3}^{[m]}(q)-\cdots-qf_0^{[m]}(q))\\&\quad=\sum_{k=0}^{n}\left[\binom{n}{k}_q+q^{k+1}\binom{n}{k+1}_q\right]f_k^{[m]}(q)\cdots f_{k+m-2}^{[m]}(q)\\&\qquad-[n+1]_q f_0^{[m]}(q)\cdots f_{m-2}^{[m]}(q)+[n]_q q f_0^{[m]}(q)\cdots f_{m-2}^{[m]}(q)\\&\quad=\sum_{k=0}^{n}\binom{n+1}{k+1}_q f_k^{[m]}(q)\cdots f_{k+m-2}^{[m]}(q)\\&\qquad-([n+1]_q-q[n]_q)f_0^{[m]}(q)\cdots f_{m-2}^{[m]}(q)\,.\end{aligned}$$

Since $[n+1]_q-q[n]_q=1$, we have the identity

$$\begin{aligned}&f_0^{[m]}(q)\cdots f_{m-3}^{[m]}(q)(f_{m-1}^{[m]}(q)-q^{m-2}f_{m-3}^{[m]}(q)-\cdots-qf_0^{[m]}(q))\\&\quad=([n+1]_q-q[n]_q)f_0^{[m]}(q)\cdots f_{m-3}^{[m]}(q)f_{m-2}^{[m]}(q)\end{aligned}$$

if and only if

$$f_{m-1}^{[m]}(q)-q^{m-2}f_{m-3}^{[m]}(q)-\cdots-qf_0^{[m]}(q)=f_{m-2}^{[m]}(q)\,,$$

that is, if and only if

$$f_{m-1}^{[m]}(q)=f_{m-2}^{[m]}(q)+q^{m-2}f_{m-3}^{[m]}(q)-\cdots-qf_0^{[m]}(q)$$

and this equation is true by recurrence (1), with $n=-1$. Hence, we have identity (6). □

Remark 2. For $m=2$, we recover identities (4) and (5). Moreover, for $m=3$, we have the identities

$$\sum_{k=0}^{n}\binom{n}{k}_q f_k^{[3]}(q)\Big(f_{k+2}^{[3]}(q)-q^{k+1}f_k^{[3]}(q)\Big)=\sum_{k=0}^{n}\binom{n+1}{k+1}_q f_k^{[3]}(q)f_{k+1}^{[3]}(q)\,,\qquad(8)$$

$$\sum_{k=0}^{n} \binom{n}{k} f_k^{[3]} \left(f_{k+2}^{[3]} - f_k^{[3]} \right) = \sum_{k=0}^{n} \binom{n+1}{k+1} f_k^{[3]} f_{k+1}^{[3]} . \tag{9}$$

Finally, for $m = 4$, we have the identities

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k}_q f_k^{[4]}(q) f_{k+1}^{[4]}(q) \left(f_{k+3}^{[4]}(q) - q^{k+2} f_{k+1}^{[4]}(q) - q^{k+1} f_k^{[4]}(q) \right) \\ &\qquad = \sum_{k=0}^{n} \binom{n+1}{k+1}_q f_k^{[4]}(q) f_{k+1}^{[4]}(q) f_{k+2}^{[4]}(q) , \end{aligned} \tag{10}$$

$$\sum_{k=0}^{n} \binom{n}{k} f_k^{[4]} f_{k+1}^{[4]} \left(f_{k+3}^{[4]} - f_{k+1}^{[4]} - f_k^{[4]} \right) = \sum_{k=0}^{n} \binom{n+1}{k+1} f_k^{[4]} f_{k+1}^{[4]} f_{k+2}^{[4]} . \tag{11}$$

In next theorem, we generalize identities (4) and (5) to the generalized Fibonacci and q-Fibonacci numbers of the second kind.

