THE ADJACENCY-PELL-HURWITZ NUMBERS

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Abstract

In this paper, we define the k-adjacency-Pell-Hurwitz numbers by using the Hurwitz matrix of order $4m$ which is obtained by the aid of the characteristic polynomial of the adjacency-Pell sequence. Firstly, we give relationships between the k-adjacency-Pell-Hurwitz numbers and the generating matrices for these sequences. Further, we obtain the Binet formula for the $(2m-1)$-adjacency-Pell-Hurwitz numbers. Also, we derive relationships between the k-adjacency-Pell-Hurwitz numbers and permanents and determinants of certain matrices. Finally, we give the combinatorial and exponential representations of the k-adjacency-Pell-Hurwitz numbers.

1. Introduction

It is well-known that the Pell sequence is defined by the following equation:

$$P_{n+1} = 2P_n + P_{n-1}$$

for $n > 0$, where $P_0 = 0$, $P_1 = 1$.

The adjacency-type sequence is defined in [3] by an mn-order recurrence equation:

$$x_{nm+k}^{n,m} = x_{nm-n+1+k}^{n,m} + x_k^{n,m}$$

for $k \geq 1$, where $x_1^{n,m} = \cdots = x_{nm-n+1}^{n,m} = 0$, $x_{nm-n+2}^{n,m} = 1$, $x_{nm-n+3}^{n,m} = \cdots = x_{nm}^{n,m} = 0$ and $n, m \geq 2$.

Karaduman and Deveci defined the adjacency-Pell sequence as follows:

$$a_{m,n}(mn+k) = 2a_{m,n}(mn-n+k+1) + a_{m,n}(k)$$

for the integers $k \geq 1$, $m \geq 2$ and $n \geq 4$, with initial constants $a_{m,n}(1) = \cdots = a_{m,n}(mn-1) = 0$ and $a_{m,n}(mn) = 1$ [5].

Consider the k-step recurrence sequence:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. Earlier, Kalman [8] derived a number of closed-form formulas for some generalized sequences via the companion matrix method as follows:

If the companion matrix A is defined by

$$A = [a_{i,j}]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geqslant 0$.

Consider f a real polynomial of degree n given by

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n.$$

Hurwitz [7] introduced the matrix $H_n = [h_{i,j}]_{n\times n}$ associated to f as follows:

$$H_n = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & & & & \vdots & \vdots & \vdots \\ 0 & a_1 & a_3 & & & & \vdots & \vdots & \vdots \\ \vdots & a_0 & a_2 & \ddots & & & 0 & \vdots & \vdots \\ \vdots & 0 & a_1 & & \ddots & & a_n & \vdots & \vdots \\ \vdots & \vdots & a_0 & & & \ddots & a_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & & & & a_{n-2} & a_n & \vdots \\ \vdots & \vdots & \vdots & & & & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{n-4} & a_{n-2} & a_n \end{bmatrix}.$$

Several suthors have used homogeneous linear recurrance relations to deduce miscellaneous properties for a plethora of sequences; see for example [4, 6, 9, 10, 13, 14, 15, 16, 17, 18, 19]. In particular, Deveci and Shannon defined the adjacency-type numbers and examined their structural properties [3]. The adjacency-Pell numbers, their miscellaneous properties and applications in groups were studied by Deveci and Karaduman in [5]. In the present paper, we define the k-adjacency-Pell-Hurwitz numbers by a recurrence relations of order $4m$, $(m \geq 2)$ and give their generating matrices, Binet formulas, permanental, determinantal, combinatorial, exponential representations, and we derive a formula for the sums of the k-adjacency-Pell-Hurwitz numbers.

2. The Main Results

For $m \geq 2$ and $n = 4$ it is clear that the characteristic polynomial of the adjacency-Pell sequence is

$$p(x) = x^{4m} - 2x^{4m-3} - 1. \tag{2.1}$$

Then by (2.1), we see that the Hurwitz matrix $H_{4m} = [h_{i,j}]_{4m\times 4m}$ associated to a polynomial p is

$$[h_{i,j}]_{4m\times 4m} = \begin{cases} 2 & \text{if } i = 2k-1 \text{ and } \ j = k+1 \text{ for } 1 \leq k \leq 2m, \\ -1 & \text{if } i = 2k \text{ and } \ j = k+2m \text{ for } 1 \leq k \leq 2m, \\ 1 & \text{if } i = 2k \text{ and } j = k \text{ for } 1 \leq k \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

We define the k-adjacency-Pell-Hurwitz numbers by using the Hurwitz matrix H_{4m} as shown:

$$x^{(k,m)}_{4m+u} = -2x^{(k,m)}_{4m-2k+1+u} + x^{(k,m)}_{4m-2k-2+u} \tag{2.2}$$

for the integers $u \geq 1$, $m \geq 2$ and $1 \leq k \leq 2m-1$, with initial constants $x_1^{(k,m)} = \cdots = x_{4m-1}^{(k,m)} = 0$ and $x_{4m}^{(k,m)} = 1$. Here $x_i^{(k,m)}$ is the ith term of the kth sequence according to the constant m.

