ETA QUOTIENTS, EISENSTEIN SERIES AND ELLIPTIC CURVES

Ayşe Alaca
School of Mathematics and Statistics, Carleton University, Ottawa, Canada
aysealaca@cunet.carleton.ca

Şaban Alaca
School of Mathematics and Statistics, Carleton University, Ottawa, Canada
sabanalaca@cunet.carleton.ca

Zafer Selcuk Aygin
School of Mathematics and Statistics, Carleton University, Ottawa, Canada
zaferaygin@cunet.carleton.ca

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Abstract

We express the newforms in $S_2(\Gamma_0(N))$ with $N = 30$, 33, 34, 35, 38, 40, 42, 44, 45 and 56 as linear combinations of eta quotients and Eisenstein series and list their corresponding strong Weil curves. Then we give generating functions for the group orders of these strong Weil curves on $\mathbb{Z}_p$, where p is a prime. At the end, we use arithmetic properties of our generating functions to deduce some beautiful congruences for these group orders.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the sets of positive integers, integers, rational numbers and complex numbers, respectively. Let $N \in \mathbb{N}$. Let $\Gamma_0(N)$ be the modular subgroup defined by

$$\Gamma_0(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}.$$

We write $M_2(\Gamma_0(N))$ to denote the space of modular forms of weight 2 for $\Gamma_0(N)$ and $S_2(\Gamma_0(N))$ to denote the subspace of cusp forms of $M_2(\Gamma_0(N))$.

The *Dedekind eta function* $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by the product formula

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}).$$

A product of the form

$$f(z) = \prod_{1 \le \delta | N} \eta^{r_\delta}(\delta z),$$

where $r_\delta \in \mathbb{Z}$, not all zero, is called an *eta quotient*. As in [6] we use the notation $q := q(z) = e^{2\pi i z}$. We set $[n]f(z) := a_n$ for $f(z) = \sum_{n \in \mathbb{Z}} a_n q^n$.

Martin and Ono [8] listed all the newforms in $S_2(\Gamma_0(N))$ that are eta quotients and gave their corresponding strong Weil curves. There are such eta quotients only for levels $N = 11$, 14, 15, 20, 24, 27, 32, 36, 48, 64, 80 and 144.

It is well-known that for modular forms we have

$$f(z) = e_f(z) + o_f(z) + n_f(z)$$

where $e_f(z)$, $o_f(z)$ and $n_f(z)$ denote the Eisenstein part, oldform part and newform part of the modular form $f(z)$, respectively. Our approach to finding newforms in $M_2(\Gamma_0(N))$ is taking an eta quotient from the space and extracting the Eisenstein part and oldform part. If the dimension of the newform space is greater than 1 then the resulting newform part might be a linear combination of newforms. Such a case occurs when $N = 38$ and 56, where we successfully manage to find the right eta quotients to isolate two different newforms in each level.

In this paper we use the method described above to express the newforms in $S_2(\Gamma_0(N))$ with $N = 30$, 33, 34 35, 38, 40, 42, 44, 45 and 56 as linear combinations of eta quotients and Eisenstein series, and give their corresponding strong Weil curves. Let $E(\mathbb{Z}_p)$ denote the group of algebraic points of an elliptic curve E over $\mathbb{Z}_p$, where p is a prime. We give generating functions for the group orders $|E(\mathbb{Z}_p)|$ of certain strong Weil curves in terms of eta quotients and Eisenstein series (see Theorem 4). We then use our generating functions to deduce some beautiful congruences for these group orders (see Corollary 1).