Theorem 3. *For every $m, n \in \mathbb{N}$, $m \geq 2$, we have the q-binomial identity*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k}_q g_k^{[m]}(q) \cdots g_{k+m-3}^{[m]}(q) g_{k+m-1}^{[m]}(q) \\ &\qquad = \sum_{k=0}^{n} \binom{n+1}{k+1}_q g_k^{[m]}(q) g_{k+1}^{[m]}(q) \cdots g_{k+m-2}^{[m]}(q) . \end{aligned} \tag{12}$$

In particular, for $q = 1$, we have the binomial identity

$$\sum_{k=0}^{n} \binom{n}{k} g_k^{[m]} \cdots g_{k+m-3}^{[m]} g_{k+m-1}^{[m]} = \sum_{k=0}^{n} \binom{n+1}{k+1} g_k^{[m]} g_{k+1}^{[m]} \cdots g_{k+m-2}^{[m]} . \tag{13}$$

Proof. From recurrence (2), we have

$$\begin{aligned} &g_{k+1}^{[m]}(q) \cdots g_{k+m-2}^{[m]}(q) g_{k+m}^{[m]}(q) \\ &\qquad = g_{k+1}^{[m]}(q) \cdots g_{k+m-2}^{[m]}(q) g_{k+m-1}^{[m]}(q) + q^{k+1} g_k^{[m]}(q) g_{k+1}^{[m]}(q) \cdots g_{k+m-2}^{[m]}(q) , \end{aligned}$$

and then

$$\begin{aligned} &\sum_{k=0}^{n-1} \binom{n}{k+1}_q g_{k+1}^{[m]}(q) \cdots g_{k+m-2}^{[m]}(q) g_{k+m}^{[m]}(q) \\ &\quad = \sum_{k=0}^{n-1} \binom{n}{k+1}_q g_{k+1}^{[m]}(q) \cdots g_{k+m-1}^{[m]}(q) + \sum_{k=0}^{n-1} \binom{n}{k+1}_q q^{k+1} g_k^{[m]}(q) \cdots g_{k+m-2}^{[m]}(q) . \end{aligned}$$

Replacing k by $k-1$ in the first two sums, we obtain

$$\sum_{k=1}^{n} \binom{n}{k}_q g_k^{[m]}(q) \cdots g_{k+m-3}^{[m]}(q) g_{k+m-1}^{[m]}(q)$$
$$= \sum_{k=1}^{n} \binom{n}{k}_q g_k^{[m]}(q) \cdots g_{k+m-2}^{[m]}(q) + \sum_{k=0}^{n-1} \binom{n}{k+1}_q q^{k+1} g_k^{[m]}(q) \cdots g_{k+m-2}^{[m]}(q)\,,$$

and then, by recurrence (3), we have

$$\sum_{k=0}^{n} \binom{n}{k}_q g_k^{[m]}(q) \cdots g_{k+m-3}^{[m]}(q) g_{k+m-1}^{[m]}(q) - g_0^{[m]}(q) \cdots g_{m-3}^{[m]}(q) g_{m-1}^{[m]}(q)$$
$$= \sum_{k=0}^{n} \left[\binom{n}{k}_q + q^{k+1} \binom{n}{k+1}_q \right] g_k^{[m]}(q) \cdots g_{k+m-2}^{[m]}(q)$$
$$- [n+1]_q g_0^{[m]}(q) \cdots g_{m-2}^{[m]}(q) + [n]_q q g_0^{[m]}(q) \cdots g_{m-2}^{[m]}(q)$$
$$= \sum_{k=0}^{n} \binom{n+1}{k+1}_q g_k^{[m]}(q) \cdots g_{k+m-2}^{[m]}(q) - ([n+1]_q - q[n]_q) g_0^{[m]}(q) \cdots g_{m-2}^{[m]}(q)\,.$$

Since $[n+1] - q[n] = 1$ and all the initial conditions are equal to 1, we have identity (12). □

