POWERS OF TWO AS SUMS OF TWO PADOVAN NUMBERS

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Abstract

Let $(P_n)_{n\geqslant 0}$ be the *Padovan sequence* given by $P_0 = 1$, $P_1 = P_2 = 0$ and the recurrence formula $P_{n+3} = P_{n+1} + P_n$, for all $n \geqslant 0$. In this note we study and completely solve the Diophantine equations $P_n = 2^a$ and $P_n + P_m = 2^a$ in non-negative integers n, a and n, m, a, respectively.

1. Introduction

We recall that the *Fibonacci sequence* $(F_n)_{n\geqslant 0}$ is given by $F_0 = 0$, $F_1 = 1$ and the recurrence formula $F_{n+2} = F_{n+1} + F_n$ for all $n \geqslant 0$. Recently, the Diophantine equation

$$F_n + F_m = 2^a \tag{1}$$

in positive integers n, m and a has been studied. Indeed, in [6] Bravo and Luca prove the following result:

Theorem 1. *The only solutions of equation* (1) *in positive integers* (n, m, a) *with* $n \geqslant m$ *are given by*

$$2F_1 = 2, \quad 2F_2 = 2, \quad 2F_3 = 4, \quad 2F_6 = 16$$

and

$$F_2 + F_1 = 2, \quad F_4 + F_1 = 4, \quad F_4 + F_2 = 4, \quad F_5 + F_4 = 8, \quad F_7 + F_4 = 16.$$

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Then, they solve the same problem with Lucas and k-Fibonacci sequences (see [5] and [4]). Also, the problem of writing powers of two as sum of three Fibonacci numbers and three Pell numbers have been studied (see [3] and [2]). Inspired by this results in this note we study the same kind of problems with the Padovan sequence.

The *Padovan sequence* $(P_n)_{n\geqslant 0}$, named after the architect R. Padovan, is a ternary recurrence sequence given by $P_0 = 1$, $P_1 = P_2 = 0$ and the recurrence formula

$$P_{n+3} = P_{n+1} + P_n, \quad \text{for all} \quad n \geqslant 0. \tag{2}$$

This is the A000931 sequence in [13]. Its few first terms are

$$1,\ 0,\ 0,\ 1,\ 0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 3,\ 4,\ 5,\ 7,\ 9,\ 12,\ 16,\ 21,\ 28,\ 37,\ 49,\ldots$$

Some arithmetical problems with the Padovan sequence have been studied. For example, in [14], Stewart asks for the intersection of the Padovan and Fibonacci sequences. In [8], De Weger solves this problem. He actually proves that the distance between Padovan and Fibonacci numbers growths exponentially. In this paper, we shall study the following arithmetical problems with the Padovan sequence, namely, we study the Diophantine equations:

$$P_n = 2^a \tag{3}$$

and

$$P_n + P_m = 2^a \tag{4}$$

in non-negative integers n, m and a. Since $P_1 = P_2 = P_4 = 0$, we will assume that $n, m \neq 1, 2$. That is, whenever we think of 0 as a member of the Padovan sequence, we think of it as being P_4. We do the same with the terms 1 and 2: for 1, $n, m \neq 0, 3, 5, 6$, and we think of 1 as being P_7; for 2, $n, m \neq 8$ and thus we think of 2 as being P_9. With these conventions our results are

Theorem 2. *All solutions of equation* (3) *in non-negative integers* (n, a) *are given by*

$$P_7 = 2^0, \quad P_9 = 2^1, \quad P_{11} = 2^2 \quad \textit{and} \quad P_{16} = 2^4.$$