By (2.2), we may write

$$M_m^{(1)} = \left[m_{i,j}^{(1)}\right]_{4m\times 4m} = \begin{bmatrix} -2 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix},$$

$M_m^{(\lambda)} = \left[m_{i,j}^{(\lambda)}\right]_{4m\times 4m} =$

$$\begin{matrix} & & & & & & & & (2\lambda+2)\,\text{th} & & & & & & \\ & & & & & & & & \downarrow & & & & & & \end{matrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & & & & & \cdots & & & & & 0 & 0 \\ 0 & 1 & 0 & & & & & & \cdots & & & & & 0 & 0 \\ 0 & 0 & 1 & 0 & & & & & \cdots & & & & & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & & & & \cdots & & & & & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & & & & & & & & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & & \vdots & & & & & & \vdots & \vdots \\ 0 & 0 & & \cdots & & 0 & 1 & 0 & 0 & 0 & & \cdots & & 0 & 0 \\ & & & & & & & 1 & 0 & 0 & 0 & \cdots & & 0 & 0 \\ 0 & 0 & & & \cdots & & & 0 & 0 & 1 & 0 & \cdots & & 0 & 0 \\ 0 & 0 & & & \cdots & & & & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & & & & & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & & & & & & & & & & \ddots & \ddots & & \vdots \\ 0 & 0 & & & & \cdots & & & & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & & & & \cdots & & & & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix},$$

for $2 \le \lambda \le 2m-2$ and

$$M_m^{(2m-1)} = \left[m_{i,j}^{(2m-1)}\right]_{4m\times 4m} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & & \cdots & & 0 & 0 \\ 0 & 1 & 0 & & \cdots & & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & & \\ 0 & & \cdots & & 0 & 1 & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We call matrix $M_m^{(\alpha)}$ the α-adjacency-Pell-Hurwitz matrix of size $4m \times 4m$.

By an inductive argument on τ, we obtain

$$\left(M_m^{(1)}\right)^{\tau} = \begin{bmatrix} x_{4m+\tau}^{(1,m)} & x_{4m+\tau-3}^{(1,m)} & x_{4m+\tau-2}^{(1,m)} & x_{4m+\tau-1}^{(1,m)} & 0 & 0 & 0 & \cdots & 0 \\ x_{4m+\tau-1}^{(1,m)} & x_{4m+\tau-4}^{(1,m)} & x_{4m+\tau-3}^{(1,m)} & x_{4m+\tau-2}^{(1,m)} & 0 & 0 & 0 & \cdots & 0 \\ x_{4m+\tau-2}^{(1,m)} & x_{4m+\tau-5}^{(1,m)} & x_{4m+\tau-4}^{(1,m)} & x_{4m+\tau-3}^{(1,m)} & 0 & 0 & 0 & \cdots & 0 \\ x_{4m+\tau-3}^{(1,m)} & x_{4m+\tau-6}^{(1,m)} & x_{4m+\tau-5}^{(1,m)} & x_{4m+\tau-4}^{(1,m)} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix},$$

for $\tau \ge 1$,

$$\left(M_m^{(\lambda)}\right)^{\tau} = \begin{bmatrix} x_{4m+\tau}^{(\lambda,m)} & x_{4m+\tau+1}^{(\lambda,m)} & \cdots & x_{4m+\tau+2\lambda-2}^{(\lambda,m)} & \\ x_{4m+\tau-1}^{(\lambda,m)} & x_{4m+\tau}^{(\lambda,m)} & \cdots & x_{4m+\tau+2\lambda-3}^{(\lambda,m)} & \\ x_{4m+\tau-2}^{(\lambda,m)} & x_{4m+\tau-1}^{(\lambda,m)} & \cdots & x_{4m+\tau+2\lambda-4}^{(\lambda,m)} & \\ \vdots & \vdots & & \vdots & \\ x_{4m+\tau-2\lambda}^{(\lambda,m)} & x_{4m+\tau-2\lambda+1}^{(\lambda,m)} & \cdots & x_{4m+\tau-2}^{(\lambda,m)} & E \\ x_{4m+\tau-2\lambda-1}^{(\lambda,m)} & x_{4m+\tau-2\lambda}^{(\lambda,m)} & \cdots & x_{4m+\tau-3}^{(\lambda,m)} & \\ 0 & 0 & \cdots & 0 & \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 0 & \end{bmatrix},$$

