ALMOST GAP BALANCING NUMBERS

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Abstract
Almost balancing numbers were introduced by Panda and Panda as a certain generalization of balancing numbers. In this paper, we extend this notion to gap balancing numbers. Additionally, we establish a balancer duality theorem which generalizes the relationship observed between balancing and cobalancing numbers by Panda and Ray.

1. Introduction
Let \( \nu \) and \( w \) be integers with \( \nu \geq 0 \). A positive integer \( B \) is called an almost gap balancing number with gap \( \nu \) and weight \( w \), or an \( A(\nu, w) \)-balancing number, if \( B \geq \nu \) and

\[
1 + 2 + 3 + \cdots + (B - \nu) + w = (B + 1) + \cdots + (B + r)
\]

for some integer \( r \geq 0 \). Panda and Panda [5] introduced almost balancing numbers as a generalization of balancing numbers [4, A001109] and cobalancing numbers [4, A053141] when studying (1) with \( \nu = 1 \) and \( w = \pm 1 \). We extend their results to \( A(\nu, w) \)-balancing numbers. Additionally, we establish a balancer duality theorem which generalizes the relationship observed between balancing and cobalancing numbers by Panda and Ray [7].
2. $A(\nu, w)$-balancing Numbers

In this section, we define almost gap balancing numbers and give several examples.

Definition 1. Let $\nu$ and $w$ be integers with $\nu \geq 0$. Define an integer $B$ to be an almost gap balancing number with gap $\nu$ and weight $w$, or an $A(\nu, w)$-balancing number, if $B \geq \nu$ and

$$1 + 2 + 3 + \cdots + (B - \nu) + w = (B + 1) + \cdots + (B + r)$$

for some integer $r \geq 0$. We refer to $r$ as the $A(\nu, w)$-balancer corresponding to the $A(\nu, w)$-balancing number $B$.

It follows from (1) that $B$ is an $A(\nu, w)$-balancing number if and only if

$$T(B - \nu) + T(B) + w = T(B + r)$$

(2)

where $T(n) = \frac{n(n+1)}{2}$ is the $n$th triangular number. Solving (2) for $r$ gives

$$r = \frac{-(2B + 1) + \sqrt{8B^2 + 8(1 - \nu)B + (2\nu - 1)^2 + 8w}}{2}$$

where we take the positive square root so that $r \geq 0$. Thus $B$ is an $A(\nu, w)$-balancing number if and only if $8B^2 + 8(1 - \nu)B + (2\nu - 1)^2 + 8w$ is a perfect square. The latter expression occurs frequently so we make the following definition.

Definition 2. Let $B$ be an $A(\nu, w)$-balancing number. Define its corresponding $A(\nu, w)$-Lucas balancing number to be

$$C = \sqrt{8B^2 + 8(1 - \nu)B + (2\nu - 1)^2 + 8w}.$$ 

For brevity we say the pair $(B, C)$ is an $A(\nu, w)$-balancing pair.

Definition 2 implies that the $A(\nu, w)$-balancing pair $(B, C)$ is a solution to the Pell-like equation

$$y^2 = 8x^2 + 8(1 - \nu)x + (2\nu - 1)^2 + 8w.$$  

(3)

Equation (3) can be rewritten as

$$y^2 - 2z^2 = 2\nu^2 + 8w - 1$$

(4)

where $z = 2x + 1 - \nu$.

Definition 3. Given integers $\nu$ and $w$ with $\nu \geq 0$, we refer to the Pell-like equation

$$y^2 - 2z^2 = 2\nu^2 + 8w - 1$$

as the $A(\nu, w)$-companion equation. For later convenience let $N(\nu, w) = 2\nu^2 + 8w - 1$. 

In particular, an integral solution \((y, z)\) to the \(A(\nu, w)\)-companion equation corresponds to an \(A(\nu, w)\)-balancing pair \((B, C)\) where \(B = \frac{z + \nu - 1}{2}\) and \(C = y\) provided \(y > 0\), \(z \geq \nu + 1\), and \(z \equiv 1 - \nu \pmod{2}\).