Theorem 3. *All solutions of equation* (4) *in non-negative integers* (n, m, a) *with* $n \geqslant m$ *are given by*

$$2P_7 = 2^1, \quad 2P_9 = 2^2, \quad 2P_{11} = 2^3, \quad 2P_{16} = 2^5;$$

$$P_7 + P_4 = 2^0, \quad P_9 + P_4 = 2^1, \quad P_{11} + P_4 = 2^2 = P_{10} + P_7,$$

$$P_{12} + P_{10} = 2^3 = P_{13} + P_7, \quad P_{14} + P_{13} = 2^4 = P_{15} + P_{11} = P_{16} + P_4,$$

$$P_{18} + P_{11} = 2^5, \quad P_{32} + P_{29} = 2^{11} = P_{33} + P_{24}.$$

2. Tools

In the proofs of Theorems 2 and 3 the first tool we need is a lower bound for linear forms in logarithms: we use a lower bound given by Matveev. Let α be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\alpha = \alpha^{(1)}, \ldots, \alpha^{(d)}$ denote its conjugates. The *logarithmic height* of α is defined as

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log\max\left\{|\alpha^{(i)}|, 1\right\}\right).$$

In particular, if $\alpha = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then

$$h(\alpha) = \log\max\{|p|, q\}.$$

The following are basic properties of the logarithmic height. For α, β algebraic numbers and $m \in \mathbb{Z}$ we have

- $h(\alpha + \beta) \leqslant h(\alpha) + h(\beta) + \log(2)$
- $h(\alpha\beta) \leqslant h(\alpha) + h(\beta)$
- $h(\alpha^m) = |m|h(\alpha)$

Now, let $\mathbb{K}$ be a real number field of degree $d_{\mathbb{K}}$, $\alpha_1, \ldots, \alpha_\ell \in \mathbb{K}$ positive elements and $b_1, \ldots, b_\ell \in \mathbb{Z} \setminus \{0\}$. Let $B \geqslant \max\{|b_1|, \ldots, |b_\ell|\}$ and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_\ell^{b_\ell} - 1.$$

Let $A_1, \ldots, A_\ell$ be real numbers with

$$A_i \geqslant \max\{d_{\mathbb{K}}\, h(\alpha_i),\ |\log\alpha_i|,\ 0.16\} \quad \text{for} \quad i = 1, 2, \ldots, \ell.$$

The following result is due to Matveev in [11] (see also Theorem 9.4 in [7]).

Theorem 4. *(Matveev's Theorem) Assume that $\Lambda \neq 0$. Then*

$$\log|\Lambda| > -1.4 \cdot 30^{\ell+3} \cdot \ell^{4.5} \cdot d_{\mathbb{K}}^2 \cdot (1 + \log d_{\mathbb{K}}) \cdot (1 + \log B) A_1 \cdots A_\ell.$$

In this note we always use $\ell = 3$. Further, $\mathbb{K} = \mathbb{Q}(\gamma)$ has degree $d_{\mathbb{K}} = 3$, where γ is defined at the begining of Section 3. Thus, once for all we fix the constant

$$C := 2.70444 \times 10^{12} > 1.4 \cdot 30^{3+3} \cdot 3^{4.5} \cdot 3^2 \cdot (1 + \log 3).$$

Our second tool is a version of the reduction method of Baker-Davenport based on Lemma in [1]. We shall use the one given by Bravo, Gómez and Luca in [4]. For a real number x, we write $\|x\|$ for the distance from x to the nearest integer.

Lemma 1. *Let M be a positive integer. Let $\tau,\ \mu,\ A>0,\ B>1$ be given real numbers. Assume that p/q is a convergent of τ such that $q>6M$ and $\varepsilon := \|\mu q\| - M\|\tau q\| > 0$. Then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < \frac{A}{B^w}$$

in positive integers u, v and w in the ranges

$$u \leqslant M \quad \textit{and} \quad w \geqslant \frac{\log(Aq/\varepsilon)}{\log(B)}.$$

Lemma 1 is a slight variation of the one given by Dujella and Pethő in [9]. Finally, the following result will be very useful. This is Lemma 7 in [10].

