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WHEN THE SMALL DIVISORS OF A NATURAL NUMBER ARE IN ARITHMETIC PROGRESSION

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Abstract

We consider the positive divisors of a natural number n that do not exceed the square root of n. We refer to these as the *small divisors* of n. We find all natural numbers whose small divisors are in arithmetic progression.

1. Introduction

For a natural number n, let $\mathcal{S}_n$ denote the set of positive divisors of n that do not exceed its square root; that is,

$$\mathcal{S}_n = \{d : d \mid n, d \leq \sqrt{n}\}. \tag{1}$$

We may say that $\mathcal{S}_n$ is the set of *small divisors* of n. Observe that $\mathcal{S}_{60} = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{S}_{45} = \{1, 3, 5\}$, and $\mathcal{S}_{1729} = \{1, 7, 13, 19\}$. Thus, 60, 45, and 1729 are examples of natural numbers whose small divisors are in arithmetic progression. In this paper, we shall determine the set of *all* such natural numbers, in Theorem 4. That is, we shall determine all n such that

$$\mathcal{S}_n = \{1,\ a+1,\ 2a+1, \ldots, (k-1)a+1\} = \{ja+1\}_{j=0}^{k-1} \tag{2}$$

for some natural numbers k and a. Since $1 \in \mathcal{S}_n$, it is clear that all elements of $\mathcal{S}_n$ are congruent to 1 modulo a.

The phrase "small divisors," as used here, is not to be confused with classical small divisors problems of mathematical physics (see, e.g., Yoccoz [7]). Otherwise, the literature appears bereft of any mention of this phrase as applied specifically to the divisors of a natural number.

While divisors of integers (rational or Gaussian) either in, or avoiding, arithmetic progressions have been studied (e.g., Banks, Friedlander, and Luca [1], Varbanec and Zarzycki [6]), there seem to be no apparent results regarding *all* such divisors of an integer, say, up to a certain magnitude (or absolute value).

As usual, we define

$$\tau(n) = \sum_{d|n} 1\,;$$

i.e., $\tau(n)$ counts the natural divisors of n.

If $n = bc$ and $b \le c$, then $b \le \sqrt{n} \le c$; hence

$$\tau(n) = \begin{cases} 2|\mathcal{S}_n|, & \text{if } n \text{ is not a square;} \\ 2|\mathcal{S}_n| - 1, & \text{if } n \text{ is a square.} \end{cases} \tag{3}$$

Furthermore $\tau(n)$ is multiplicative. Thus, for the k distinct primes $p_1 < p_2 < \cdots < p_k$, and natural numbers $a_1, a_2, \ldots, a_k$,

$$\tau(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1). \tag{4}$$

2. Small Cases of $|\mathcal{S}_n|$

We begin by considering natural numbers n such that $|\mathcal{S}_n|$ does not exceed 6.

Lemma 1. *If $\mathcal{S}_n$ is given as in (2) and $k \le 6$ then one of the following is true:*

(i) $n = 1$, *hence* $\mathcal{S}_n = \{1\}$.

(ii) $n = p$ *for some prime* p, *hence* $\mathcal{S}_n = \{1\}$.

(iii) $n = p^2$ *for some prime* p, *hence* $\mathcal{S}_n = \{1, p\}$.

(iv) $n = p^3$ *for some prime* p, *hence* $\mathcal{S}_n = \{1, p\}$.

(v) $n = pq$ *for some primes* $p < q$, *hence* $\mathcal{S}_n = \{1, p\}$.

(vi) $n = p^2 q$ *for primes* p *and* $q = 2p - 1$, *hence* $\mathcal{S}_n = \{1, p, q\}$.

(vii) $n = pq^2$ *for primes* p *and* $q = 2p - 1$, *hence* $\mathcal{S}_n = \{1, p, q\}$.

(viii) $n = 24$, *hence* $\mathcal{S}_n = \{1, 2, 3, 4\}$.

(ix) $n = pqr$ *for primes* p, $q = 2p - 1$, $r = 3p - 2$, *hence* $\mathcal{S}_n = \{1, p, q, r\}$.

(x) $n = 60$, *hence* $\mathcal{S}_n = \{1, 2, 3, 4, 5, 6\}$.