for $\tau \geq 1$ and $2 \leq \lambda \leq 2m-2$, where E is the following $4m \times (4m-2\lambda+1)$ matrix:

$$E = \begin{bmatrix} x_{4m+\tau-3}^{(\lambda,m)} & x_{4m+\tau-2}^{(\lambda,m)} & x_{4m+\tau-1}^{(\lambda,m)} & 0 & 0 & 0 & \cdots & 0 \\ x_{4m+\tau-4}^{(\lambda,m)} & x_{4m+\tau-3}^{(\lambda,m)} & x_{4m+\tau-2}^{(\lambda,m)} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & 0 & \cdots & 0 \\ x_{4m+\tau-2\lambda-3}^{(\lambda,m)} & x_{4m+\tau-2\lambda-2}^{(\lambda,m)} & x_{4m+\tau-2\lambda-1}^{(\lambda,m)} & 0 & 0 & 0 & \cdots & 0 \\ x_{4m+\tau-2\lambda-4}^{(\lambda,m)} & x_{4m+\tau-2\lambda-3}^{(\lambda,m)} & x_{4m+\tau-2\lambda-2}^{(\lambda,m)} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

and

$$\left(M_m^{(2m-1)}\right)^{\tau} =$$

$$\begin{bmatrix} x_{4m+\tau}^{(2m-1,m)} & x_{4m+\tau+1}^{(2m-1,m)} & \cdots & x_{8m+\tau-4}^{(2m-1,m)} & x_{4m+\tau-3}^{(2m-1,m)} & x_{4m+\tau-2}^{(2m-1,m)} & x_{4m+\tau-1}^{(2m-1,m)} \\ x_{4m+\tau-1}^{(2m-1,m)} & x_{4m+\tau}^{(2m-1,m)} & \cdots & x_{8m+\tau-5}^{(2m-1,m)} & x_{4m+\tau-4}^{(2m-1,m)} & x_{4m+\tau-3}^{(2m-1,m)} & x_{4m+\tau-2}^{(2m-1,m)} \\ x_{4m+\tau-2}^{(2m-1,m)} & x_{4m+\tau-1}^{(2m-1,m)} & \cdots & x_{8m+\tau-6}^{(2m-1,m)} & x_{4m+\tau-5}^{(2m-1,m)} & x_{4m+\tau-4}^{(2m-1,m)} & x_{4m+\tau-3}^{(2m-1,m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ x_{\tau+3}^{(2m-1,m)} & x_{\tau+4}^{(2m-1,m)} & \cdots & x_{4m+\tau-1}^{(2m-1,m)} & x_{\tau}^{(2m-1,m)} & x_{\tau+1}^{(2m-1,m)} & x_{\tau+2}^{(2m-1,m)} \\ x_{\tau+2}^{(2m-1,m)} & x_{\tau+3}^{(2m-1,m)} & \cdots & x_{4m+\tau-2}^{(2m-1,m)} & x_{\tau-1}^{(2m-1,m)} & x_{\tau}^{(2m-1,m)} & x_{\tau+1}^{(2m-1,m)} \\ x_{\tau+1}^{(2m-1,m)} & x_{\tau+2}^{(2m-1,m)} & \cdots & x_{4m+\tau-3}^{(2m-1,m)} & x_{\tau-2}^{(2m-1,m)} & x_{\tau-1}^{(2m-1,m)} & x_{\tau}^{(2m-1,m)} \end{bmatrix},$$

for $\tau \geq 3$. Let $m \geq 2$ and let $S_t^{(k,m)} = \sum_{i=1}^{t} x_i^{(k,m)}$ such that $1 \leq k \leq 2m-1$. We introduce matrix $H(k,m)$ by

$$H(k,m) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & M_m^{(k)} & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

for $1 \leq k \leq 2m-1$. Note that $H(k,m)$ is a square matrix of size $(4m+1) \times$