Lemma 2. *If $m \geqslant 1$, $T > (4m^2)^m$ and $T > x/(\log x)^m$. Then*

$$x < 2^m T(\log T)^m.$$

3. Proofs

We start with some basic properties of the Padovan sequence. For a complex number z, we write $\overline{z}$ for its complex conjugate. Let $\omega \neq 1$ be a cubic root of 1. Put

$$\gamma := \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}, \quad \delta := \omega\sqrt[3]{\frac{9+\sqrt{69}}{18}} + \overline{\omega}\sqrt[3]{\frac{9-\sqrt{69}}{18}}.$$

We see that $\gamma, \delta, \overline{\delta}$ are the three roots of the $\mathbb{Q}$-irreducible polynomial $X^3 - X - 1$. It is plain to see, by induction for example, the Binet formula

$$P_n = c_1\gamma^n + c_2\delta^n + c_3\overline{\delta}^n, \quad \text{for} \quad \text{all} \quad n \geqslant 0, \tag{5}$$

where

$$c_1 = \frac{1}{2\gamma+3}, \quad c_2 = \frac{1}{2\delta+3}, \quad c_3 = \overline{c_2}.$$

Formula (5) follows from the general theorem on linear recurrence sequences, since the above polynomial is the characteristic polynomial of the Padovan sequence (see for example Theorem C.1 in [12]). We note that

$$\gamma = 1.32471\ldots,\ |\delta| = 0.86883\ldots,\ c_1 = 0.17700\ldots,\ |c_2| = 0.49560\ldots$$

Further, we have the inequalities

$$\gamma^{n-7} \leqslant P_n < \gamma^{n-1} \tag{6}$$

which hold for all $n \geqslant 5$. These can be proved by induction.

Proof of Theorem 2. Let us start with the study of equation (3) in non-negative integers (n, a), where, as we have said, $n \neq 0, 1, 2, 3, 5, 6, 8$. We will assume $n \geqslant 7$. Then from (6) we have

$$\gamma^{n-7} \leqslant P_n = 2^a \qquad \text{and} \qquad \gamma^{n-1} > P_n = 2^a.$$

Thus

$$(n-7)\frac{\log \gamma}{\log 2} \leqslant a < (n-1)\frac{\log \gamma}{\log 2}.$$

Now, since $\dfrac{1}{3} < \dfrac{\log \gamma}{\log 2} < \dfrac{1}{2}$ we get that

$$\frac{2(n-7)}{3} < 2a < n. \tag{7}$$

If $n \leqslant 200$ then $a \leqslant 99$. Running a *Mathematica* program in the range $7 \leqslant n \leqslant 200$, $0 \leqslant a \leqslant 99$, with our conventions, we get all the posibilities listed in Theorem 2.

From now on we assume $n > 200$. Then from (7) we have $a > 64$. From (5) we rewrite (3) as

$$|2^a - c_1\gamma^n| \leqslant 2|c_2||\delta|^n < 1.$$

Dividing through by $c_1\gamma^n$ we obtain

$$|2^a\gamma^{-n}c_1^{-1} - 1| < \frac{1}{c_1\gamma^n} < \frac{1}{\gamma^{n-7}}, \tag{8}$$

since $1 < c_1\gamma^7$. Put $\Lambda := 2^a\gamma^{-n}c_1^{-1} - 1$. We claim that $\Lambda \neq 0$. To see this, we consider the $\mathbb{Q}-$automorphism σ of the Galois extension $\mathbb{Q}(\gamma, \delta)$ over $\mathbb{Q}$ defined by $\sigma(\gamma) = \delta$ and $\sigma(\delta) = \gamma$. We note $\sigma(\overline{\delta}) = \overline{\delta}$. If $\Lambda = 0$ then $\sigma(\Lambda) = 0$. Thus

$$2^a = \sigma(c_1\gamma^n) = c_2\delta^n.$$

By taking absolute values and since $|c_2|,\ |\delta| < 1$, $a > 64$, we obtain

$$1 < 2^a = |c_2||\delta|^n < 1,$$

which is a contradiction. Thus $\Lambda \neq 0$. Now, we will apply Matveev's Theorem to Λ. To do this, we take

$$\alpha_1 = 2,\ \alpha_2 = \gamma,\ \alpha_3 = c_1 \in \mathbb{K}, \quad b_1 = a,\ b_2 = -n,\ b_3 = -1.$$

