#A79 INTEGERS 18 (2018)

ALMOST GAP BALANCING NUMBERS

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Received: 12/27/17, Accepted: 9/27/18, Published: 10/5/18

Abstract

Almost balancing numbers were introduced by Panda and Panda as a certain generalization of balancing numbers. In this paper, we extend this notion to gap balancing numbers. Additionally, we establish a balancer duality theorem which generalizes the relationship observed between balancing and cobalancing numbers by Panda and Ray.

1. Introduction

Let ν and w be integers with $\nu \geq 0$. A positive integer B is called an almost gap balancing number with gap ν and weight w, or an $A(\nu, w)$-balancing number, if $B \geq \nu$ and

$$1 + 2 + 3 + \cdots + (B - \nu) + w = (B + 1) + \cdots + (B + r) \tag{1}$$

for some integer $r \geq 0$. Panda and Panda [5] introduced almost balancing numbers as a generalization of balancing numbers [4, A001109] and cobalancing numbers [4, A053141] when studying (1) with $\nu = 1$ and $w = \pm 1$. We extend their results to $A(\nu, w)$-balancing numbers. Additionally, we establish a balancer duality theorem which generalizes the relationship observed between balancing and cobalancing numbers by Panda and Ray [7].

2. $A(\nu, w)$-balancing Numbers

In this section, we define almost gap balancing numbers and give several examples.

Definition 1. Let ν and w be integers with $\nu \geq 0$. Define an integer B to be an *almost gap balancing number with gap ν and weight w*, or an *$A(\nu, w)$-balancing number*, if $B \geq \nu$ and

$$1 + 2 + 3 + \cdots + (B - \nu) + w = (B + 1) + \cdots + (B + r)$$

for some integer $r \geq 0$. We refer to r as the *$A(\nu, w)$-balancer* corresponding to the $A(\nu, w)$-balancing number B.

It follows from (1) that B is an $A(\nu, w)$-balancing number if and only if

$$T(B - \nu) + T(B) + w = T(B + r) \tag{2}$$

where $T(n) = \frac{n(n+1)}{2}$ is the nth triangular number. Solving (2) for r gives

$$r = \frac{-(2B + 1) + \sqrt{8B^2 + 8(1 - \nu)B + (2\nu - 1)^2 + 8w}}{2}$$

where we take the positive square root so that $r \geq 0$. Thus B is an $A(\nu, w)$-balancing number if and only if $8B^2 + 8(1 - \nu)B + (2\nu - 1)^2 + 8w$ is a perfect square. The latter expression occurs frequently so we make the following definition.

Definition 2. Let B be an $A(\nu, w)$-balancing number. Define its corresponding *$A(\nu, w)$-Lucas balancing number* to be

$$C = \sqrt{8B^2 + 8(1 - \nu)B + (2\nu - 1)^2 + 8w}.$$

For brevity we say the pair (B, C) is an *$A(\nu, w)$-balancing pair*.

Definition 2 implies that the $A(\nu, w)$-balancing pair (B, C) is a solution to the Pell-like equation

$$y^2 = 8x^2 + 8(1 - \nu)x + (2\nu - 1)^2 + 8w. \tag{3}$$

Equation (3) can be rewritten as

$$y^2 - 2z^2 = 2\nu^2 + 8w - 1 \tag{4}$$

where $z = 2x + 1 - \nu$.

Definition 3. Given integers ν and w with $\nu \geq 0$, we refer to the Pell-like equation

$$y^2 - 2z^2 = 2\nu^2 + 8w - 1$$

as the *$A(\nu, w)$-companion equation*. For later convenience let $N(\nu, w) = 2\nu^2 + 8w - 1$.

In particular, an integral solution (y, z) to the $A(\nu, w)$-companion equation corresponds to an $A(\nu, w)$-balancing pair (B, C) where $B = \frac{z+\nu-1}{2}$ and $C = y$ provided $y > 0$, $z \geq \nu + 1$, and $z \equiv 1 - \nu \pmod{2}$.