**Example 1.** Since \(T(6) + T(6) + 3 = T(9)\) and \(T(38) + T(38) + 3 = T(54)\), the numbers 6 and 38 are \(A(0, 3)\)-balancing numbers with balancers 3 and 16, respectively. The corresponding \(A(0, 3)\)-Lucas balancing numbers are 19 and 109. Similarly, 3 and 16 are \(A(1, -3)\)-balancing numbers with balancers 0 and 6 since \(T(2) + T(3) - 3 = T(3)\) and \(T(15) + T(16) - 3 = T(22)\). Their \(A(1, -3)\)-Lucas balancing numbers are 7 and 45. Additional examples of \(A(1, -3)\)- and \(A(0, 3)\)-balancing numbers are given in Tables 1 and 2, respectively.

**Example 2.** The \(A(\nu, 0)\)-balancing numbers are the upper gap balancing numbers [1]. In particular, the \(A(0, 0)\) and \(A(1, 0)\)-balancing numbers are cobalancing [7] and balancing numbers [2], respectively. The \(A(1, 1)\) and \(A(1, -1)\)-balancing numbers are the almost balancing numbers of the first and second kind, respectively, studied by Panda and Panda [5]. Lastly, the \(A(1, -k^2)\)-balancing numbers are the \(k\)-circular balancing numbers [6].

**Example 3.** For \(\nu \geq 0\), a class of \(A(\nu, -1)\)-balancing numbers is generated from the seed \((1, 2\nu - 3)\). The initial \(A(1, -1)\)-balancing pairs are \((1, 1)\), \((4, 11)\), and \((23, 65)\). For \(\nu \neq 1\), the initial \(A(\nu, -1)\)-balancing pairs are \((\nu + 1, 2\nu + 3)\), \((4\nu + 7, 10\nu + 21)\), and \((21\nu + 43, 58\nu + 123)\).

The last example is a special case of the following result.

**Theorem 1.** Let \(k \geq 0\) be an integer and \(w = -T(k)\). Then there exists \(A(\nu, w)\)-balancing numbers for every \(\nu \geq 0\).

**Proof.** Observe \(B = \nu + k\) is an \(A(\nu, w)\)-balancing number with balancer \(r = 0\).

**Remark 1.** The existence of \(A(\nu, w)\)-balancing numbers is not always guaranteed. More specifically, \(A(\nu, w)\)-balancing numbers exist if and only if the \(A(\nu, w)\)-companion equation \(y^2 - 2z^2 = N(\nu, w)\) has integral solutions. From Pell equation theory [3, pp. 205–207] the latter can be determined by searching for solutions in a certain finite interval. For example, there do not exist \(A(1, 2)\)-balancing numbers since its companion equation, \(y^2 - 2z^2 = 15\), does not have any integral solutions. In fact, this argument proves that there are no \(A(\nu, w)\)-balancing numbers whenever \(N(\nu, w) = 15\).

### 3. Functions Involving \(A(\nu, w)\)-balancing Numbers

We present several functions which generate \(A(\nu, w)\)-balancing numbers from known \(A(\nu, w)\)-balancing numbers. These functions are derived using \(A(\nu, w)\)-companion
equations and linear substitutions.

3.1. Functions Generating $A(\nu, w)$-balancing Numbers

From the theory of Pell equations, integral solutions to (4), if they exist, occur in a finite number of cyclic classes. That is, if $(y_i, z_i)$ is a solution of the $A(\nu, w)$-companion equation corresponding to an $A(\nu, w)$-balancing pair, then so is $(y_{i+1}, z_{i+1})$ where

$$y_{i+1} + z_{i+1} \sqrt{2} = (3 + 2\sqrt{2})(y_i + z_i \sqrt{2})$$

or equivalently in matrix form

$$V : \begin{bmatrix} y_{i+1} \\ z_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_i \\ z_i \end{bmatrix}. \tag{5}$$

To see this, suppose $(y_i, z_i)$ satisfies $y_i > 0$, $z_i \geq \nu + 1$, and $z_i \equiv 1 - \nu \pmod{2}$. Clearly $y_{i+1} > 0$ and $z_{i+1} \geq \nu + 1$. Lastly $z_{i+1} = z_i = 1 - \nu \pmod{2}$ which establishes the claim. Throughout this paper we freely use functional notation $V(y, z)$ for the map given in (5) and use similar notation for other such maps.