Thus $B = n$. Further, $h(\alpha_1) = \log 2$, $h(\alpha_2) = \log \gamma/3$. For α_3 we use the properties of the height and we deduce that

$$h(\alpha_3) \leqslant \frac{\log \gamma}{3} + 4\log 2.$$

Then we choose $A_1 = 2.1, A_2 = 0.3$ and $A_3 = 8.6$. From Theorem 4 we get

$$\log|\Lambda| > -C \cdot (1+\log n) \cdot 2.1 \cdot 0.3 \cdot 8.6 > -1.46528 \times 10^{13}(1+\log n),$$

which compared with (8) yields

$$(n-7)\log\gamma < 1.46528 \times 10^{13}(2\log n).$$

Thus $n < 1.04217 \times 10^{14} \log n$ and from Lemma 2 we obtain

$$n < 6.72773 \times 10^{15}. \tag{9}$$

Now we will reduce this upper bound on n. To do this, we consider

$$\Gamma = a\log 2 - n\log\gamma + \log(1/c_1)$$

and we go to (8). Note that $e^{\Gamma} - 1 = \Lambda$. Since $\Lambda \neq 0$, $\Gamma \neq 0$. If $\Gamma > 0$ then we obtain

$$0 < \Gamma \leqslant e^{\Gamma} - 1 = |e^{\Gamma} - 1| = |\Lambda| < \frac{1}{\gamma^{n-7}}.$$

If, for another hand $\Gamma < 0$ we have that $1 - e^{\Gamma} = |e^{\Gamma} - 1| = |\Lambda| < 1/2$, since $n > 200$. Then $e^{|\Gamma|} < 2$. Thus

$$0 < |\Gamma| < e^{|\Gamma|} - 1 = e^{|\Gamma|}|\Lambda| < \frac{2}{\gamma^{n-7}}.$$

So, in both cases we have that

$$0 < |\Gamma| < \frac{2}{\gamma^{n-7}}$$

Replacing Γ in the above inequality by its formula and dividing through by $\log\gamma$ we get

$$0 < |a\tau - n + \mu| < \frac{51}{\gamma^n}, \tag{10}$$

where

$$\tau := \frac{\log 2}{\log\gamma} \quad \text{and} \quad \mu := \frac{\log(1/c_1)}{\log\gamma}.$$

Put $M := 3.36387 \times 10^{15}$. From (9) we see that M is the upper bound on a since $2a < n$. With a *Mathematica* program we find that the denominator of the convergent

$$\frac{p_{42}}{q_{42}} = \frac{1814208205674503586}{735997475682980473}$$

of τ satisfies $q_{42} > 6M$ and that $\varepsilon = \|q_{42}\,\mu\| - M\|q_{42}\,\tau\| = 0.264486 > 0$. Thus, from Lemma 1 applied to (10) with $A := 51, B := \gamma$ we have that

$$n < \frac{\log(51\,q_{42}/\varepsilon)}{\log\gamma} < 166,$$

which contradicts our assumption on n. This finishes the proof of Theorem 2. $\square$