Remark 4. For $m = 2$, we recover identities (4) and (5) once again. For $m = 3$, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k}_q g_k^{[3]}(q) g_{k+2}^{[3]}(q) = \sum_{k=0}^{n} \binom{n+1}{k+1}_q g_k^{[3]}(q) g_{k+1}^{[3]}(q)\,, \tag{14}$$

$$\sum_{k=0}^{n} \binom{n}{k} g_k^{[3]} g_{k+2}^{[3]} = \sum_{k=0}^{n} \binom{n+1}{k+1} g_k^{[3]} g_{k+1}^{[3]}\,, \tag{15}$$

and, for $m = 4$, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k}_q g_k^{[4]}(q) g_{k+1}^{[4]}(q) g_{k+3}^{[4]}(q) = \sum_{k=0}^{n} \binom{n+1}{k+1}_q g_k^{[4]}(q) g_{k+1}^{[4]}(q) g_{k+2}^{[4]}(q)\,, \tag{16}$$

$$\sum_{k=0}^{n} \binom{n}{k} g_k^{[4]} g_{k+1}^{[4]} g_{k+3}^{[4]} = \sum_{k=0}^{n} \binom{n+1}{k+1} g_k^{[4]} g_{k+1}^{[4]} g_{k+2}^{[4]}\,. \tag{17}$$

3. Other Identities

Theorems 1 and 3 give a natural generalization of the original shifting property for the Fibonacci and q-Fibonacci numbers, given by identities (4) and (5). The following Theorems 5 and 7 give a different, but equally natural, generalization of this property. For the generalized Fibonacci and q-Fibonacci numbers of the first kind, we have the next theorem.

Theorem 5. *For every* $m, n \in \mathbb{N}$, $m \geq 2$, *we have the q-binomial identity*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k f_k^{[m]}(q)\cdots f_{k+m-2}^{[m]}(q)B_k^{[m]}(q)\\ =\sum_{k=0}^{n}\binom{n+1}{k+1}_q(-1)^k f_k^{[m]}(q)f_{k+1}^{[m]}(q)\cdots f_{k+m-1}^{[m]}(q)\end{aligned}\tag{18}$$

where

$$B_k^{[m]}(q)=f_{k+m-2}^{[m]}(q)+q^{k+m-2}f_{k+m-3}^{[m]}(q)+\cdots+q^{k+1}f_k^{[m]}(q)\,.$$

In particular, for $q=1$, *we have the binomial identity*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^k f_k^{[m]}\cdots f_{k+m-2}^{[m]}\Big(f_{k+m-2}^{[m]}+f_{k+m-3}^{[m]}+\cdots+f_k^{[m]}\Big)\\ =\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k f_k^{[m]}f_{k+1}^{[m]}\cdots f_{k+m-1}^{[m]}\end{aligned}\tag{19}$$

Proof. From recurrence (1), by replacing n by k, we have

$$f_{k+m-1}^{[m]}(q)+q^{k+m-1}f_{k+m-2}^{[m]}(q)+\cdots+q^{k+2}f_{k+1}^{[m]}(q)=f_{k+m}^{[m]}(q)-q^{k+1}f_k^{[m]}(q)$$

from which we obtain

$$\begin{aligned}&f_{k+1}^{[m]}(q)\cdots f_{k+m-1}^{[m]}(q)(f_{k+m-1}^{[m]}(q)+q^{k+m-1}f_{k+m-2}^{[m]}(q)+\cdots+q^{k+2}f_{k+1}^{[m]}(q))\\ &\quad=f_{k+1}^{[m]}(q)\cdots f_{k+m-1}^{[m]}(q)f_{k+m}^{[m]}(q)-q^{k+1}f_k^{[m]}(q)f_{k+1}^{[m]}(q)\cdots f_{k+m-1}^{[m]}(q)\,.\end{aligned}$$

Then, we have

$$\begin{aligned}&\sum_{k=0}^{n-1}\binom{n}{k+1}_q(-1)^{k+1}f_{k+1}^{[m]}(q)\cdots f_{k+m-1}^{[m]}(q)(f_{k+m-1}^{[m]}(q)+\cdots+q^{k+2}f_{k+1}^{[m]}(q))\\ &=\sum_{k=0}^{n-1}\binom{n}{k+1}_q(-1)^{k+1}f_{k+1}^{[m]}(q)\cdots f_{k+m}^{[m]}(q)\\ &\quad-\sum_{k=0}^{n-1}\binom{n}{k+1}_q(-1)^{k+1}q^{k+1}f_k^{[m]}(q)\cdots f_{k+m-1}^{[m]}(q)\,.\end{aligned}$$