$(4m+1)$, and it can be shown by induction that:

$$(H(1,m))^{\tau} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ S_{\tau+4m-1}^{(1,m)} & & & & \\ S_{\tau+4m-2}^{(1,m)} & & & & \\ S_{\tau+4m-3}^{(1,m)} & & & & \\ S_{\tau+4m-4}^{(1,m)} & & \left(M_m^{(1)}\right)^{\tau} & & \\ 0 & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \; for\ \tau \geq 1,$$

$$(H(\lambda,m))^{\tau} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ S_{\tau+4m-1}^{(\lambda,m)} & & & & \\ S_{\tau+4m-2}^{(\lambda,m)} & & & & \\ \vdots & & & & \\ S_{\tau+4m-2\lambda-2}^{(\lambda,m)} & & \left(M_m^{(\lambda)}\right)^{\tau} & & \\ 0 & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \text{ for } \tau \geq 1 \text{ and } 2 \leq \lambda \leq 2m-2$$

and

$$(H(2m-1,m))^{\tau} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ S_{\tau+4m-1}^{(2m-1,m)} & & & & \\ S_{\tau+4m-2}^{(2m-1,m)} & & & & \\ \vdots & & \left(M_m^{(2m-1)}\right)^{\tau} & & \\ S_{\tau+2}^{(2m-1,m)} & & & & \\ S_{\tau+1}^{(2m-1,m)} & & & & \\ S_{\tau}^{(2m-1,m)} & & & & \end{bmatrix} \; (\tau \geq 3).$$

Lemma 2.1. *The equation $x^{4m}+2x^3-1=0$ does not have multiple roots for any integer $m \geq 2$.*

Proof. Let $q(x) = x^{4m}+2x^3-1$ and suppose v is a multiple root of $q(x)$. Since $q(0) \neq 0$, it follows that $v \neq 0$. Then, the hypotheses $q(v)=0$ and $q'(v)=0$ imply $v^{4m-3} = -\frac{3}{2m}$ and $v^3 = \frac{2m}{4m-3}$, respectively. It follows that $v^3>0$ and $v^{4m-3}<0$, inequalities that cannot hold simultaneously for $m \geq 2$. This is a contradiction resulting from our assumption that v is a multiple root, which concludes the proof of the lemma. □

Let $q(v)$ be the characteristic polynomial of matrix $M_m^{(2m-1)}$. Then, $q(v) = v^{4m} + 2v^3 - 1$, a clear fact because $M_m^{(2m-1)}$ is a companion matrix.

Let v_1, v_2, …,v_{4m} be the eigenvalues of $M_m^{(2m-1)}$. By Lemma 2.1, we know that these are $4m$ distinct numbers. Let $V_m^{(2m-1)}$ be the following Vandermonde matrix:

$$V_m^{(2m-1)} = \begin{bmatrix} (v_1)^{4m-1} & (v_2)^{4m-1} & \cdots & (v_{4m})^{4m-1} \\ (v_1)^{4m-2} & (v_2)^{4m-2} & \cdots & (v_m)^{4m-2} \\ \vdots & \vdots & & \vdots \\ v_1 & v_2 & & v_{4m} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Denote by $V_m^{(2m-1)}(i,j)$ the matrix obtained from $V_m^{(2m-1)}$ by replacing the jth column by

$$C_m^{(2m-1)}(i,j) = \begin{bmatrix} v_1^{\tau+4m-i} \\ v_2^{\tau+4m-i} \\ \vdots \\ v_{4m}^{\tau+4m-i} \end{bmatrix}.$$

We can give the generalized Binet formula for the $(2m-1)$-adjacency-Pell-Hurwitz numbers with the following theorem.

Theorem 2.1. *For the matrix* $\left(M_m^{(2m-1)}\right)^\tau = \left[m_{i,j}^{(2m-1,\tau)}\right]_{4m\times 4m}$, *for* $\tau \geq 3$,

$$m_{i,j}^{(2m-1,\tau)} = \frac{\det V_m^{(2m-1)}(i,j)}{\det V_m^{(2m-1)}}. \tag{2.3}$$