Example 1. Since $T(6) + T(6) + 3 = T(9)$ and $T(38) + T(38) + 3 = T(54)$, the numbers 6 and 38 are $A(0, 3)$-balancing numbers with balancers 3 and 16, respectively. The corresponding $A(0, 3)$-Lucas balancing numbers are 19 and 109. Similarly, 3 and 16 are $A(1, -3)$-balancing numbers with balancers 0 and 6 since $T(2)+T(3)-3 = T(3)$ and $T(15)+T(16)-3 = T(22)$. Their $A(1, -3)$-Lucas balancing numbers are 7 and 45. Additional examples of $A(1, -3)$- and $A(0, 3)$-balancing numbers are given in Tables 1 and 2, respectively.

Example 2. The $A(\nu, 0)$-balancing numbers are the upper gap balancing numbers [1]. In particular, the $A(0, 0)$- and $A(1, 0)$-balancing numbers are cobalancing [7] and balancing numbers [2], respectively. The $A(1, 1)$- and $A(1, -1)$-balancing numbers are the almost balancing numbers of the first and second kind, respectively, studied by Panda and Panda [5]. Lastly, the $A(1, -k^2)$-balancing numbers are the k-circular balancing numbers [6].

Example 3. For $\nu \geq 0$, a class of $A(\nu, -1)$-balancing numbers is generated from the seed $(1, |2\nu - 3|)$. The initial $A(1, -1)$-balancing pairs are $(1, 1)$, $(4, 11)$, and $(23, 65)$. For $\nu \neq 1$, the initial $A(\nu, -1)$-balancing pairs are $(\nu + 1, 2\nu + 3)$, $(4\nu + 7, 10\nu + 21)$, and $(21\nu + 43, 58\nu + 123)$.

The last example is a special case of the following result.

Theorem 1. *Let $k \geq 0$ be an integer and $w = -T(k)$. Then there exists $A(\nu, w)$-balancing numbers for every $\nu \geq 0$.*

Proof. Observe $B = \nu + k$ is an $A(\nu, w)$-balancing number with balancer $r = 0$. $\square$

Remark 1. The existence of $A(\nu, w)$-balancing numbers is not always guaranteed. More specifially, $A(\nu, w)$-balancing numbers exist if and only if the $A(\nu, w)$-companion equation $y^2 - 2z^2 = N(\nu, w)$ has integral solutions. From Pell equation theory [3, pp. 205–207] the latter can be determined by searching for solutions in a certain finite interval. For example, there do not exist $A(1, 2)$-balancing numbers since its companion equation, $y^2 - 2z^2 = 15$, does not have any integral solutions. In fact, this argument proves that there are no $A(\nu, w)$-balancing numbers whenever $N(\nu, w) = 15$.

3. Functions Involving $A(\nu, w)$-balancing Numbers

We present several functions which generate $A(\nu, w)$-balancing numbers from known $A(\nu, w)$-balancing numbers. These functions are derived using $A(\nu, w)$-companion equations and linear substitutions.

3.1. Functions Generating $A(\nu, w)$-balancing Numbers

From the theory of Pell equations, integral solutions to (4), if they exist, occur in a finite number of cyclic classes. That is, if (y_i, z_i) is a solution of the $A(\nu, w)$-companion equation corresponding to an $A(\nu, w)$-balancing pair, then so is (y_{i+1}, z_{i+1}) where

$$y_{i+1} + z_{i+1}\sqrt{2} = (3 + 2\sqrt{2})(y_i + z_i\sqrt{2})$$

or equivalently in matrix form

$$V : \begin{bmatrix} y_{i+1} \\ z_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_i \\ z_i \end{bmatrix}. \tag{5}$$

To see this, suppose (y_i, z_i) satisfies $y_i > 0$, $z_i \geq \nu + 1$, and $z_i \equiv 1 - \nu \pmod 2$. Clearly $y_{i+1} > 0$ and $z_{i+1} \geq \nu + 1$. Lastly $z_{i+1} \equiv z_i \equiv 1 - \nu \pmod 2$ which establishes the claim. Throughout this paper we freely use functional notation $V(y, z)$ for the map given in (5) and use similar notation for other such maps.