The relations $y_i = C_i$ and $z_i = 2B_i + 1 - \nu$ can be expressed as the affine transformations

$$S_{\nu, w} : \begin{bmatrix} y_i \\ z_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} B_i \\ C_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - \nu \end{bmatrix}, \tag{6}$$

and

$$S_{\nu, w}^{-1} : \begin{bmatrix} B_i \\ C_i \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_i \\ z_i \end{bmatrix} + \begin{bmatrix} \nu - 1/2 \\ 0 \end{bmatrix}. \tag{7}$$

Using (6) and (7), equation (5) can be expressed in terms of $A(\nu, w)$-balancing pairs as

$$J_{\nu, w} = S_{\nu, w}^{-1}V S_{\nu, w} : \begin{bmatrix} B_{i+1} \\ C_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} B_i \\ C_i \end{bmatrix} + \begin{bmatrix} 1 - \nu \\ 4 - 4\nu \end{bmatrix}. \tag{8}$$

Noting $(3 + 2\sqrt{2})^{-1} = 3 - 2\sqrt{2}$ we similarly see

$$y_{i-1} + z_{i-1} \sqrt{2} = (3 - 2\sqrt{2})(y_i + z_i \sqrt{2})$$

and

$$J_{\nu, w}^{-1} = S_{\nu, w}^{-1}V^{-1} S_{\nu, w} : \begin{bmatrix} B_{i-1} \\ C_{i-1} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} B_i \\ C_i \end{bmatrix} + \begin{bmatrix} 1 - \nu \\ 4\nu - 4 \end{bmatrix}. \tag{9}$$

We summarize the observations above in the following result.

**Theorem 2.** If $(x, y)$ is an $A(\nu, w)$-balancing pair, then so is $J_{\nu, w}(x, y)$ where

$$J_{\nu, w} : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 - \nu \\ 4 - 4\nu \end{bmatrix}. \tag{10}$$
Recall $y = \sqrt{8x^2 + 8(1-\nu)x + (2\nu - 1)^2 + 8u}$. Since the components of $J_{\nu,w}$ viewed as functions of $x$ are strictly increasing on $[\nu, \infty)$, their inverses exist and

$$J^{-1}_{\nu,w}[x \ y] = \begin{bmatrix} 3 & -1 \\ -8 & 3 \end{bmatrix} [x \ y] + \begin{bmatrix} 1 - \nu \\ 4\nu - 4 \end{bmatrix}.$$ 

Given an $A(\nu, w)$-balancing pair $(x_0, y_0)$ the functions $J_{\nu,w}$ and $J^{-1}_{\nu,w}$ can be applied iteratively to form a class of solutions $((x_i, y_i))_{i \in \mathbb{Z}}$ to (3) via $(x_{i+1}, y_{i+1}) = J_{\nu,w}(x_i, y_i)$. It is always possible to reindex so that the minimal $A(\nu, w)$-balancing pair corresponds to $(x_0, y_0)$. Then $((x_i, y_i))_{i \in \mathbb{Z}}$ is a class of solutions to (3) such that the nonnegative indexed terms correspond to $A(\nu, w)$-balancing pairs. We tacitly assume that a class of $A(\nu, w)$-balancing pairs are indexed in this manner unless stated otherwise.

**Example 4.** There are two classes of $A(1, -3)$-balancing pairs whose initial terms are $(3, 7)$ and $(9, 25)$, respectively. The initial $A(1, -3)$-balancing numbers and associated sequences are given in Table 1. There are also two classes of $A(0, 3)$-balancing pairs and their initial terms are $(0, 5)$ and $(3, 11)$, respectively. The initial $A(0, 3)$-balancing numbers and associated sequences are given in Table 2.