Proof. Consider the integer $\tau \geq 3$ to be fixed. Since $v_1, v_2, \ldots, v_{4m}$ are distinct, the matrix $M_m^{(2m-1)}$ is diagonalizable. Then, $M_m^{(2m-1)} V_m^{(2m-1)} = V_m^{(2m-1)} D_m$, where $D_m = (v_1, v_2, \ldots, v_{4m})$. Since $\det V_m^{(2m-1)} \neq 0$, we can write

$$\left(V_m^{(2m-1)}\right)^{-1} M_m^{(2m-1)} V_m^{(2m-1)} = D_m.$$

Then, the matrix $M_m^{(2m-1)}$ is similar to D_m and so

$$\left(M_m^{(2m-1)}\right)^\tau V_m^{(2m-1)} = V_m^{(2m-1)} (D_m)^\tau .$$

We can now easily establish the following linear system of equations:

$$\begin{cases} m_{i,1}^{(2m-1,\tau)} (v_1)^{4m-1} + m_{i,2}^{(2m-1,\tau)} (v_1)^{4m-2} + \cdots + m_{i,4m}^{(2m-1,\tau)} = (v_1)^{n+4m-i} \\ m_{i,1}^{(2m-1,\tau)} (v_2)^{4m-1} + m_{i,2}^{(2m-1,\tau)} (v_2)^{4m-2} + \cdots + m_{i,4m}^{(2m-1,\tau)} = (v_2)^{n+4m-i} \\ \vdots \\ m_{i,1}^{(2m-1,\tau)} (v_{4m})^{4m-1} + m_{i,2}^{(2m-1,\tau)} (v_{4m})^{4m-2} + \cdots + m_{i,4m}^{(2m-1,\tau)} = (v_{4m})^{n+4m-i} \end{cases}.$$

The numbers in formula (2.3) are solutions of the last linear system. □

Theorem 2.1 gives immediately:

Corollary 2.2. *Let $x_\tau^{(2m-1,m)}$ be the τth element of the $(2m-1)$-adjacency-Pell-Hurwitz sequence, then*

$$x_\tau^{(2m-1,m)} = \frac{\det V_m^{(2m-1)}(4m,4m)}{\det V_m^{(2m-1)}} = \frac{\det V_m^{(2m-1)}(4m-1,4m-1)}{\det V_m^{(2m-1)}} = \frac{\det V_m^{(2m-1)}(4m-2,4m-2)}{\det V_m^{(2m-1)}}\ .$$

Now we consider the permanental representations of the k-adjacency-Pell-Hurwitz numbers.

Definition 2.1. *Let $M = [m_{i,j}]$ be $u \times v$ real matrix and let $r^1, r^2, \ldots, r^u$ and $c^1, c^2, \ldots, c^v$ be respectively, the row and column vectors of M. If r^α contains exactly two non-zero entries, then M is contractible on row α . Similarly, M is contractible on column β provided c^β contains exactly two non-zero entries.*

Let $x_1, x_2, \ldots, x_u$ be row vectors of the matrix M and let M be contractible in the α^{th} column with $m_{i,\alpha} \neq 0, m_{j,\alpha} \neq 0$ and $i \neq j$. Then the $(u-1) \times (v-1)$ matrix $M_{ij:\alpha}$ obtained from M by replacing the i^{th} row with $m_{i,\alpha}x_j + m_{j,\alpha}x_i$ and deleting the j^{th} row and the α^{th} column is called the contraction in the α^{th} column relative to the i^{th} row and the j^{th} row.

The permanent of a u-square matrix $A = [a_{i,j}]$ is defined by

$$\operatorname{per}(A) = \sum_{\sigma \in S_u} \prod_{i=1}^{u} a_{i,\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_u.

In [1], Brualdi and Gibson showed that $\operatorname{per}(A) = \operatorname{per}(B)$ if A is a real matrix of order $u > 1$ and B is a contraction of A.

Let $n \geq 4m$ and let $X(k,(m,n)) = \left[x_{i,j}^{m,n,k}\right]$, $1 \leq k \leq 2m-1$, be the $n \times n$ super-diagonal matrices defined using the following cases:

1. $i = \gamma$ and $j = \gamma + 2k - 2$ for $1 \leq \gamma \leq n - 2k - 1$,
2. $i = \gamma$, $j = \gamma + 2k + 1$ for $1 \leq \gamma \leq n - 2k - 1$, and $j = \gamma - 1$ for $2 \leq \gamma \leq n$,
3. otherwise.

$$x_{i,j}^{m,n,k} \begin{cases} -2 & \text{if case (1) applies} \\ 1 & \text{if case (2) applies} \\ 0 & \text{if case(3) applies} \end{cases},$$

where m is as in the definition of the k-adjacency-Pell-Hurwitz numbers. Then we have the following theorem.