The relations $y_i = C_i$ and $z_i = 2B_i + 1 - \nu$ can be expressed as the affine transformations

$$S_{\nu,w} : \begin{bmatrix} y_i \\ z_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} B_i \\ C_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - \nu \end{bmatrix} \tag{6}$$

and

$$S^{-1}_{\nu,w} : \begin{bmatrix} B_i \\ C_i \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_i \\ z_i \end{bmatrix} + \begin{bmatrix} \frac{\nu - 1}{2} \\ 0 \end{bmatrix}. \tag{7}$$

Using (6) and (7), equation (5) can be expressed in terms of $A(\nu, w)$-balancing pairs as

$$J_{\nu,w} = S^{-1}_{\nu,w} V S_{\nu,w} : \begin{bmatrix} B_{i+1} \\ C_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} B_i \\ C_i \end{bmatrix} + \begin{bmatrix} 1 - \nu \\ 4 - 4\nu \end{bmatrix}. \tag{8}$$

Noting $(3 + 2\sqrt{2})^{-1} = 3 - 2\sqrt{2}$ we similarly see

$$y_{i-1} + z_{i-1}\sqrt{2} = (3 - 2\sqrt{2})(y_i + z_i\sqrt{2})$$

and

$$J^{-1}_{\nu,w} = S^{-1}_{\nu,w} V^{-1} S_{\nu,w} : \begin{bmatrix} B_{i-1} \\ C_{i-1} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} B_i \\ C_i \end{bmatrix} + \begin{bmatrix} 1 - \nu \\ 4\nu - 4 \end{bmatrix}.$$

We summarize the observations above in the following result.

Theorem 2. *If (x, y) is an $A(\nu, w)$-balancing pair, then so is $J_{\nu,w}(x, y)$ where*

$$J_{\nu,w} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 - \nu \\ 4 - 4\nu \end{bmatrix}.$$

i	0	1	2	3	4	5	6	7
B	3	9	16	52	93	303	542	1766
C	7	25	45	147	263	857	1533	4995
r	0	3	6	21	38	125	224	731
$\hat{r}$	5	11	19	61	109	355	635	2069
m	3	12	22	73	131	428	766	2497

Table 1: Initial $A(1,-3)$-balancing numbers and associated sequences.

i	0	1	2	3	4	5	6	7
B	0	3	6	21	38	125	224	731
C	5	11	19	61	109	355	635	2069
r	2	2	3	9	16	52	93	303
$\hat{r}$	3	3	7	25	45	147	263	857
m	2	5	9	30	54	177	317	1034

Table 2: Initial $A(0,3)$-balancing numbers and associated sequences.

Recall $y = \sqrt{8x^2 + 8(1-\nu)x + (2\nu-1)^2 + 8w}$. Since the components of $J_{\nu,w}$ viewed as functions of x are strictly increasing on $[\nu, \infty)$, their inverses exist and

$$J_{\nu,w}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1-\nu \\ 4\nu - 4 \end{bmatrix}.$$

Given an $A(\nu,w)$-balancing pair (x_0, y_0) the functions $J_{\nu,w}$ and $J_{\nu,w}^{-1}$ can be applied iteratively to form a class of solutions $((x_i, y_i))_{i\in\mathbb{Z}}$ to (3) via $(x_{i+1}, y_{i+1}) = J_{\nu,w}(x_i, y_i)$. It is always possible to reindex so that the minimal $A(\nu,w)$-balancing pair corresponds to (x_0, y_0). Then $((x_i, y_i))_{i\in\mathbb{Z}}$ is a class of solutions to (3) such that the nonnegative indexed terms correspond to $A(\nu,w)$-balancing pairs. We tacitly assume that a class of $A(\nu,w)$-balancing pairs are indexed in this manner unless stated otherwise.

Example 4. There are two classes of $A(1,-3)$-balancing pairs whose initial terms are $(3,7)$ and $(9,25)$, respectively. The initial $A(1,-3)$-balancing numbers and associated sequences are given in Table 1. There are also two classes of $A(0,3)$-balancing pairs and their initial terms are $(0,5)$ and $(3,11)$, respectively. The initial $A(0,3)$-balancing numbers and associated sequences are given in Table 2.

The next two theorems are generalizations of results of upper gap balancing numbers [1] extended to $A(\nu,w)$-balancing numbers using Theorem 2. The proofs are straightforward modifications and omitted here.