Proof. We present the proof in the form of a list, with each item corresponding to each k, $1 \le k \le 6$. In each item of the proof for k, we itemize further with subordinate lists if necessitated by cases, or by subcases of the cases, and so forth.

$\boldsymbol{k = 1}$ **:** If $k = 1$, then by (3) we have $\tau(n) = 1$ (if n is a square) or $\tau(n) = 2$ (otherwise). Therefore by (4) $n = 1$ or n is a prime, hence $\mathcal{S}_n = \{1\}$. This corresponds to items *(i)* and *(ii)* of the statement of the lemma.

$k = 2$: Suppose $k = 2$. By (3), either $\tau(n) = 3$ or $\tau(n) = 4$.

$\tau(n) = 3$: In this case, by (4) $n = p^2$ for some prime p, hence $\mathcal{S}_n = \{1, p\}$, corresponding to case *(iii)* of the statement of the lemma.

$\tau(n) = 4$: Here, by (4), either $n = p^3$ or $n = pq$ for primes $p < q$. It is clear that $\mathcal{S}_n = \{1, p\}$ in each of these two subcases, which correspond respectively to items *(iv)* and *(v)* of the statement of the lemma.

$k = 3$: Suppose $k = 3$. By (3), either $\tau(n) = 5$ or $\tau(n) = 6$.

$\tau(n) = 5$: This case is impossible: by (4) necessarily $n = p^4$, but for all primes p, $\{1, p, p^2\}$ is not in arithmetic progression.

$\tau(n) = 6$: By (4), this case yields $n = p^2q$ or $n = pq^2$ for primes $p < q$. Clearly $p = a + 1$ in either subcase. The next smallest divisor of n is either p^2 or q. It cannot be p^2 (again 1, p, and p^2 are not in arithmetic progression), hence in either subcase $\mathcal{S}_n = \{1, p, q\}$. Since the elements of $\mathcal{S}_n$ are in arithmetic progression, it is necessary that $q = 2p - 1$. Thus these two subcases correspond respectively to items *(vi)* and *(vii)* of the statement of the lemma.

$k = 4$: Suppose $k = 4$. By (3), $\tau(n) = 7$ or $\tau(n) = 8$. However, n is not the power of a prime, because (again) for all primes p, it is clear that 1, p, and p^2 are not in arithmetic progression. Therefore by (4), $\tau(n) = 8$ and either $n = p^3q$, or $n = pqr$ for distinct primes p, q, and r.

$n = p^3q$: The case $n = p^3q$ admits two subcases, $p < q$ and $q < p$.

$p < q$: In this subcase, clearly $p = a + 1$ and $q = 2a + 1$ (as we cannot have $p^2 = 2a+1$), hence $p^2 = 3a+1$. This implies $p = 2$, thus $q = 3$, hence $n = 24$ and $\mathcal{S}_n = \{1, 2, 3, 4\}$. This corresponds to item *(viii)* of the statement of the lemma.

$q < p$: In this subcase, we have $q = a + 1$, $p = 2a + 1$, $pq = 3a + 1$, which is impossible.

$n = pqr$: In this case we may assume without loss of generality that $p < q < r$. Two subcases apply: either $r > pq$ or $r < pq$.

$r > pq$: This subcase is impossible because 1, p, q, and pq are not in arithmetic progression for all distinct primes p and q.

$r < pq$: This subcase yields $p = a+1$, $q = 2a+1$, $r = 3a+1$. Therefore $q = 2p - 1$ and $r = 3p - 2$, and we have $\mathcal{S}_n = \{1, p, q, r\}$. This corresponds to item *(ix)* of the statement of the lemma.

$k = 5$: Suppose $k = 5$. By (3), either $\tau(n) = 9$ or $\tau(n) = 10$. Again, n cannot be the power of a prime. Thus by (4) two cases occur, either $n = p^2q^2$, or $n = p^4q$, for distinct primes p and q.

$n = p^2q^2$: In this case we may assume $p < q$, and therefore $q < p^2$ (again, 1, p, and p^2 are not in arithmetic progression). Hence $p = a+1$, $q = 2a+1$, $p^2 = 3a+1$, and $pq = 4a+1$; clearly this is impossible.