Theorem 2.3. *For $1 \leq k \leq 2m-1$, we have*

$$\operatorname{per}(X\left(k,(m,n)\right)) = x_{4m+n}^{(k,m)}.$$

Proof. Consider the matrix $X\left(1,(m,n)\right)$. We will use induction on n. Assume the equation holds for $n \geq 4m$. Then we must show that the equation $\operatorname{per}(X\left(1,(m,n)\right)) = x_{4m+n}^{(1,m)}$ holds for $n+1$. If we expand $\operatorname{per}(X\left(1,(m,n)\right))$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\operatorname{per}(X\left(1,(m,n+1)\right)) = -2\operatorname{per}(X\left(1,(m,n)\right)) + \operatorname{per}(X\left(1,(m,n-3)\right)).$$

Since $\operatorname{per}(X\left(1,(m,n-3)\right)) = x_{4m+n-3}^{(1,m)}$, we obtain

$$\operatorname{per}(X\left(1,(m,n+1)\right)) = -2x_{4m+n}^{(1,m)} + x_{4m+n-3}^{(1,m)} = x_{4m+n+1}^{(1,m)}.$$

The proofs for $2 \leq k \leq 2m-1$ are similar to the above and are omitted. ☐

Let $n \geq 4m$. Now we define the $n \times n$ matrices $Y\left(k,(m,n)\right) = \left[y_{i,j}^{m,n,k}\right]$, $1 \leq k \leq 2m-1$, using the following cases:

1. $i = \delta$ and $j = \delta + 2k - 2$ for $1 \leq \delta \leq n - 2k - 1$,
2. $i = \delta$, $j = \delta + 2k - 2$ for $1 \leq \delta \leq n - 2k - 1$, and $j = \delta - 1$ for $2 \leq \delta \leq n$,
3. otherwise,

With these three cases in mind we now define the desired matrix:

$$y_{i,j}^{m,n,k} = \begin{cases} -2 & \text{if case (1) applies} \\ 1 & \text{if case (2) applies} \\ 0 & \text{if case(3) applies} \end{cases},$$

where m is as in the definition of the k-adjacency-Pell-Hurwitz numbers. In the next theorem we obtain another permanental representation.

Theorem 2.4. *For $1 \leq k \leq 2m-1$, we have*

$$\operatorname{per}(Y\left(k,(m,n)\right)) = x_{4m-2k-2+n}^{(k,m)}.$$

Proof. Consider matrices $Y\left(\lambda,(m,n)\right) = \left[y_{i,j}^{m,n,\lambda}\right]$, $2 \leq \lambda \leq 2m-2$. We will use induction on n. Suppose that the equation holds for $n \geq 4m$. Then we must show that the equation holds for $n+1$. If we expand the $perY(\lambda,(m,n))$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\begin{aligned}\operatorname{per}(Y(\lambda,(m,n+1))) &= -2\operatorname{per}(Y(\lambda,(m,n-2\lambda+2)))+\operatorname{per}(Y(\lambda,(m,n-2\lambda-1))) \\ &= -2x^{(\lambda,m)}_{4m-4\lambda+n}+x^{(\lambda,m)}_{4m-4\lambda+n-3}=x^{(\lambda,m)}_{4m-2\lambda+n-1}\end{aligned}$$

for $2\le\lambda\le 2m-2$. Thus, the conclusion is obtained.

The proofs for the matrices $Y(1,(m,n))$ and $Y(2m-1,(m,n))$ are similar. □

We now consider the sums of the k-adjacency-Pell-Hurwitz numbers by using their permanental representations. Let $n>4m$ and suppose that $Z(1,(m,n))$, $Z(\lambda,(m,n))$, $(2\le\lambda\le 2m-2)$ and $Z(2m-1,(m,n))$ are the $n\times n$ matrices defined by

$$Z(1,(m,n))=\begin{bmatrix} 1 & \cdots & \underset{\uparrow\,(n-4)\text{th}}{1} & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & Y(1,(m,n-1)) & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix}, \tag{2.4}$$

$$Z(\lambda,(m,n))=\begin{bmatrix} 1 & \cdots & \underset{\uparrow\,(n-\lambda-4)\text{th}}{1} & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & Y(\lambda,(m,n-1)) & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix} \tag{2.5}$$

and

$$Z(2m-1,(m,n))=\begin{bmatrix} 1 & \cdots & \underset{\uparrow\,(n-6)\text{th}}{1} & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & Y(2m-1,(m,n-1)) & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix}. \tag{2.6}$$

Then we have the following theorem.