The next two theorems are generalizations of results of upper gap balancing numbers [1] extended to $A(\nu, w)$-balancing numbers using Theorem 2. The proofs are straightforward modifications and omitted here.

**Theorem 3.** Suppose $((B_i, C_i))_{i \geq 0}$ is a class of $A(\nu, w)$-balancing pairs. Then for $i \geq 1$

$$B_{i+1} = 6B_i - B_{i-1} + 2 - 2\nu$$

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>9</td>
<td>16</td>
<td>52</td>
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<td>303</td>
<td>542</td>
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<td>25</td>
<td>45</td>
<td>147</td>
<td>263</td>
<td>857</td>
<td>1533</td>
<td>4995</td>
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<tr>
<td>$r$</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>21</td>
<td>38</td>
<td>125</td>
<td>224</td>
<td>731</td>
</tr>
<tr>
<td>$\tilde{r}$</td>
<td>5</td>
<td>11</td>
<td>19</td>
<td>61</td>
<td>109</td>
<td>355</td>
<td>635</td>
<td>2069</td>
</tr>
<tr>
<td>$m$</td>
<td>3</td>
<td>12</td>
<td>22</td>
<td>73</td>
<td>131</td>
<td>428</td>
<td>766</td>
<td>2497</td>
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Table 1: Initial $A(1, -3)$-balancing numbers and associated sequences.

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
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<td>0</td>
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<td>$r$</td>
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</tr>
<tr>
<td>$\tilde{r}$</td>
<td>3</td>
<td>3</td>
<td>7</td>
<td>25</td>
<td>45</td>
<td>147</td>
<td>263</td>
<td>857</td>
</tr>
<tr>
<td>$m$</td>
<td>2</td>
<td>5</td>
<td>9</td>
<td>30</td>
<td>54</td>
<td>177</td>
<td>317</td>
<td>1034</td>
</tr>
</tbody>
</table>

Table 2: Initial $A(0, 3)$-balancing numbers and associated sequences.
and
\[ C_{i+1} = 6C_i - C_{i-1}. \]

**Theorem 4.** Let \(((B_i, C_i))_{i \geq 0}\) be a class of \(A(\nu, w)\)-balancing pairs. The generating function for its \(A(\nu, w)\)-balancing numbers is
\[
G(s) = \frac{(2 - 2\nu - B_1 + 6B_0)s^2 + (B_1 - 7B_0)s + B_0}{(1 - s)(1 - 6s + s^2)}.
\]

**Example 5.** There are two classes of \(A(0, 3)\)-balancing numbers whose generating functions are
\[
G_1(s) = \frac{-4s^2 + 6s}{(1 - s)(1 - 6s + s^2)} \quad \text{and} \quad G_2(s) = \frac{-s^2 + 3}{(1 - s)(1 - 6s + s^2)}.
\]

These can be combined to obtain the generating function for all \(A(0, 3)\)-balancing numbers, namely
\[
G(s) = \frac{s^4 + 3s^3 - 3s^2 - 3s}{(s - 1)(s^2 - 2s - 1)(s^2 + 2s - 1)}.
\]

### 3.2. Transition Functions

In this section, we present functions which map \(A(\nu, w)\)-balancing numbers to \(A(\nu', w')\)-balancing numbers.

**Lemma 1.** Let \(N\) and \(N'\) be integers. Suppose \((y_0, z_0)\) and \((y'_0, z'_0)\) are solutions to the equations \(y^2 - 2z^2 = N\) and \(y'^2 - 2z'^2 = N'\), respectively. Let
\[
H = \frac{1}{N} \begin{bmatrix} y_0 & 2z_0 \\ -z_0 & -y_0 \end{bmatrix} \begin{bmatrix} y'_0 & 2z'_0 \\ -z'_0 & -y'_0 \end{bmatrix}.
\]
Then
\[
\begin{bmatrix} y'_0 \\ z'_0 \end{bmatrix} = H \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}.
\]