$n = p^4q$: This case admits two subcases, $p < q$ and $q < p$.

$p < q$: As above, this subcase yields $p = a+1$, $q = 2a+1$, $p^2 = 3a+1$, and $pq = 4a+1$, which is impossible.

$q < p$: This subcase yields $q = a+1$, $p = 2a+1$, and $pq = 3a+1$, which is impossible.

$k = 6$: Finally, suppose $k = 6$. As before, by (3) and (4), we have three cases, $n = p^5q$, $n = p^3q^2$, or $n = p^2qr$, where p, q, and r are distinct primes.

$n = p^5q$: In this case, we have two subcases, $q < p$ and $p < q$.

$q < p$: This subcase yields $q = a+1$, $p = 2a+1$, $pq = 3a+1$, which is impossible.

$p < q$: In this subcase, we have (as before) $q < p^2$, yielding $p = a+1$, $q = 2a+1$, $p^2 = 3a+1$, $pq = 4a+1$, which is impossible.

$n = p^3q^2$: This case also admits two subcases, either $q < p$ or $p < q$.

$q < p$: In this subcase, we have $p < q^2$, hence $q = a+1$, $p = 2a+1$, $q^2 = 3a+1$, $pq = 4a+1$, which is impossible.

$p < q$: In this subcase we have $q < p^2$, hence $p = a+1$, $q = 2a+1$, $p^2 = 3a+1$, $pq = 4a+1$, which is impossible.

$n = p^2qr$: This leaves the third case, $n = p^2qr$. Without loss of generality, we have three subcases, either $p < q < r$, or $q < p < r$, or $q < r < p$.

$p < q < r$: In the subcase $p < q < r$, we have (as before) $q < p^2$. This admits two further possibilities, $r < p^2$ and $r > p^2$.

$r < p^2$: Here, we have $p = a+1$, $q = 2a+1$, $r = 3a+1$, $p^2 = 4a+1$, and $pq = 5a+1$, which is impossible.

$r > p^2$: This possibility yields $p = a+1$, $q = 2a+1$, $p^2 = 3a+1$, which implies $a = 1$. Hence $p = 2$ and $q = 3$. Thus $pq = 6 = 5a+1$, hence $r = 4a+1 = 5$; this implies $n = 60$, hence $\mathcal{S}_n = \{1, 2, 3, 4, 5, 6\}$. This corresponds to item *(x)* of the statement of the lemma.

$q < p < r$: In this subcase, we have two further possibilities, $r > pq$ or $r < pq$.

$r > pq$: Here, $q = a+1$, $p = 2a+1$, $pq = 3a+1$, which is impossible.

$r < pq$: This possibility yields $p = a+1$, $q = 2a+1$, $r = 3a+1$, $pq = 4a+1$, which is impossible.

$\boldsymbol{q < r < p:}$ In this third and final subcase, we have $p < qr$ (as q, r, and qr are not in arithmetic progression). Thus $q = a + 1$, $r = 2a + 1$, $p = 3a + 1$, and $qr = 4a + 1$, which is impossible.

□

3. Small Cases of the Common Difference a

The next step is to show that it is impossible to have $|\mathcal{S}_n| > 6$ for all n as given in (2). We do this first for the two smallest values of a.

Lemma 2. *If $\mathcal{S}_n$ is given as in (2) with $a = 1$ or $a = 2$, then $k \le 6$.*

Proof. We use *reductio ad absurdum*, hence we assume $k \ge 7$. First suppose $a = 1$. Then $\mathcal{S}_n = \{1, 2, \ldots, k\}$ for some $k \ge 7$. Let

$$M = [1,\, 2,\, \ldots,\, k];$$

that is, M denotes the least common multiple of the natural numbers up to k. Note that $M \mid n$. Since $k \ge 7$ we have $420 \mid M$, hence $420 \mid n$. Since $20 \cdot 21 = 420$, we have $20 \mid n$, $20 < \sqrt{M} \le \sqrt{n}$, hence $20 \in \mathcal{S}_n$. Therefore $k \ge 20$; in particular, $19 \in \mathcal{S}_n$.