Replacing k by $k-1$ in the first two sums, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n}\binom{n}{k}_q(-1)^k f_k^{[m]}(q)\cdots f_{k+m-2}^{[m]}(q)(f_{k+m-2}^{[m]}(q)+\cdots+q^{k+1}f_k^{[m]}(q))\\
&\quad=\sum_{k=1}^{n}\binom{n}{k}_q(-1)^k f_k^{[m]}(q)\cdots f_{k+m-1}^{[m]}(q)\\
&\qquad+\sum_{k=0}^{n-1}\binom{n}{k+1}_q(-1)^k q^{k+1} f_k^{[3]}(q)f_{k+1}^{[3]}(q)f_{k+2}^{[3]}(q)\,,
\end{aligned}$$

and then, by recurrence (3), we have

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k f_k^{[m]}(q)\cdots f_{k+m-2}^{[m]}(q)(f_{k+m-2}^{[m]}(q)+\cdots+q^{k+1}f_k^{[m]}(q))\\
&\quad-f_0^{[m]}(q)\cdots f_{m-2}^{[m]}(q)(f_{m-2}^{[m]}(q)+\cdots+qf_0^{[m]}(q))\\
&\qquad=\sum_{k=0}^{n}\left[\binom{n}{k}_q+q^{k+1}\binom{n}{k+1}_q\right](-1)^k f_k^{[m]}(q)\cdots f_{k+m-1}^{[m]}(q)\\
&\qquad\quad-[n+1]_q f_0^{[m]}(q)\cdots f_{m-1}^{[m]}(q)+[n]_q q f_0^{[3]}(q)f_0^{[m]}(q)\cdots f_{m-1}^{[m]}(q)\\
&\qquad=\sum_{k=0}^{n}\binom{n+1}{k+1}_q(-1)^k f_k^{[m]}(q)\cdots f_{k+m-1}^{[m]}(q)\\
&\qquad\quad-([n+1]_q-q[n]_q)f_0^{[m]}(q)\cdots f_{m-1}^{[m]}(q)\,.
\end{aligned}$$

Since $[n+1]-q[n]=1$, we have the identity

$$f_0^{[m]}(q)\cdots f_{m-2}^{[m]}(q)(f_{m-2}^{[m]}(q)+\cdots+qf_0^{[m]}(q))=([n+1]_q-q[n]_q)f_0^{[m]}(q)\cdots f_{m-1}^{[m]}(q)$$

if and only if

$$f_{m-1}^{[m]}(q)=f_{m-2}^{[m]}(q)+q^{m-2}f_{m-3}^{[m]}(q)+\cdots+qf_0^{[m]}(q)\,,$$

and this equation is true by recurrence (1), with $n=-1$. Hence, we have identity (18). □

Remark 6. For $m=2$, we have the identities

$$\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k f_k(q)^2=\sum_{k=0}^{n}\binom{n+1}{k+1}_q(-1)^k f_k(q)f_{k+1}(q)\,, \tag{20}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k f_k^2=\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k f_k f_{k+1}\,. \tag{21}$$

For $m=3$, we have the identities

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k f_k^{[3]}(q)f_{k+1}^{[3]}(q)\big(f_{k+1}^{[3]}(q)+q^{k+1}f_k^{[3]}(q)\big)\\ =\sum_{k=0}^{n}\binom{n+1}{k+1}_q(-1)^k f_k^{[3]}(q)f_{k+1}^{[3]}(q)f_{k+2}^{[3]}(q)\,,\end{aligned} \tag{22}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k f_k^{[3]}f_{k+1}^{[3]}\big(f_{k+1}^{[3]}+f_k^{[3]}\big)=\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k f_k^{[3]}f_{k+1}^{[3]}f_{k+2}^{[3]}\,. \tag{23}$$