**Proof.** Since \(y_0^2 - 2z_0^2 = N\) we see
\[
H \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \frac{1}{N} \begin{bmatrix} y'_0(y_0^2 - 2z_0^2) \\ z'_0(y_0^2 - 2z_0^2) \end{bmatrix} = \begin{bmatrix} y'_0 \\ z'_0 \end{bmatrix}.
\]

**Lemma 2.** Let \(H\) be as defined in Lemma 1. Then \(HV = VH\).

**Proof.** Suppress the naughts in \(H\) and observe
\[
HV = \frac{1}{N} \begin{bmatrix} 3yy' - 6zz' + 4yz' - 4y'z \\ 2yy' + 3yz' - 3zy' - 4z'z \end{bmatrix} = VH.
\]
Using Lemmas 1 and 2 we construct a function $T$ which maps a given class of $A(\nu, w)$-balancing numbers $((B_i, C_i))_{i \geq 0}$ to a given class of $A(\nu', w')$-balancing numbers $((B'_i, C'_i))_{i \geq 0}$. In terms of the notation used above define

$$T := S^{-1}_{\nu', w'} HS_{\nu, w}$$

as the composition of affine transformations where $(y_0, z_0)$ and $(y'_0, z'_0)$ used in $H$ are solutions of the companion equations corresponding to the balancing pairs $(B_0, C_0)$ and $(B'_0, C'_0)$, respectively.

Observe that $T$ is an affine transformation and $(B_0, C_0) \mapsto (B'_0, C'_0)$ by Lemma 1 and construction. To see that $(B_i, C_i) \mapsto (B'_i, C'_i)$ for all $i$, observe using (8) and Lemma 2 that

$$J_{\nu', \nu} T = S^{-1}_{\nu', w'} V S_{\nu', w} S^{-1}_{\nu', w'} H S_{\nu, w} = S^{-1}_{\nu', w'} V H S_{\nu, w} = S^{-1}_{\nu', w'} H S_{\nu, w} S^{-1}_{\nu, w} V S_{\nu, w} = T J_{\nu, w}.$$ 

It follows that $J_{\nu', \nu}^i T = T J_{\nu, w}^i$, which implies $(B_i, C_i) \mapsto (B'_i, C'_i)$ for all $i$.

It is straightforward to give the following explicit presentation of $T$.

**Theorem 5.** Let $((B_i, C_i))_{i \geq 0}$ and $((B'_i, C'_i))_{i \geq 0}$ be classes of $A(\nu, w)$ and $A(\nu', w')$-balancing pairs, respectively. Then the transition function $T$ given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ 8b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ (4 - 4\nu)b \end{bmatrix}$$

where

$$a = -\frac{8B_0B'_0 + 4(1 - \nu')B_0 + 4(1 - \nu)B'_0 + 2(1 - \nu)(1 - \nu') - C_0C'_0}{2\nu^2 - 1 + 8w}$$

$$b = \frac{2(C_0B'_0 - B_0C'_0) + (1 - \nu')C_0 - (1 - \nu)C'_0}{2(2\nu^2 - 1 + 8w)}$$

$$c = \frac{(1 - \nu) [C_0C'_0 - 8B_0B'_0 - 4(1 - \nu)B'_0] + (1 - \nu') [8B'_0 + 4(1 - \nu)B_0 - C'_0]}{2(2\nu^2 - 1 + 8w)}$$

maps $(B_i, C_i)$ to $(B'_i, C'_i)$ for all $i$.

**Example 6.** Consider the transition function from the $A(1, 0)$-balancing pairs $((B_i, C_i))_{i \geq 0}$ to the $A(0, 0)$-balancing pairs $((B'_i, C'_i))_{i \geq 0}$. Recall $(B_0, C_0) = (1, 3)$
and \((B_0', C_0') = (0, 1)\). Then \((y_0, z_0) = (3, 2), \ (y_0', z_0') = (1, 1)\),

\[
H = \begin{bmatrix}
-1 & 2 \\
1 & -1
\end{bmatrix}
\]

and

\[
T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix}
-1 & 1/4 \\
1/2 & -1
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0
\end{bmatrix}.
\]

Observe \(H^2 = V^{-1}\). This plays a key role in the balancer duality result in section 5.