If we enumerate the sequence of prime numbers so that $p_1 = 2$, $p_2 = 3$, and so forth, then

$$M = p_1^{a_1} p_2^{a_2} \cdots p_j^{a_j} \tag{5}$$

for some $j \ge 1$ and natural numbers a_i, $1 \le i \le j$. Since $19 \in \mathcal{S}_n$, we have $19 \mid M$ and hence $j \ge 8$.

Note that for each i, $1 \le i \le j$, we have

$$a_i = \max\{\nu : p_i^{\nu} \le k\} = [\log_{p_i} k] > \log_{p_i} k - 1,$$

where, as usual, $[x]$ denotes the integer part of the real number x. Thus by (5)

$$M > \frac{k}{p_1} \cdot \frac{k}{p_2} \cdots \frac{k}{p_j} = \frac{k^j}{p_1 p_2 \cdots p_j} \ge \frac{k^j}{M},$$

hence $k^j < M^2$. Therefore

$$\sqrt{n} \ge \sqrt{M} > k^{j/4} \ge k^2 > k(k-1).$$

However, $k(k-1) = [k-1, k]$, which divides M, hence $k(k-1) \mid n$, hence $k(k-1) \in \mathcal{S}_n$. This contradicts the maximality of k in $\mathcal{S}_n$. This contradiction shows that $a \neq 1$.

The proof that $a \neq 2$ is identical to that of $a \neq 1$. Here, we have $\mathcal{S}_n = \{1, 3, 5, \ldots, 2k-1\}$ for some $k \geq 7$, and we let

$$M = [1, 3, 5, \ldots, 2k-1].$$

Therefore $3465 \mid M$, hence $55 \mid M$ and $55 < \sqrt{M}$. Since $M \mid n$, we have $55 \mid n$, $55 < \sqrt{n}$, hence $53 \in \mathcal{S}_n$. Enumerating the odd primes as $q_1 = 3$, $q_2 = 5$, etc., we have $M = q_1^{b_1} q_2^{b_2} \cdots q_j^{b_j}$, where $j \geq 15$ (as $53 = q_{15}$). Then,

$$b_i = \max\{\nu : q_i^{\nu} \leq 2k-1\} = [\log_{q_i}(2k-1)] > \log_{q_i}(2k-1) - 1,$$

hence

$$M > \frac{2k-1}{q_1} \cdot \frac{2k-1}{q_2} \cdots \frac{2k-1}{q_j} = \frac{(2k-1)^j}{q_1 q_2 \cdots q_j} \geq \frac{(2k-1)^j}{M},$$

hence $(2k-1)^j < M^2$. As $j \geq 15$, we have $\sqrt{n} \geq \sqrt{M} > (2k-1)^{15/4} > (2k-3)(2k-1)$. Now, $(2k-3)(2k-1) = [2k-3, 2k-1]$, which divides M, hence $(2k-3)(2k-1) \in \mathcal{S}_n$; this contradicts the maximality of $2k-1$ in $\mathcal{S}_n$. □

4. The Proof of the Theorem Completed

It remains to show that Lemma 1 gives *all* natural numbers whose small divisors are in arithmetic progression.

Lemma 3. *If n is given as in* (2) *with $a > 2$ and $k \geq 7$ then $k \geq a + 3$.*

Proof. Suppose otherwise. That is, $a > 2$ and $k \geq 7$, but $k \leq a + 2$. We claim that $ja + 1$ is prime for all j, $1 \leq j \leq k-1$. Otherwise, if $ja + 1$ is composite, then its nontrivial proper divisors all belong to $\mathcal{S}_n$. Hence there exist b, c, such that $1 \leq b < j$, $1 \leq c < j$, and

$$(k-1)a + 1 \geq ja + 1 = (ba+1)(ca+1) \geq (a+1)^2 = (a+2)a + 1 > (k-1)a + 1,$$

a contradiction. Therefore n is divisible by at least $k-1$ distinct primes, hence by (4) $\tau(n) \geq 2^{k-1}$, hence by (3) $k = |\mathcal{S}_n| \geq 2^{k-2}$. Clearly this is false as $k \geq 7$. □

Theorem 4. *If n is given as in* (2), *then n is one of the values given in the statement of Lemma 1. That is, Lemma 1 lists all natural numbers n such that the elements of $\mathcal{S}_n$ are in arithmetic progression.*

Proof. By Lemmata 1 and 2, we may assume $k \geq 7$ and $a > 2$. Clearly a is even, hence $a \geq 4$. By Bertrand's postulate, there exists a prime q such that $a/2 < q < a$. Note that $q \notin \mathcal{S}_n$, as $1 < q < a + 1$.