Theorem 3. *Suppose $((B_i, C_i))_{i\geq 0}$ is a class of $A(\nu,w)$-balancing pairs. Then for $i \geq 1$*

$$B_{i+1} = 6B_i - B_{i-1} + 2 - 2\nu$$

and

$$C_{i+1} = 6C_i - C_{i-1}.$$

Theorem 4. *Let $((B_i, C_i))_{i\geq 0}$ be a class of $A(\nu, w)$-balancing pairs. The generating function for its $A(\nu, w)$-balancing numbers is*

$$G(s) = \frac{(2 - 2\nu - B_1 + 6B_0)s^2 + (B_1 - 7B_0)s + B_0}{(1-s)(1-6s+s^2)}.$$

Example 5. There are two classes of $A(0, 3)$-balancing numbers whose generating functions are

$$G_1(s) = \frac{-4s^2 + 6s}{(1-s)(1-6s+s^2)} \quad \text{and } G_2(s) = \frac{-s^2+3}{(1-s)(1-6s+s^2)}.$$

These can be combined to obtain the generating function for all $A(0, 3)$-balancing numbers, namely

$$G(s) = \frac{s^4 + 3s^3 - 3s^2 - 3s}{(s-1)(s^2-2s-1)(s^2+2s-1)}.$$

3.2. Transition Functions

In this section, we present functions which map $A(\nu, w)$-balancing numbers to $A(\nu', w')$-balancing numbers.

Lemma 1. *Let N and N' be integers. Suppose (y_0, z_0) and (y_0', z_0') are solutions to the equations $y^2 - 2z^2 = N$ and $y^2 - 2z^2 = N'$, respectively. Let*

$$H = \frac{1}{N}\begin{bmatrix} y_0 & 2z_0 \\ -z_0 & -y_0 \end{bmatrix}\begin{bmatrix} y_0' & 2z_0' \\ -z_0' & -y_0' \end{bmatrix}.$$

Then

$$\begin{bmatrix} y_0' \\ z_0' \end{bmatrix} = H \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}.$$

Proof. Since $y_0^2 - 2z_0^2 = N$ we see

$$H\begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \frac{1}{N}\begin{bmatrix} y_0'(y_0^2 - 2z_0^2) \\ z_0'(y_0^2 - 2z_0^2) \end{bmatrix} = \begin{bmatrix} y_0' \\ z_0' \end{bmatrix}.$$

□

Lemma 2. *Let H be as defined in Lemma 1. Then $HV = VH$.*

Proof. Suppress the naughts in H and observe

$$HV = \frac{1}{N}\begin{bmatrix} 3yy' - 6zz' + 4yz' - 4zy' & 6yz' - 6zy' + 4yy' - 8zz' \\ 2yy' + 3yz' - 3zy' - 4zz' & 3yy' + 4yz' - 4zy' - 6zz' \end{bmatrix} = VH.$$

□

Using Lemmas 1 and 2 we construct a function T which maps a given class of $A(\nu, w)$-balancing numbers $((B_i, C_i))_{i\geq 0}$ to a given class of $A(\nu', w')$-balancing numbers $((B'_i, C'_i))_{i\geq 0}$. In terms of the notation used above define

$$T := S^{-1}_{\nu',w'} H S_{\nu,w}$$

as the composition of affine transformations where (y_0, z_0) and (y'_0, z'_0) used in H are solutions of the companion equations corresponding to the balancing pairs (B_0, C_0) and (B'_0, C'_0), respectively.

Observe that T is an affine transformation and $(B_0, C_0) \mapsto (B'_0, C'_0)$ by Lemma 1 and construction. To see that $(B_i, C_i) \mapsto (B'_i, C'_i)$ for all i, observe using (8) and Lemma 2 that

$$\begin{aligned} J_{\nu',w'} T &= S^{-1}_{\nu',w'} V S_{\nu',w'} S^{-1}_{\nu',w'} H S_{\nu,w} \\ &= S^{-1}_{\nu',w'} V H S_{\nu,w} \\ &= S^{-1}_{\nu',w'} H V S_{\nu,w} \\ &= S^{-1}_{\nu',w'} H S_{\nu,w} S^{-1}_{\nu,w} V S_{\nu,w} \\ &= T J_{\nu,w}. \end{aligned}$$

It follows that $J^i_{\nu',w'} T = T J^i_{\nu,w}$ which implies $(B_i, C_i) \mapsto (B'_i, C'_i)$ for all i.

It is straightforward to give the following explicit presentation of T.