4. Balancers, Related Sequences, and Their Functions

In this section we investigate the sequence of \(A(\nu, w)\)-balancers associated to \(A(\nu, w)\)-balancing numbers and related sequences. Recall that an integer \(r \geq 0\) is a \(A(\nu, w)\)-balancer of the \(A(\nu, w)\)-balancing number \(B\) provided it satisfies \(T(B - \nu) + T(B) + w = T(B + r)\). Solving for \(B\) gives

\[
B = \frac{(2r + 2\nu - 1) + \sqrt{8r^2 + 8\nu r - 8w + 1}}{2}
\]

where we take the positive square root so that \(B \geq 0\). Thus \(r\) is an \(A(\nu, w)\)-balancer if and only if \(8r^2 + 8\nu r - 8w + 1\) is a perfect square. Due to the latter expression we make the following definition.

**Definition 4.** Let \(r\) be an \(A(\nu, w)\)-balancer. Define its \(A(\nu, w)\)-Lucas balancer to be

\[
\hat{r} = \sqrt{8r^2 + 8\nu r - 8w + 1}.
\]

We refer to \((r, \hat{r})\) as an \(A(\nu, w)\)-balancer pair.

Equation (9) can be reformulated as \(\hat{r} = 2B - 2r - 2\nu + 1\). Definition 4 implies that the \(A(\nu, w)\)-balancer pair \((r, \hat{r})\) is a solution to the equation

\[
y^2 = 8x^2 + 8\nu x - 8w + 1.
\]

(10)

Alternatively (10) can be rewritten as

\[
y^2 - 2z^2 = -N(\nu, w)
\]

(11)

where \(y = \hat{r}\) and \(z = 2r + \nu\). We refer to (11) as the \(A(\nu, w)\)-balancer companion equation. An integral solution of (11) corresponds to an \(A(\nu, w)\)-balancer pair \((r, \hat{r})\) where \(r = \frac{z - \nu}{2}\) and \(\hat{r} = y\) provided \(z \equiv \nu \pmod{2}\), \(z \geq \nu\), and \(y > 0\).
Again applying the theory of Pell equations, solutions to (11), if they exist, occur in a finite number of cyclic classes. Moreover, if \((y_i, z_i)\) is a solution of the \(A(\nu, w)\)-balancer companion equation corresponding to an \(A(\nu, w)\)-balancer pair, then it is straightforward using techniques as before to see that \((y_{i+1}, z_{i+1}) = V(y_i, z_i)\) is also a solution.

The relations \(y_i = \hat{r}_i\) and \(z_i = 2r_i + \nu\) can be expressed as the affine transformations

\[
M_{\nu, w} : \begin{bmatrix} y_i \\ z_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} r_i \\ \hat{r}_i \end{bmatrix} + \begin{bmatrix} 0 \\ \nu \end{bmatrix}
\]

and

\[
M_{\nu, w}^{-1} : \begin{bmatrix} r_i \\ \hat{r}_i \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_i \\ z_i \end{bmatrix} + \begin{bmatrix} \frac{-\nu}{2} \\ 0 \end{bmatrix}.
\]

Using (12) and (13), \((y_{i+1}, z_{i+1}) = V(y_i, z_i)\) can be expressed as the affine transformation

\[
L_{\nu, w} = M_{\nu, w}^{-1} V M_{\nu, w} : \begin{bmatrix} r_{i+1} \\ \hat{r}_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} r_i \\ \hat{r}_i \end{bmatrix} + \begin{bmatrix} \nu \\ 4\nu \end{bmatrix}.
\]

We summarize these observations in the following result.