For $m=4$, we have the identities

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k f_k^{[4]}(q)f_{k+1}^{[4]}(q)f_{k+2}^{[4]}(q)\big(f_{k+2}^{[4]}(q)+q^{k+2}f_{k+1}^{[4]}(q)+q^{k+1}f_k^{[4]}(q)\big)\\ =\sum_{k=0}^{n}\binom{n+1}{k+1}_q(-1)^k f_k^{[4]}(q)f_{k+1}^{[4]}(q)f_{k+2}^{[4]}(q)f_{k+3}^{[4]}(q)\,,\end{aligned} \tag{24}$$

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^k f_k^{[4]}(q)f_{k+1}^{[4]}f_{k+2}^{[4]}\big(f_{k+2}^{[4]}+f_{k+1}^{[4]}+f_k^{[4]}\big)\\ =\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k f_k^{[4]}f_{k+1}^{[4]}f_{k+2}^{[4]}f_{k+3}^{[4]}\,.\end{aligned} \tag{25}$$

Finally, for the generalized Fibonacci and q-Fibonacci numbers of the second kind, we have the following theorem.

Theorem 7. *For every* $m,n\in\mathbb{N}$, $m\geq 2$, *we have the* q*-binomial identity*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k g_k^{[m]}(q)g_{k+1}^{[m]}(q)\cdots g_{k+m-3}^{[m]}(q)g_{k+m-2}^{[m]}(q)^2\\ =\sum_{k=0}^{n}\binom{n+1}{k+1}_q(-1)^k g_k^{[m]}(q)g_{k+1}^{[m]}(q)g_{k+2}^{[m]}(q)\cdots g_{k+m-1}^{[m]}(q)\,.\end{aligned} \tag{26}$$

In particular, for $q=1$, *we have the binomial identity*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^k g_k^{[m]}g_{k+1}^{[m]}\cdots g_{k+m-3}^{[m]}\big(g_{k+m-2}^{[m]}\big)^2\\ =\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k g_k^{[m]}g_{k+1}^{[m]}g_{k+2}^{[m]}\cdots g_{k+m-1}^{[m]}\,.\end{aligned} \tag{27}$$

Proof. From recurrence (2), we have

$$g_{k+m-1}^{[m]}(q)=g_{k+m}^{[m]}(q)-q^{k+1}g_k^{[m]}(q)\,.$$

Hence, we have

$$g_{k+1}^{[m]}(q)\cdots g_{k+m-1}^{[m]}(q)^2 = g_{k+1}^{[m]}(q)\cdots g_{k+m-1}^{[m]}(q)g_{k+m}^{[m]}(q) - q^{k+1}g_k^{[m]}(q)g_{k+1}^{[m]}(q)\cdots g_{k+m-1}^{[m]}(q)\,,$$

and then

$$\begin{aligned}\sum_{k=0}^{n-1}\binom{n}{k+1}_q&(-1)^{k+1}g_{k+1}^{[m]}(q)\cdots g_{k+m-1}^{[m]}(q)^2\\ &=\sum_{k=0}^{n-1}\binom{n}{k+1}_q(-1)^{k+1}g_{k+1}^{[m]}(q)\cdots g_{k+m-1}^{[m]}(q)g_{k+m}^{[m]}(q)\\ &\quad-\sum_{k=0}^{n-1}\binom{n}{k+1}_q(-1)^{k+1}q^{k+1}g_k^{[m]}(q)g_{k+1}^{[m]}(q)\cdots g_{k+m-1}^{[m]}(q)\,.\end{aligned}$$