Proof of Theorem 3. We now study equation (4) in non-negative integers (n, m, a), where, $n, m \neq 0, 1, 2, 3, 5, 6, 8$. If $n = m$, we have $P_n = 2^{a-1}$. Then from Theorem 2 we get the first row of the solutions listed in Theorem 3. Thus we assume $n > m$. Since the case $m = 4$ corresponds to those of Theorem 2, from now on we assume that $n > m \geqslant 7$. From (6) we get

$$\gamma^{n-7} < P_n < P_n + P_m = 2^a \qquad \text{and} \qquad 2^n > 2\gamma^{n-1} > 2P_n > P_n + P_m = 2^a,$$

since $2 > \gamma$. Then

$$\gamma^{n-7} < 2^a < 2^n.$$

Again, since $\frac{1}{3} < \frac{\log \gamma}{\log 2}$ we have

$$\frac{(n-7)}{3} < a < n. \tag{11}$$

If $n \leqslant 350$ then we get $a \leqslant 349$. We ran a *Mathematica* program in the range $7 \leqslant m < n \leqslant 350$, $0 \leqslant a \leqslant 349$, with our conventions, we obtained the remainder solutions listed in Theorem 3.

From now on we assume $n > 350$. Then from (11) we have $a > 114$. From the Binet formula (5) we rewrite (4) as

$$|2^a - c_1\gamma^n| < 2|c_2||\delta|^n + \gamma^{m-1} < 1 + \gamma^{m-1} < 2\gamma^{m-1} < \gamma^{m+2},$$

where we use $2|c_2|, |\delta| < 1$ and $2 < \gamma^3$. Dividing through by $c_1\gamma^n$ we get

$$|2^a\gamma^{-n}c_1^{-1} - 1| < \frac{1}{\gamma^{n-m-9}}, \tag{12}$$

since $\gamma^2 < c_1\gamma^9$. Put $\Lambda_1 := 2^a\gamma^{-n}c_1^{-1} - 1$. We note that $\Lambda_1 = \Lambda$ where Λ is the one given in the proof of Theorem 2. Since in this case $a > 114$, we also have $\Lambda_1 \neq 0$. Actually, since we are studying equation (4) we can prove that $\Lambda_1 > 0$. Indeed, by rewriting (5) we get

$$c_1\gamma^n = |c_1\gamma^n| = |P_n - c_2\delta^n - c_3\overline{\delta}^n| \leqslant P_n + 2|c_2||\delta|^n < P_n + 1 \leqslant P_n + P_m = 2^a,$$

since, $m \geqslant 7$. As we have noted $\Lambda = \Lambda_1$, we use the same $\alpha_1,\ \alpha_2,\ \alpha_3 \in \mathbb{K}$ and the same $b_1,\ b_2,\ b_3,\ A_1,\ A_2,\ A_3$ to apply Matveev's theorem to Λ_1. Thus we obtain

$$(n-m)\log\gamma < 1.46528 \times 10^{13}(1 + \log n). \tag{13}$$

Now we get a bound on n. From the Binet formula (5) we rewrite our equation (4) as

$$\left|2^a - c_1(\gamma^{n-m}+1)\gamma^m\right| = |c_2\delta^n + c_3\overline{\delta}^n + c_2\delta^m + c_3\overline{\delta}^m| < 4|c_2||\delta|^7 < 1$$

Dividing through by $c_1(\gamma^{n-m}+1)\gamma^m$ we get

$$\left|2^a\gamma^{-m}\left(c_1(\gamma^{n-m}+1)\right)^{-1}-1\right| < \frac{1}{c_1(\gamma^n+\gamma^m)} < \frac{1}{\gamma^{n-7}}, \tag{14}$$

since $1 < c_1\gamma^7$. Put $\Lambda_2 := 2^a\gamma^{-m}\left(c_1(\gamma^{n-m}+1)\right)^{-1}-1$. We claim that $\Lambda_2 \neq 0$. To see this we consider the same $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{Q}(\gamma,\delta)$ over $\mathbb{Q}$ given in the proof of Theorem 2. If $\Lambda_2 = 0$ then $\sigma(\Lambda_2) = 0$. Thus

$$2^a = |c_2\left(\delta^n+\delta^m\right)| \leqslant 2|c_2||\delta|^7 < 1$$

which is absurd since $a > 114$. Now we apply Matveev's Theorem to Λ_2. To do this, we take

$$\alpha_1 = 2,\ \alpha_2 = \gamma,\ \alpha_3 = c_1(\gamma^{n-m}+1) \in \mathbb{K}, \quad b_1 = a,\ b_2 = -m,\ b_3 = -1.$$