**Theorem 6.** If \((x, y)\) is an \(A(\nu, w)\)-balancer pair, then so is \(L_{\nu, w}(x, y)\) where

\[
L_{\nu, w} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \nu \\ 4\nu \end{bmatrix}.
\]

Put \(y = \sqrt{8x^2 + 8
u x - 8w + 1}\). Since the component functions of \(L_{\nu, w}\) viewed as a function of \(x\) are strictly increasing on \([\nu, \infty)\), their inverses exist and

\[
L_{\nu, w}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \nu \\ -4\nu \end{bmatrix}.
\]

We also have the following results which extends the analogous results for upper gap balancing numbers [1]. We omit the proofs since each is a straightforward adaptation.

**Theorem 7.** If \((r, \hat{r})\) is the \(A(\nu, w)\)-balancer pair for the \(A(\nu, w)\)-balancing pair \((B, C)\), then \(L_{\nu, w}(r, \hat{r})\) is the \(A(\nu, w)\)-balancer pair for the \(A(\nu, w)\)-balancing pair \(L_{\nu, w}(B, C)\).

**Definition 5.** The counterbalancer \(m\) of an \(A(\nu, w)\) balancing number \(B\) with \(A(\nu, w)\)-balancer \(r\) is defined to be \(m = B + r\).

**Theorem 8.** Suppose \((B, C)\) is an \(A(\nu, w)\)-balancing pair with \((r, \hat{r})\) its associated \(A(\nu, w)\)-balancer pair and \(m\) its counterbalancer. Then

(a) \(r = \frac{-2B + C - 1}{2}\).
(b) \( \hat{r} = 2B - 2r - 2\nu + 1 \);
(c) \( \hat{r} = 4B - C + 2 - 2\nu \);
(d) \( m = \frac{C-1}{2} \).

**Theorem 9.** Let \( ((B_i, C_i))_{i \geq 0} \) be a class of \( A(\nu, w) \)-balancing pairs, \( ((\hat{r}_i, \hat{r}_i))_{i \geq 0} \) its \( A(\nu, w) \)-balancer, and \( (m_i)_{i \geq 0} \) the associated counterbalancers. Then

(a) \( r_{i+1} = 6r_i - r_{i-1} + 2\nu \);
(b) \( \hat{r}_{i+1} = 6\hat{r}_i - \hat{r}_{i-1} \);
(c) \( m_{i+1} = 6m_i - m_{i-1} + 2 \).

**Theorem 10.** Let \( ((B_i, C_i))_{i \geq 0} \) be a class of \( A(\nu, w) \)-balancing pairs, \( ((\hat{r}_i, \hat{r}_i))_{i \geq 0} \) its \( A(\nu, w) \)-balancer, and \( (m_i)_{i \geq 0} \) the associated counterbalancers. Then

\[
\lim_{i \to \infty} \frac{B_{i+1}}{B_i} = \lim_{i \to \infty} \frac{C_{i+1}}{C_i} = \lim_{i \to \infty} \frac{r_{i+1}}{r_i} = \lim_{i \to \infty} \frac{\hat{r}_{i+1}}{\hat{r}_i} = \lim_{i \to \infty} \frac{m_{i+1}}{m_i} = 3 + \sqrt{8}.
\]

5. \( A(\nu, w) \)-balancer Duality

Panda and Ray [7, p. 1196] showed that the balancer of an \( A(1, 0) \)-balancing number is an \( A(0, 0) \)-balancing number, and the balancer of an \( A(0, 0) \)-balancing number is an \( A(1, 0) \)-balancing number. Moreover, the balancer of a balancer of an \( A(\nu, 0) \)-balancing number is the previous \( A(\nu, 0) \)-balancing number for \( \nu = 0, 1 \). We refer to this phenomena as balancer duality and say \( A(1, 0) \)-balancing numbers and \( A(0, 0) \)-balancing numbers are balancer dual to each other. In this section we extend this result to \( A(\nu, w) \)-balancing numbers.