Replacing k by $k-1$ in the first two sums, we obtain

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}_q&(-1)^k g_k^{[m]}(q)\cdots g_{k+m-2}^{[m]}(q)^2\\ &=\sum_{k=1}^{n}\binom{n}{k}_q(-1)^k g_k^{[m]}(q)\cdots g_{k+m-1}^{[m]}(q)\\ &\quad+\sum_{k=0}^{n-1}\binom{n}{k+1}_q(-1)^k q^{k+1}g_k^{[m]}(q)\cdots g_{k+m-1}^{[m]}(q)\,,\end{aligned}$$

and then, by recurrence (3), we have

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}_q&(-1)^k g_k^{[m]}(q)\cdots g_{k+m-2}^{[m]}(q)^2 - g_0^{[m]}(q)\cdots g_{m-2}^{[m]}(q)^2\\ &=\sum_{k=1}^{n}\left[\binom{n}{k}_q+q^{k+1}\binom{n}{k+1}_q\right](-1)^k g_k^{[m]}(q)\cdots g_{k+m-1}^{[m]}(q)\\ &\quad-[n+1]_q g_0^{[m]}(q)\cdots g_{m-1}^{[m]}(q)+[n]_q q g_0^{[m]}(q)\cdots g_{m-1}^{[m]}(q)\\ &=\sum_{k=1}^{n}\binom{n+1}{k+1}_q(-1)^k g_k^{[m]}(q)\cdots g_{k+m-1}^{[m]}(q)\\ &\quad-([n+1]_q-q[n]_q)g_0^{[m]}(q)\cdots g_{m-1}^{[m]}(q)\,.\end{aligned}$$

Finally, since $[n+1]-q[n]=1$ and the initial conditions are all equal to 1, we have identity (26). □

Remark 8. For $m=2$, we recover identities (20) and (21). For $m=3$, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k g_k^{[3]}(q) g_{k+1}^{[3]}(q)^2 = \sum_{k=0}^{n} \binom{n+1}{k+1}_q (-1)^k g_k^{[3]}(q) g_{k+1}^{[3]}(q) g_{k+2}^{[3]}(q)\,, \tag{28}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k g_k^{[3]} \big(g_{k+1}^{[3]}\big)^2 = \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k g_k^{[3]} g_{k+1}^{[3]} g_{k+2}^{[3]}\,. \tag{29}$$

For $m = 4$, we have the identities

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k g_k^{[4]}(q) g_{k+1}^{[4]}(q) g_{k+2}^{[4]}(q)^2 \\ &\quad = \sum_{k=0}^{n} \binom{n+1}{k+1}_q (-1)^k g_k^{[4]}(q) g_{k+1}^{[4]}(q) g_{k+2}^{[4]}(q) g_{k+3}^{[4]}(q)\,, \end{aligned} \tag{30}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k g_k^{[4]} g_{k+1}^{[4]} (g_{k+2}^{[4]})^2 = \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k g_k^{[4]} g_{k+1}^{[4]} g_{k+2}^{[4]} g_{k+3}^{[4]}\,. \tag{31}$$

4. Final Remarks

Identities (4) and (5) can be extended to many other sequences similar to the Fibonacci sequence. In this final section, we consider the following polynomial sequences. The *Fibonacci polynomials* $F_n(x)$ defined by the recurrence $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$ with the initial values $F_0(x) = 1$ and $F_1(x) = x$. The *Pell polynomials* $P_n(x)$, [6], defined by the recurrence $P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x)$ with the initial values $P_0(x) = 1$ and $P_1(x) = 2x$. The *Jacobsthal polynomials* $J_n(x)$, [7], defined by the recurrence $J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x)$ with the initial values $J_0(x) = J_1(x) = 1$. The *Chebyshev polynomials of the second kind* $U_n(x)$, [1, 15], defined by the recurrence $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$ with the initial values $U_0(x) = 1$ and $U_1(x) = 2x$. The *Fermat polynomials* $\varphi_n(x)$, [5], defined by the recurrence $\varphi_{n+2}(x) = x\varphi_{n+1}(x) - 2\varphi_n(x)$ with the initial values $\varphi_0(x) = 1$ and $\varphi_1(x) = x$. The *Morgan-Voyce polynomials* $B_n(x)$, [9], defined by the recurrence $B_{n+2}(x) = (x+2)B_{n+1}(x) - B_n(x)$ with the initial values $B_0(x) = 1$ and $B_1(x) = x+2$. The *q-Fibonacci polynomials* $\Phi_n(q;x)$, [3], defined by the recurrence