Let

$$Q = \{a+1,\, 2a+1,\, 3a+1, \ldots,\, qa+1\}.$$

Since $q < a$, it follows from Lemma 3 that $Q \subset \mathcal{S}_n$. Since $q \nmid a$, it follows that no two elements of this set are congruent modulo q. Hence there exists j such that $1 \le j \le q$ and $q \mid ja + 1$. Thus $q \mid n$ and $q < \sqrt{n}$, hence $q \in \mathcal{S}_n$. This, of course, contradicts the fact that $q \notin \mathcal{S}_n$, thus completing the proof. $\square$

5. Remarks

Cases *(i)–(v)* of Lemma 1 are trivial because any set of either one or two natural numbers has its elements in arithmetic progression. Clearly cases *(ii)–(v)* occur infinitely often. We can count how often they occur up to a bound $x > 0$ by means of the prime counting function $\pi(x)$, which gives the number of primes p such that $p \le x$. For example, the number of $n \le x$, either of form p, p^2, or p^3 for a prime p, is

$$\pi(x) + \pi\left(x^{\frac{1}{2}}\right) + \pi\left(x^{\frac{1}{3}}\right). \tag{6}$$

The number of $n \le x$ of the form pq for primes $p < q$ is

$$\sum_{p<\sqrt{x}} \left(\pi\left(\frac{x}{p}\right) - \pi(p)\right). \tag{7}$$

This is because $p < \sqrt{x}$ for all $n \le x$ such that $n = pq$ (as $p < q$). Thus for each fixed such p, we have $p < q \le x/p$.

We may obtain asymptotic formulas for these values. Recall that two functions $f(x)$ and $g(x)$ are said to be *asymptotic* if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

In this case we write $f(x) \sim g(x)$. The prime number theorem [3, Thm. 6] says

$$\pi(x) \sim \frac{x}{\ln x}.$$

Thus it follows that

$$\pi(x^{1/2}) \sim \frac{2x^{1/2}}{\ln x}, \qquad \pi(x^{1/3}) \sim \frac{3x^{1/3}}{\ln x}.$$

Landau [4, § 56] applied the prime number theorem to show

$$\sum_{p<\sqrt{x}} \left(\pi\left(\frac{x}{p}\right) - \pi(p)\right) \sim \frac{x \ln\ln x}{\ln x},$$

and hence by (6) and (7) we have the following: if $f(x)$ denotes the number of $n \le x$ of either of the forms given in cases *(i)–(v)* of Lemma 1, then

$$f(x) \sim \frac{x \ln\ln x}{\ln x}. \tag{8}$$

The cases *(vi)–(vii)*, as well as case *(ix)*, are a different matter. It is unknown if $2p-1$ is prime infinitely often if p runs through all the primes. *A fortiori* it is also unknown if the pair $2p-1$ and $3p-2$ simultaneously assume prime values infinitely often as p runs through all the primes. These two questions, as that of the twin prime problem, are special cases of Dickson's conjecture [2]. This says that if m is a natural number, and if $\{a_k\}_{k=1}^m$ and $\{b_k\}_{k=1}^m$ are natural numbers such that $(a_k, b_k) = 1$ for $1 \le k \le m$, then the linear forms

$$a_1 n + b_1, \quad a_2 n + b_2, \quad \ldots, \quad a_m n + b_m$$

simultaneously assume prime values for infinitely many natural numbers n.