As motivation, we recall the similarities between (4) and (11) as well as observe the transition function from Example 3.2 sends an \( A(1, 0) \)-balancing pair to its \( A(1, 0) \)-balancer pair. The choice of \( H \) in this particular transition function plays a key role in balancer duality, so we designate it as

\[
H_0 = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.
\]

Observe that solutions of \( y^2 - 2z^2 = M \) and \( y^2 - 2z^2 = N \) can be used to form a solution of \( y^2 - 2z^2 = MN \) using Brahmagupta’s identity

\[
(a^2 - 2b^2)(c^2 - 2d^2) = (ac + 2bd)^2 - 2(ad + bc)^2.
\]

In particular, take a solution \( (y_i, z_i) \) to \( y^2 - 2z^2 = N \) where \( N = N(\nu, w) \) and the solution \( (-1, 1) \) to \( y^2 - 2z^2 = M \) where \( M = -1 \). Then we see that

\[
(y_i + z_i \sqrt{2})(-1 + \sqrt{2}) = (-y_i + 2z_i) + (y_i - z_i) \sqrt{2}
\]
which shows \((-y_i + 2z_i, y_i - z_i) = H_0(y_i, z_i)\) is a solution to \(y^2 - 2z^2 = -N(\nu, w)\).

Define the affine transformation \(D_{\nu,w} = S_{\nu,w}^{-1}H_0S_{\nu,w}\) and let \((B_i, C_i)\) be an \(A(\nu, w)\)-balancing pair. To extend balancer duality between \(A(1, 0)\)-balancing pairs and \(A(0, 0)\)-balancing pairs, we would like \(D_{\nu,w}D_{\nu,w} = J_{\nu,w}^{-1}\). This requires \(\nu' = 1 - \nu\) and \(w' = -\frac{\nu^2 - w}{2} - w\). Using the matrix representations, it is straightforward to see with these choices of \(\nu'\) and \(w'\) that

\[
D_{\nu,w} \begin{bmatrix} B_i \\ C_i \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} \\ 4 & -1 \end{bmatrix} \begin{bmatrix} B_i \\ C_i \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 2 - 2\nu \end{bmatrix}
\]

so that

\[
D_{\nu,w}(B_i) = -B_i + \frac{C_i - 1}{2} = -B_i + m_i = r_i
\]

and

\[
D_{\nu,w}(C_i) = 4B_i - C_i + 2 - 2\nu = \hat{r}_i.
\]

This sets the stage for the next theorem.

**Theorem 11 (\(A(\nu, w)\)-Balancer duality).** Let \(D_{\nu,w} = S_{\nu,w}^{-1}H_0S_{\nu,w}\) where \(\nu' = 1 - \nu\) and \(w' = -\frac{\nu^2 - w}{2} - w\). Then \(D_{\nu,w}D_{\nu,w} = J_{\nu,w}^{-1}\).

**Proof.** The discussion above shows that \(D_{\nu,w}\) maps an \(A(\nu, w)\)-balancing pair to its \(A(\nu, w)\)-balancer pair. To complete the proof, note that

\[
\nu'' = (\nu')' = 1 - (1 - \nu) = \nu
\]

and

\[
w'' = (w')' = -\frac{(1 - \nu)^2 - (1 - \nu)}{2} - \left(-\frac{\nu^2 - \nu}{2} - w\right) = w.
\]

We have

\[
D_{\nu',w'}D_{\nu,w} = S_{\nu',w'}^{-1}H_0S_{\nu',w'}S_{\nu,w}^{-1}H_0S_{\nu,w} = S_{\nu',w'}^{-1}H_0S_{\nu,w} = S_{\nu,w}^{-1}V^{-1}S_{\nu,w} = J_{\nu,w}^{-1}.
\]

\(\square\)

Since \(\nu \geq 0\) we can summarize Theorem 11 as follows.

**Corollary 1.** The \(A(1, w)\)-balancing numbers and \(A(0, -w)\)-balancing numbers are balancer dual to each other.

**Example 7.** The \(A(1, -3)\)- and \(A(0, 3)\)-balancing numbers are balancer dual to each other. This is illustrated in Tables 1 and 2. Note that the \(A(0, 3)\)-balancer pairs of the initial \(A(0, 3)\)-balancing pairs of each class correspond to the pair obtained by applying \(J_{1,-3}\) to the initial \(A(1, -3)\)-balancing pair in the corresponding class. This offset always occurs for the initial pairs.
References