Theorem 5. *Let $((B_i, C_i))_{i\geq 0}$ and $((B'_i, C'_i))_{i\geq 0}$ be classes of $A(\nu, w)$ and $A(\nu', w')$-balancing pairs, respectively. Then the transition function T given by*

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ 8b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ (4-4\nu)b \end{bmatrix}$$

where

$$a = -\frac{8B_0B'_0 + 4(1-\nu')B_0 + 4(1-\nu)B'_0 + 2(1-\nu)(1-\nu') - C_0C'_0}{2\nu^2 - 1 + 8w}$$

$$b = \frac{2(C_0B'_0 - B_0C'_0) + (1-\nu')C_0 - (1-\nu)C'_0}{2(2\nu^2 - 1 + 8w)}$$

$$c = \frac{(1-\nu)\left[C_0C'_0 - 8B_0B'_0 - 4(1-\nu)B'_0\right] + (1-\nu')\left[8B_0^2 + 4(1-\nu)B_0 - C_0^2\right]}{2(2\nu^2 - 1 + 8w)}.$$

maps (B_i, C_i) to (B'_i, C'_i) for all i.

Example 6. Consider the transition function from the $A(1, 0)$-balancing pairs $((B_i, C_i))_{i\geq 0}$ to the $A(0, 0)$-balancing pairs $((B'_i, C'_i))_{i\geq 0}$. Recall $(B_0, C_0) = (1, 3)$

and $(B'_0, C'_0) = (0, 1)$. Then $(y_0, z_0) = (3, 2)$, $(y'_0, z'_0) = (1, 1)$,

$$H = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

and

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}.$$

Observe $H^2 = V^{-1}$. This plays a key role in the balancer duality result in section 5.

4. Balancers, Related Sequences, and Their Functions

In this section we investigate the sequence of $A(\nu, w)$-balancers associated to $A(\nu, w)$-balancing numbers and related sequences. Recall that an integer $r \geq 0$ is a $A(\nu, w)$-balancer of the $A(\nu, w)$-balancing number B provided it satisfies $T(B-\nu)+T(B)+w = T(B+r)$. Solving for B gives

$$B = \frac{(2r + 2\nu - 1) + \sqrt{8r^2 + 8\nu r - 8w + 1}}{2} \tag{9}$$

where we take the positive square root so that $B \geq 0$. Thus r is an $A(\nu, w)$-balancer if and only if $8r^2 + 8\nu r - 8w + 1$ is a perfect square. Due to the latter expression we make the following definition.

Definition 4. Let r be an $A(\nu, w)$-balancer. Define its $A(\nu, w)$-*Lucas balancer* to be

$$\hat{r} = \sqrt{8r^2 + 8\nu r - 8w + 1}.$$

We refer to $(r, \hat{r})$ as an $A(\nu, w)$-*balancer pair.*

Equation (9) can be reformulated as $\hat{r} = 2B - 2r - 2\nu + 1$. Definition 4 implies that the $A(\nu, w)$-balancer pair $(r, \hat{r})$ is a solution to the equation

$$y^2 = 8x^2 + 8\nu x - 8w + 1. \tag{10}$$

Alternatively (10) can be rewritten as

$$y^2 - 2z^2 = -N(\nu, w) \tag{11}$$

where $y = \hat{r}$ and $z = 2r + \nu$. We refer to (11) as the $A(\nu, w)$-balancer companion equation. An integral solution of (11) corresponds to an $A(\nu, w)$-balancer pair $(r, \hat{r})$ where $r = \frac{z-\nu}{2}$ and $\hat{r} = y$ provided $z \equiv \nu \pmod 2$, $z \geq \nu$, and $y > 0$.

Again applying the theory of Pell equations, solutions to (11), if they exist, occur in a finite number of cyclic classes. Moreover, if (y_i, z_i) is a solution of the $A(\nu, w)$-balancer companion equation corresponding to an $A(\nu, w)$-balancer pair, then it is straightforward using techniques as before to see that $(y_{i+1}, z_{i+1}) = V(y_i, z_i)$ is also a solution.