Thus $B = n$. Now, $h(\alpha_1)$, $h(\alpha_2)$ are already calculated. For α_3 we use the height properties and (13) to conclude that

$$h(\alpha_3) \leqslant \frac{1.46529\times 10^{13}}{3}(1+\log n).$$

Thus, we take A_1, A_2 as above and $A_3 = 1.46529\times 10^{13}(1+\log n)$. From Matveev's theorem we obtain

$$\log|\Lambda_2| > -C\cdot(1+\log n)\cdot 2.1\cdot 0.3\cdot\left(1.46529\times 10^{13}(1+\log n)\right),$$

which compared with (14) yields $n < 3.5513\times 10^{26}\,(\log n)^2$ and from Lemma 2 we get

$$n < 5.30909\times 10^{30}.$$

Now we will reduce the upper bound on $n-m$. To do this we go to equation (12) and consider

$$\Gamma_1 = a\log 2 - n\log\gamma + \log(1/c_1).$$

Note that $e^{\Gamma_1}-1 = \Lambda_1 > 0$. Then $\Gamma_1 > 0$ and we have

$$0 < \Gamma_1 < e^{\Gamma_1}-1 = |\Lambda_1| < \frac{1}{\gamma^{n-m-9}}.$$

Dividing through by $\log\gamma$ we get

$$0 < |a\tau - n + \mu| < \frac{45}{\gamma^{n-m}}, \tag{15}$$

where

$$\tau := \frac{\log 2}{\log\gamma} \quad \text{and} \quad \mu := \frac{\log(1/c_1)}{\log\gamma}.$$

Let $M := 5.30909 \times 10^{30}$. Since $a < n$, $a < M$. With a *Mathematica* program, we find that the denominator of the convergent

$$\frac{p_{80}}{q_{80}} = \frac{361887494861952880596116858035559963}{146812412085088870866736031482146999}$$

of τ satisfies that $q_{80} > 6M$ and that $\varepsilon = \|q_{80}\,\mu\| - M\|q_{80}\,\tau\| = 0.231411 > 0$. Now, from Lemma 1 applied to inequality (15) with $A := 45$, $B := \gamma$ we get

$$n - m < \frac{\log(q_{80}\,45/\varepsilon)}{\log\gamma} < 299.$$

We will reduce the upper bound on n. To do this we go to equation (14) and consider

$$\Gamma_2 = a\log 2 - m\log\gamma + \log(1/c_1(\gamma^{n-m}+1)).$$

Note that $e^{\Gamma_2} - 1 = \Lambda_2 \neq 0$. Thus $\Gamma_2 \neq 0$. If $\Gamma_2 > 0$, then we obtain

$$0 < \Gamma_2 \leqslant e^{\Gamma_2} - 1 = |\Lambda_2| < \frac{1}{\gamma^{n-7}}.$$

From another hand, if $\Gamma_2 < 0$ then we have that $1 - e^{\Gamma_2} = |e^{\Gamma_2} - 1| = |\Lambda_2| < 1/2$ since $n > 350$. Then $e^{|\Gamma_2|} < 2$. Thus

$$0 < |\Gamma_2| < e^{|\Gamma_2|} - 1 = e^{|\Gamma_2|}|\Lambda_2| < \frac{2}{\gamma^{n-7}}.$$

So, in both cases we have that

$$0 < |\Gamma_2| < \frac{2}{\gamma^{n-7}}.$$

Replacing Γ_2 in the above inequality by its formula and dividing through by $\log\gamma$ we get

$$0 < |a\tau - m + \mu| < \frac{51}{\gamma^n}, \tag{16}$$

where τ is as above and

$$\mu := \frac{\log(1/c_1(\gamma^{n-m}+1))}{\log\gamma}.$$

Now, we use Lemma 1 again. Consider

$$\mu_\ell = \frac{\log(1/c_1(\gamma^\ell+1))}{\log\gamma}, \quad \ell = 1, 2, \ldots, 298.$$