$$\Phi_{n+2}(q;x) = \Phi_{n+1}(q;x) + q^{n+1}x\,\Phi_n(q;x) \tag{32}$$

with the initial values $\Phi_0(q;x) = \Phi_1(q;x) = 1$. All these polynomials are particular instances of the *q-Fibonacci polynomials* $F_n(q;x,y)$, [11], defined by the recurrence

$$F_{n+2}(q;x,y) = xF_{n+1}(q;x,y) + q^{n+1}yF_n(q;x,y) \tag{33}$$

with the initial values $F_0(q;x,y) = 1$ and $F_1(q;x,y) = x$. In particular, we have $F_n(1;x,1) = F_n(x)$, $F_n(1;2x,1) = P_n(x) = F_n(2x)$, $F_n(1;1,2x) = \Phi_n(q;2x) = J_n(x)$, $F_n(1;2x,-1) = U_n(x)$, $F_n(1;x,-2) = \varphi_n(x)$, $F_n(1;x+2,-1) = B_n(x)$ and $F_n(q;1,x) = \Phi_n(q;x)$. We also have $F_n(q;x,y) = x^n\Phi_n(q;y/x^2)$.

Theorem 9. *We have the q-binomial identity*

$$\sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k} F_{k+1}(q;x,y) = \sum_{k=0}^{n} \binom{n+1}{k+1}_q x^{k+1} y^{n-k} F_k(q;x,y). \tag{34}$$

Proof. From recurrence (33), with n replaced by k, we have the identity

$$\begin{aligned}\sum_{k=0}^{n-1} &\binom{n}{k+1}_q \frac{x^{k+1}}{y^{k+1}} F_{k+2}(q;x,y) \\ &= \sum_{k=0}^{n-1} \binom{n}{k+1}_q \frac{x^{k+2}}{y^{k+1}} F_{k+1}(q;x,y) + \sum_{k=0}^{n-1} \binom{n}{k+1}_q \frac{x^{k+1}}{y^{k}} q^{k+1} F_k(q;x,y),\end{aligned}$$

that is,

$$\begin{aligned}\sum_{k=1}^{n} &\binom{n}{k}_q \frac{x^{k}}{y^{k}} F_{k+1}(q;x,y) \\ &= \sum_{k=1}^{n} \binom{n}{k}_q \frac{x^{k+1}}{y^{k}} F_{k}(q;x,y) + \sum_{k=0}^{n-1} \binom{n}{k+1}_q \frac{x^{k+1}}{y^{k}} q^{k+1} F_k(q;x,y),\end{aligned}$$

that is,

$$\begin{aligned}\sum_{k=0}^{n} &\binom{n}{k}_q \frac{x^{k}}{y^{k}} F_{k+1}(q;x,y) - F_1(q;x,y) \\ &= \sum_{k=0}^{n} \left[\binom{n}{k}_q + q^{k+1} \binom{n}{k+1}_q \right] \frac{x^{k+1}}{y^{k}} F_k(q;x,y) - xF_0(q;x,y).\end{aligned}$$

By the initial values $F_0(q;x,y) = 1$ and $F_1(q;x,y) = x$ and by recurrence (3), we obtain the identity

$$\sum_{k=0}^{n} \binom{n}{k}_q \frac{x^{k}}{y^{k}} F_{k+1}(q;x,y) = \sum_{k=0}^{n} \binom{n+1}{k+1}_q \frac{x^{k+1}}{y^{k}} F_k(q;x,y)$$

which is equivalent to identity (34). □

Remark 10. As particular instances of identity (34), we have the identities

$$\sum_{k=0}^{n} \binom{n}{k} x^k F_{k+1}(x) = \sum_{k=0}^{n} \binom{n+1}{k+1} x^{k+1} F_k(x), \tag{35}$$

$$\sum_{k=0}^{n}\binom{n}{k}(2x)^k P_{k+1}(x)=\sum_{k=0}^{n}\binom{n+1}{k+1}(2x)^{k+1}P_k(x)\,, \tag{36}$$