The relations $y_i = \hat{r}_i$ and $z_i = 2r_i + \nu$ can be expressed as the affine transformations

$$M_{\nu,w} : \begin{bmatrix} y_i \\ z_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} r_i \\ \hat{r}_i \end{bmatrix} + \begin{bmatrix} 0 \\ \nu \end{bmatrix} \tag{12}$$

and

$$M_{\nu,w}^{-1} : \begin{bmatrix} r_i \\ \hat{r}_i \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_i \\ z_i \end{bmatrix} + \begin{bmatrix} -\frac{\nu}{2} \\ 0 \end{bmatrix}. \tag{13}$$

Using (12) and (13), $(y_{i+1}, z_{i+1}) = V(y_i, z_i)$ can be expressed as the affine transformation

$$L_{\nu,w} = M_{\nu,w}^{-1} V M_{\nu,w} : \begin{bmatrix} r_{i+1} \\ \hat{r}_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} r_i \\ \hat{r}_i \end{bmatrix} + \begin{bmatrix} \nu \\ 4\nu \end{bmatrix}.$$

We summarize these observations in the following result.

Theorem 6. *If (x, y) is an $A(\nu, w)$-balancer pair, then so is $L_{\nu,w}(x, y)$ where*

$$L_{\nu,w} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \nu \\ 4\nu \end{bmatrix}.$$

Put $y = \sqrt{8x^2 + 8\nu x - 8w + 1}$. Since the component functions of $L_{\nu,w}$ viewed as a function of x are strictly increasing on $[\nu, \infty)$, their inverses exist and

$$L_{\nu,w}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \nu \\ -4\nu \end{bmatrix}.$$

We also have the following results which extends the analogous results for upper gap balancing numbers [1]. We omit the proofs since each is a straightforward adaptation.

Theorem 7. *If $(r, \hat{r})$ is the $A(\nu, w)$-balancer pair for the $A(\nu, w)$-balancing pair (B, C), then $L_{\nu,w}(r, \hat{r})$ is the $A(\nu, w)$-balancer pair for the $A(\nu, w)$-balancing pair $J_{\nu,w}(B, C)$.*

Definition 5. The *counterbalancer* m of an $A(\nu, w)$ balancing number B with $A(\nu, w)$-balancer r is defined to be $m = B + r$.

Theorem 8. *Suppose (B, C) is an $A(\nu, w)$-balancing pair with $(r, \hat{r})$ its associated $A(\nu, w)$-balancer pair and m its counterbalancer. Then*

(a) $r = \frac{-2B+C-1}{2}$;

(b) $\hat{r} = 2B - 2r - 2\nu + 1$;

(c) $\hat{r} = 4B - C + 2 - 2\nu$;

(d) $m = \frac{C-1}{2}$.

Theorem 9. *Let $((B_i, C_i))_{i\geq 0}$ be a class of $A(\nu, w)$-balancing pairs, $((r_i, \hat{r}_i))_{i\geq 0}$ its $A(\nu, w)$-balancer pairs, and $(m_i)_{i\geq 0}$ the associated counterbalancers. Then*

(a) $r_{i+1} = 6r_i - r_{i-1} + 2\nu$;

(b) $\hat{r}_{i+1} = 6\hat{r}_i - \hat{r}_{i-1}$;

(c) $m_{i+1} = 6m_i - m_{i-1} + 2$.

Theorem 10. *Let $((B_i, C_i))_{i\geq 0}$ be a class of $A(\nu, w)$-balancing pairs, $((r_i, \hat{r}_i))_{i\geq 0}$ its $A(\nu, w)$-balancer pairs, and $(m_i)_{i\geq 0}$ the associated counterbalancers. Then*

$$\lim_{i\to\infty} \frac{B_{i+1}}{B_i} = \lim_{i\to\infty} \frac{C_{i+1}}{C_i} = \lim_{i\to\infty} \frac{r_{i+1}}{r_i} = \lim_{i\to\infty} \frac{\hat{r}_{i+1}}{\hat{r}_i} = \lim_{i\to\infty} \frac{m_{i+1}}{m_i} = 3 + \sqrt{8}.$$

5. $A(\nu, w)$-balancer Duality

Panda and Ray [7, p. 1196] showed that the balancer of an $A(1, 0)$-balancing number is an $A(0, 0)$-balancing number, and the balancer of an $A(0, 0)$-balancing number is an $A(1, 0)$-balancing number. Moreover, the balancer of a balancer of an $A(\nu, 0)$-balancing number is the previous $A(\nu, 0)$-balancing number for $\nu = 0, 1$. We refer to this phenomena as balancer duality and say $A(1, 0)$-balancing numbers and $A(0, 0)$-balancing numbers are balancer dual to each other. In this section we extend this result to $A(\nu, w)$-balancing numbers.