With *Mathematica* we find that the same 80-th convergent above of τ well works for all values of ℓ, except to the case $\ell = 11$. That is, $q_{80} > 6\,M$ and $\varepsilon_\ell > 0.00485795 > 0$ for all $\ell = 1, 2 \ldots, 298$ except to the case $\ell = 11$. With $A := 51$, $B := \gamma$ we calculated

$\log(q_{80}\,51/\varepsilon_\ell)/\log\gamma$ for each of these ε_ℓ, and we found that the maximum value of them is less than or equal to 312.

The problem in the case $\ell = 11$ is that ε_{11} is always less than 0. The reason for this is that we have the identity

$$\frac{1}{\gamma^5} = \frac{2\gamma+3}{\gamma^{11}+1}.$$

Thus inequality (16) is

$$0 < |a\tau - (m+5)| < \frac{51}{\gamma^n}$$

and we use the theory of continued fractions to study it. Since $n > 350$ it follows that $\gamma^n > 102M > 102a$. Thus from Legendre's theorem we have that $(m+5)/a$ is a i-th convergent of τ and therefore

$$\frac{1}{a^2(a_{i+1}+2)} < \left|\tau - \frac{m+5}{a}\right|.$$

A quick computation with *Mathematica* reveals that $q_{70} \leqslant M < q_{71}$ and that $b := \max\{a_1, \ldots, a_{71}\} = 80$. In particular $b \geqslant a_{i+1}$. Thus by combining the above inequalities we get

$$\gamma^n < M \cdot 51 \cdot 82,$$

which implies $n < 282$. Thus, by combining the above result with this remaining case we conclude that $n \leqslant 312$. This contradicts our assumption on n and finishes the proof of Theorem 3. $\square$

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References

[1] A. Baker and H. Davenport, The equations $3X^2 - 2 = Y^2$ and $8X^2 - 7 = Z^2$, *Q. J. Math.* **20** (2) (1969), 129–137.

[2] E. F. Bravo and J.J. Bravo, Powers of two as sums of three Fibonacci numbers, *Lith. Math. J.* **55** (3) (2015), 301–311.

[3] J.J. Bravo, B. Faye, and F. Luca, Powers of two as sums of three Pell numbers, *Taiwanese J. Math.* **21** (4) (2017), 739–751.

[4] J.J. Bravo, C. A. Gómez, and F. Luca, Powers of two as sums of two k-Fibonacci numbers, *Miskolc Math. Notes* **17** (1) (2016), 85–100.

[5] J.J. Bravo and F. Luca, Powers of two as sums of two Lucas numbers, *J. Integer Seq.* **17** (2014) Article 14.8.3, 12 pp.

[6] J.J. Bravo and F. Luca, On the Diophantine equation $F_n + F_m = 2^a$, *Quaest. Math.* **39** (3) (2016), 391–400.

[7] Y. Bugeaud, M. Mignotte, and S. Siksek, Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers, *Ann. of Math. (2)* **163** (3) (2006), 969–1018.

[8] B.M.M. De Weger, Padua and Pisa are exponentially far apart, *Publ. Mat.* **41** (1997), 631–651.

[9] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Q. J. Math.* **49** (3) (1998), 291–306.

[10] S. Guzmán Sánchez and F. Luca, Linear combinations of factorials and S-units in a binary recurrence sequence, *Ann. Math. Qué.* **38** (2014), 169–188.

[11] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (6) (2000), 125–180. Traslation in *Izv. Math.* **64** (6) (2000), 1217–1269.

[12] T.N. Shorey and R. Tijdeman, *Exponential Diophantine equations*, Cambridge University Press, 1986.

[13] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/

[14] I. Stewart, Mathematical Recreations: Tales of a neglected number, *Scientific American* **274** (1996), 92–93.