Theorem 2.5. *For $1 \le k \le 2m-1$, let the $n \times n$ matrices $Z(k,(m,n))$ be as in (2.4),(2.5) and (2.6). Then*

$$\mathrm{per}(Z(1,(m,n))) = \sum_{\varepsilon=1}^{n+4m-1} x_{\varepsilon}^{(1,m)},$$

$$\mathrm{per}(Z(\lambda,(m,n))) = \sum_{\varepsilon=1}^{n+4m-2\lambda-3} x_{\varepsilon}^{(\lambda,m)}, \ (2 \le \lambda \le 2m-2),$$

and

$$\mathrm{per}(Z(2m-1,(m,n))) = \sum_{\varepsilon=1}^{n-1} x_{\varepsilon}^{(2m-1,m)}.$$

Proof. Consider the matrices $Z(2m-1,(m,n))$. Expanding $\mathrm{per}(Z(2m-1,(m,n)))$ with respect to the first row, we have

$$\mathrm{per}(Z(2m-1,(m,n))) = \mathrm{per}(Z(2m-1,(m,n-1))) + \mathrm{per}(Y(2m-1,(m,n-1))).$$

By Theorem 2.3 and the inductive argument on n, we easily complete the proof.

The proofs for $1 \le k \le 2m-2$ are similar to the above and are omitted. □

A matrix M is called *convertible* if there is an $n \times n$ $(1,-1)$-matrix K such that $perM = \det(M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K.

Let $n > 4m$ and let W be the $n \times n$ matrix defined by

$$W = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

It is easy to see that

$$\mathrm{per}(X(k,(m,n)) = \det(X(k,(m,n)) \circ W)),$$

$$\mathrm{per}(Y(k,(m,n)) = \det(Y(k,(m,n)) \circ W))$$

and

$$\mathrm{per}(Z(k,(m,n)) = \det(Z(k,(m,n)) \circ W))$$

for $n > 4m$ and $1 \le k \le 2m-1$.

Consider the $n \times n$ matrix

$$C = C(c_1, c_2, \ldots, c_n) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For detailed information about the companion matrix, see [11, p.69] and [12, p.284].

Theorem 2.6. *(Chen and Louck [2]).The (i,j) entry $c_{i,j}^{(\tau)}(c_1,c_2,\ldots,c_n)$ in the matrix $C^{\tau}(c_1,c_2,\ldots,c_n)$ is given by the following formula:*

$$c_{i,j}^{(\tau)}(c_1,c_2,\ldots,c_n)=\sum_{(t_1,t_2,\ldots,t_n)}\frac{t_j+t_{j+1}+\cdots+t_n}{t_1+t_2+\cdots+t_n}\times\binom{t_1+\cdots+t_n}{t_1,\ldots,t_n}c_1^{t_1}\cdots c_n^{t_n} \tag{2.7}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+nt_n=\tau-i+j$, $\binom{t_1+\cdots+t_n}{t_1,\ldots,t_n}=\frac{(t_1+\cdots+t_n)!}{t_1!\ldots t_n!}$ is a multinomial coefficient, and the coefficients in (2.7) are defined to be 1 if $\tau=i-j$.

Now we give the combinatorial representations for the $(2m-1)$-adjacency-Pell-Hurwitz sequence by the following Corollary.

Corollary 2.7. *Let $x_{\tau}^{(2m-1,m)}$ be the τth element of the $(2m-1)$-adjacency-Pell-Hurwitz sequence such that $\tau\geq 3$ and $m\geq 2$. Then*

$$\begin{aligned}
x_{\tau}^{(2m-1,m)} &= \sum_{(t_1,t_2\ldots,t_{4m})}\frac{t_{4m-2}+t_{4m-1}+t_{4m}}{t_1+t_2+\cdots+t_{4m}}\times\binom{t_1+\cdots+t_{4m}}{t_1,\ldots,t_{4m}}(-2)^{t_{4m-3}}\\
&= \sum_{(t_1,t_2\ldots,t_{4m})}\frac{t_{4m-1}+t_{4m}}{t_1+t_2+\cdots+t_{4m}}\times\binom{t_1+\cdots+t_{4m}}{t_1,\ldots,t_{4m}}(-2)^{t_{4m-3}}\\
&= \sum_{(t_1,t_2\ldots,t_{4m})}\frac{t_{4m}}{t_1+t_2+\cdots+t_{4m}}\times\binom{t_1+\cdots+t_{4m}}{t_1,\ldots,t_{4m}}(-2)^{t_{4m-3}},
\end{aligned}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(4m)\,t_{4m}=\tau$.