As motivation, we recall the similarities between (4) and (11) as well as observe the transition function from Example 3.2 sends an $A(1, 0)$-balancing pair to its $A(1, 0)$-balancer pair. The choice of H in this particular transition function plays a key role in balancer duality, so we designate it as

$$H_0 = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Observe that solutions of $y^2 - 2z^2 = M$ and $y^2 - 2z^2 = N$ can be used to form a solution of $y^2 - 2z^2 = MN$ using Brahmagupta's identity

$$(a^2 - 2b^2)(c^2 - 2d^2) = (ac + 2bd)^2 - 2(ad + bc)^2.$$

In particular, take a solution (y_i, z_i) to $y^2 - 2z^2 = N$ where $N = N(\nu, w)$ and the solution $(-1, 1)$ to $y^2 - 2z^2 = M$ where $M = -1$. Then we see that

$$(y_i + z_i\sqrt{2})(-1 + \sqrt{2}) = (-y_i + 2z_i) + (y_i - z_i)\sqrt{2}$$

which shows $(-y_i + 2z_i, y_i - z_i) = H_0(y_i, z_i)$ is a solution to $y^2 - 2z^2 = -N(\nu, w)$.

Define the affine transformation $D_{\nu,w} = S^{-1}_{\nu',w'} H_0 S_{\nu,w}$ and let (B_i, C_i) be an $A(\nu, w)$-balancing pair. To extend balancer duality between $A(1, 0)$-balancing pairs and $A(0, 0)$-balancing pairs, we would like $D_{\nu',w'} D_{\nu,w} = J^{-1}_{\nu,w}$. This requires $\nu' = 1 - \nu$ and $w' = -\frac{\nu^2-\nu}{2} - w$. Using the matrix representations, it is straightforward to see with these choices of ν' and w' that

$$D_{\nu,w} \begin{bmatrix} B_i \\ C_i \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} \\ 4 & -1 \end{bmatrix} \begin{bmatrix} B_i \\ C_i \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 2 - 2\nu \end{bmatrix}$$

so that

$$D_{\nu,w}(B_i) = -B_i + \frac{C_i - 1}{2} = -B_i + m_i = r_i$$

and

$$D_{\nu,w}(C_i) = 4B_i - C_i + 2 - 2\nu = \hat{r}_i.$$

This sets the stage for the next theorem.

Theorem 11 ($A(\nu, w)$-Balancer duality). *Let $D_{\nu,w} = S^{-1}_{\nu',w'} H_0 S_{\nu,w}$ where $\nu' = 1 - \nu$ and $w' = -\frac{\nu^2-\nu}{2} - w$. Then $D_{\nu',w'} D_{\nu,w} = J^{-1}_{\nu,w}$.*

Proof. The discussion above shows that $D_{\nu,w}$ maps an $A(\nu, w)$-balancing pair to its $A(\nu, w)$-balancer pair. To complete the proof, note that

$$\nu'' = (\nu')' = 1 - (1 - \nu) = \nu$$

and

$$w'' = (w')' = -\frac{(1-\nu)^2 - (1-\nu)}{2} - \left(-\frac{\nu^2 - \nu}{2} - w\right) = w.$$

We have

$$\begin{aligned} D_{\nu',w'} D_{\nu,w} &= S^{-1}_{\nu'',w''} H_0 S_{\nu',w'} S^{-1}_{\nu',w'} H_0 S_{\nu,w} \\ &= S^{-1}_{\nu'',w''} H_0^2 S_{\nu,w} \\ &= S^{-1}_{\nu,w} V^{-1} S_{\nu,w} \\ &= J^{-1}_{\nu,w}. \end{aligned}$$

□

Since $\nu \geq 0$ we can summarize Theorem 11 as follows.

Corollary 1. *The $A(1, w)$-balancing numbers and $A(0, -w)$-balancing numbers are balancer dual to each other.*

Example 7. The $A(1, -3)$- and $A(0, 3)$-balancing numbers are balancer dual to each other. This is illustrated in Tables 1 and 2. Note that the $A(0, 3)$-balancer pairs of the initial $A(0, 3)$-balancing pairs of each class correspond to the pair obtained by applying $J_{1,-3}$ to the initial $A(1, -3)$-balancing pair in the corresponding class. This offset always occurs for the initial pairs.

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