$$\sum_{k=0}^{n}\binom{n}{k}(2x)^{n-k} J_{k+1}(x)=\sum_{k=0}^{n}\binom{n+1}{k+1}(2x)^{n-k}J_k(x)\,, \tag{37}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(2x)^k U_{k+1}(x)=\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^{n-k}(2x)^{k+1}U_k(x)\,, \tag{38}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-2)^{n-k}x^k\varphi_{k+1}(x)=\sum_{k=0}^{n}\binom{n+1}{k+1}(-2)^{n-k}x^{k+1}\varphi_k(x)\,, \tag{39}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(x+2)^k B_{k+1}(x)=\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^{n-k}(x+2)^{k+1}B_k(x)\,, \tag{40}$$

$$\sum_{k=0}^{n}\binom{n}{k}_q x^{n-k}\Phi_{k+1}(q;x)=\sum_{k=0}^{n}\binom{n+1}{k+1}_q x^{n-k}\Phi_k(q;x)\,. \tag{41}$$

Theorem 11. *We have the q-binomial identity*

$$\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k x^{n-k}\Phi_k(q;x)^2=\sum_{k=0}^{n}\binom{n+1}{k+1}_q(-1)^k x^{n-k}\Phi_k(q;x)\Phi_{k+1}(q;x)\,. \tag{42}$$

Proof. From the recurrence of the q-polynomials $\Phi_k(q;x)$, we obtain the identity

$$\Phi_{k+1}(q;x)^2=\Phi_{k+1}(q;x)\Phi_{k+2}(q;x)-q^{k+1}x\,\Phi_k(q;x)\Phi_{k+1}(q;x)\,,$$

and consequently the identity

$$\begin{aligned}&\sum_{k=0}^{n-1}\binom{n}{k+1}_q\frac{(-1)^{k+1}}{x^{k+1}}\;\Phi_{k+1}(q;x)^2\\&\quad=\sum_{k=0}^{n-1}\binom{n}{k+1}_q\frac{(-1)^{k+1}}{x^{k+1}}\;\Phi_{k+1}(q;x)\Phi_{k+2}(q;x)\\&\qquad-\sum_{k=0}^{n-1}\binom{n}{k+1}_q\frac{(-1)^{k+1}}{x^{k}}\;q^{k+1}\,\Phi_k(q;x)\Phi_{k+1}(q;x)\,,\end{aligned}$$

that is,

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}_q\frac{(-1)^k}{x^k}\;\Phi_k(q;x)^2&=\sum_{k=1}^{n}\binom{n}{k}_q\frac{(-1)^k}{x^k}\;\Phi_k(q;x)\Phi_{k+1}(q;x)\\&\quad+\sum_{k=0}^{n-1}\binom{n}{k+1}_q\frac{(-1)^k}{x^k}\;q^{k+1}\,\Phi_k(q;x)\Phi_{k+1}(q;x)\,,\end{aligned}$$

that is,

$$\sum_{k=0}^{n} \binom{n}{k}_q \frac{(-1)^k}{x^k}\, \Phi_k(q;x)^2 - \Phi_0(q;x)^2 =$$
$$= \sum_{k=0}^{n} \left[\binom{n}{k}_q + q^{k+1} \binom{n}{k+1}_q \right] \frac{(-1)^k}{x^k}\, \Phi_k(q;x)\Phi_{k+1}(q;x) - \Phi_0(q;x)\Phi_1(q;x)\,.$$

By the initial values $\Phi_0(q;x) = \Phi_1(q;x) = 1$ and by recurrence (3), we obtain the identity

$$\sum_{k=0}^{n} \binom{n}{k}_q \frac{(-1)^k}{x^k}\, \Phi_k(q;x)^2 = \sum_{k=0}^{n} \binom{n+1}{k+1}_q \frac{(-1)^k}{x^k}\, \Phi_k(q;x)\Phi_{k+1}(q;x)$$

which is equivalent to identity (42). □

Remark 12. As a special case of identity (42), we have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (2x)^{n-k} J_k(x)^2 = \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k (2x)^{n-k} J_k(x) J_{k+1}(x)\,. \qquad (43)$$

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