Proof. In Theorem 2.6, if we choose $n=4m$ and $i=j$ such that $4m-2\leq i,j\leq 4m$, then the proof follows from (2). □

Now we give the generating function of the k-adjacency-Pell-Hurwitz numbers.

Let

$$g^{(k,m)}(y)=x_{4m}^{(k,m)}+x_{4m+1}^{(k,m)}y+x_{4m+2}^{(k,m)}y^2+\cdots+x_{4m+u-1}^{(k,m)}y^u+x_{4m+u}^{(k,m)}y^{u+1}+\cdots.$$

Then

$$\begin{aligned}
2y^{2k-1}g^{(k,m)}(y) &= x_{4m}^{(k,m)}2y^{2k-1}+x_{4m+1}^{(k,m)}2y^{2k}+x_{4m+2}^{(k,m)}2y^{2k+1}\\
&\quad+\cdots+x_{4m+u-1}^{(k,m)}2y^{2k-1+u}+x_{4m+u}^{(k,m)}2y^{2k+u}+\cdots
\end{aligned}$$

and

$$y^{2k+2}g^{(k,m)}(y)=x_{4m}^{(k,m)}y^{2k+2}+x_{4m+1}^{(k,m)}y^{2k+3}+x_{4m+2}^{(k,m)}y^{2k+4}$$

$$+\cdots+x_{4m+u-1}^{(k,m)}y^{2k+2+u}+x_{4m+u}^{(k,m)}y^{2k+2+u+1}+\cdots.$$

Thus, we have

$$g^{(k,m)}\left(y\right)+2y^{2k-1}g^{(k,m)}\left(y\right)-y^{2k+2}g^{(k,m)}\left(y\right)=x_{4m}^{(k,m)}y^{4m-1}.$$

By the definition of the k-adjacency-Pell-Hurwitz numbers, we obtain

$$g^{(k,m)}\left(y\right)=\frac{y^{4m-1}}{1+2y^{2k-1}-y^{2k+2}},$$

where $1\leq k\leq 2m-1$ and $0\leqslant y^{2k+2}-2y^{2k-1}<1$.

Now we give an exponential representation for the k-adjacency-Pell-Hurwitz numbers.

Theorem 2.8. *For $1\leq k\leq 2m-1$ and $0\leqslant y^{2k+2}-2y^{2k-1}<1$, the k-adjacency-Pell-Hurwitz numbers have the following exponential representation:*

$$g^{(k,m)}\left(y\right)=y^{4m-1}\exp\left(\sum_{i=1}^{\infty}\frac{\left(y^{2k-1}\right)^{i}}{i}\left(y^{3}-2\right)^{i}\right).$$

Proof. Since

$$\ln g^{(k,m)}\left(y\right)=\ln\frac{y^{4m-1}}{1+2y^{2k-1}-y^{2k+2}}=\ln y^{4m-1}-\ln\left(1+2y^{2k-1}-y^{2k+2}\right)$$

and

$$\begin{aligned}\ln\left(1+2y^{2k-1}-y^{2k+2}\right) &= \ln\left(1-\left(y^{2k+2}-2y^{2k-1}\right)\right)\\ &= -[\left(y^{2k-1}\right)\left(y^{3}-2\right)+\frac{1}{2}\left(y^{2k-1}\right)^{2}\left(y^{3}-2\right)^{2}\\ &\quad +\cdots+\frac{1}{i}\left(y^{2k-1}\right)^{i}\left(y^{3}-2\right)^{i}+\cdots]\\ &= -\left(\sum_{i=1}^{\infty}\frac{\left(y^{2k-1}\right)^{i}}{i}\left(y^{3}-2\right)^{i}\right),\end{aligned}$$

we have

$$\ln g^{(k,m)}\left(y\right)-\ln y^{4m-1}=\sum_{i=1}^{\infty}\frac{\left(y^{2k-1}\right)^{i}}{i}\left(y^{3}-2\right)^{i}.$$

Therefore, we obtain

$$\ln\frac{g\left(x\right)}{y^{4m-1}}=\sum_{i=1}^{\infty}\frac{\left(y^{2k-1}\right)^{i}}{i}\left(y^{3}-2\right)^{i}.$$

The last formula implies the one in the text of the theorem, thus concluding the proof. □

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