At any rate, the number of natural numbers $n \le x$ such that $n = p^2 q$ or $n = pq^2$ (where p and $q = 2p-1$ are primes) is certainly bounded above by

$$\pi\left(x^{1/3}\right) + \pi\left(x^{1/3}\right) \sim \frac{6x^{1/3}}{\ln x},$$

as is the number of $n \le x$ such that $n = pqr$ (where p, $q = 2p-1$, and $r = 3p-2$ are primes) bounded above by

$$\pi\left(x^{1/3}\right) \sim \frac{3x^{1/3}}{\ln x}.$$

Hence, in light of (8), if $g(x)$ denotes the number of $n \le x$ of *any* form given in Lemma 1, then

$$g(x) \sim \frac{x \ln\ln x}{\ln x}.$$

There are numbers, e.g., 6 and 12, not only with all small divisors in arithmetic progression, but with an additional such divisor as well. Thus the elements of $\{1, 2, 3\}$ are divisors of 6 and those of $\{1, 2, 3, 4\}$ are divisors of 12. Are there any more such numbers? To answer this question, it suffices to go through cases *(i)–(x)* in Lemma 1, and see where the next larger divisor beyond those in $\mathcal{S}_n$ lies.

Case *(i)* is vacuous. Case *(ii)* is trivial since *all* divisors of a prime p are in arithmetic progression (as there are only two of them). Cases *(iii)* and *(iv)* certainly do not produce such numbers as 1, p and p^2 are not in arithmetic progression for all primes p. In case *(v)*, $n = pq$ for primes $p < q$. The three divisors 1, p, and q are in arithmetic progression if and only if $q = 2p - 1$. The next (and final) divisor in the sequence of divisors, namely $pq = 2p^2 - p$, is easily seen not to be in arithmetic progression after 1, p, and q. Hence we have as further examples n being 6, 15, 91, 703, and so on, where $|\mathcal{S}_n| = 2$, yet the *three* smallest divisors of n are in arithmetic progression.

Case *(vi)* admits the special case $n = 12$, mentioned above, but this is the only such number in this case. For, as $q = 2p - 1$, the fourth smallest divisor of n is p^2, which follows in arithmetic progression only if $p = 2$ (hence $q = 3$ and $n = 12$).

Case *(vii)* admits no possibilities, since here the fourth smallest divisor of n is pq, and this does not follow the three divisors 1, p, q, in arithmetic progression for all primes p and q such that $q = 2p - 1$.

It is clear that $n = 24$ and $n = 60$ do not have the desired property, hence we are left with case *(ix)*. Here $\mathcal{S}_n = \{1, p, q, r\}$ with $q = 2p-1$ and $r = 3p-2$ (p, q, and r of course being prime). Hence the fifth smallest divisor is $pq = 2p^2 - p$, and this is easily seen not to be in arithmetic progression after the four smallest divisors.

Thus, aside from the trivial cases (1 and the primes), the only numbers with its smallest divisors, *beyond* merely the small divisors, in arithmetic progression are 12, and pq where p and q are primes such that $q = 2p - 1$. These numbers pq form the sequence A129521 in the OEIS (Sloane [5]).

We end with a final remark regarding case *(ix)* of Lemma 1, which is particularly interesting when a is divisible by 6. For, in this case, writing $a = 6t$, we have $p - 1 = 6t$, $q - 1 = 12t$, $r - 1 = 18t$, and

$$pqr - 1 = 6t(216t^2 + 66t + 6) = 12t(108t^2 + 33t + 3) = 18t(72t^2 + 22t + 2).$$

Thus, letting $n = pqr$, we obtain, for a natural number k divisible neither by p, q, nor r, the system of congruences

$$\begin{aligned} k^{n-1} &\equiv k^{(p-1)(216t^2+66t+6)} \equiv 1 \pmod{p}, \\ k^{n-1} &\equiv k^{(q-1)(108t^2+33t+3)} \equiv 1 \pmod{q}, \\ k^{n-1} &\equiv k^{(r-1)(72t^2+22t+2)} \equiv 1 \pmod{r}. \end{aligned}$$

The Chinese remainder theorem implies $k^{n-1} \equiv 1 \pmod{n}$. Therefore n is a Carmichael number. These numbers form sequence A033502 in the OEIS (Sloane [5]). The smallest member of this sequence is $1729 = 7 \cdot 13 \cdot 19$, Ramanujan's famous taxi cab number.

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