2. Preliminary Results

Appealing to [9, Theorem 1.64, p. 18] and [6, Corollary 2.3, p. 37] (see also [1, 5, 7]), one can show that

$$\frac{\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z)}{\eta(2z)\eta(5z)\eta(6z)\eta(15z)} \in M_2(\Gamma_0(30)), \tag{1}$$

$$\frac{\eta^3(3z)\eta^3(33z)}{\eta(z)\eta(11z)}, \eta^2(3z)\eta^2(33z), \eta(z)\eta(3z)\eta(11z)\eta(33z) \in M_2(\Gamma_0(33)),$$

$$\frac{\eta^3(5z)\eta^3(7z)}{\eta(z)\eta(35z)}, \frac{\eta^3(z)\eta^3(35z)}{\eta(5z)\eta(7z)} \in M_2(\Gamma_0(35)),$$

$$\frac{\eta^4(2z)\eta^4(38z)}{\eta^2(z)\eta^2(19z)}, \frac{\eta^3(2z)\eta^3(19z)}{\eta(z)\eta(38z)}, \frac{\eta^3(z)\eta^3(38z)}{\eta(2z)\eta(19z)}, \frac{\eta^4(z)\eta^4(19z)}{\eta^2(2z)\eta^2(38z)} \in M_2(\Gamma_0(38)),$$
$$\frac{\eta^2(z)\eta^2(5z)\eta^2(8z)\eta^2(40z)}{\eta(2z)\eta(4z)\eta(10z)\eta(20z)} \in M_2(\Gamma_0(40)),$$
$$\frac{\eta^2(2z)\eta^2(3z)\eta^2(14z)\eta^2(21z)}{\eta(z)\eta(6z)\eta(7z)\eta(42z)}, \frac{\eta^2(z)\eta^2(6z)\eta^2(7z)\eta^2(42z)}{\eta(2z)\eta(3z)\eta(14z)\eta(21z)} \in M_2(\Gamma_0(42)),$$
$$\frac{\eta^3(4z)\eta^3(11z)}{\eta(z)\eta(44z)}, \frac{\eta^3(z)\eta^3(44z)}{\eta(4z)\eta(11z)} \in M_2(\Gamma_0(44)),$$
$$\frac{\eta(3z)\eta^2(5z)\eta^2(9z)\eta(15z)}{\eta(z)\eta(45z)}, \frac{\eta^2(z)\eta(3z)\eta(15z)\eta^2(45z)}{\eta(5z)\eta(9z)} \in M_2(\Gamma_0(45)),$$
$$\frac{\eta(4z)\eta^3(14z)\eta(28z)}{\eta(2z)}, \frac{\eta^{10}(4z)\eta^4(56z)}{\eta^4(2z)\eta^4(8z)\eta^2(28z)}, \frac{\eta^3(4z)\eta(8z)\eta^2(14z)\eta(56z)}{\eta^2(2z)\eta(28z)},$$
$$\frac{\eta(2z)\eta^2(4z)\eta^6(28z)}{\eta(8z)\eta^3(14z)\eta(56z)} \in M_2(\Gamma_0(56)),$$
$$\frac{\eta^4(4z)\eta^4(68z)}{\eta^2(2z)\eta^2(34z)}, \frac{\eta^{10}(2z)\eta^{10}(34z)}{\eta^4(z)\eta^4(4z)\eta^4(17z)\eta^4(68z)} \in M_2(\Gamma_0(68)).$$

The Eisenstein series $L(z)$ is defined as

$$L(z) := -\frac{1}{24} + \sum_{n>0} \sigma(n)q^n,$$

where $\sigma(n) = \sum_{0<m|n} m$ is the sum of divisors function. By [11, Theorem 5.8] we have

$$L_t(z) := L(z) - tL(tz) \in M_2(\Gamma_0(N)) \text{ for all } 1 \leq t \mid N. \tag{2}$$

Below we state the Sturm theorem specialized for $M_2(\Gamma_0(N))$. The following theorem can be used to show the equality of given modular forms.

Theorem 1. [5, Theorem 3.13] *Let $f(z)$, $g(z) \in M_2(\Gamma_0(N))$ have the Fourier series expansions*

$$f(z) = \sum_{n=0}^{\infty} a_n q^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n q^n.$$

The Sturm bound $S(N)$ for the modular space $M_2(\Gamma_0(N))$ is given by

$$S(N) = \frac{N}{6} \prod_{p|N} \big(1 + 1/p\big), \tag{3}$$

and so if $a_n = b_n$ for all $n \leq S(2k)$ then $f(z) = g(z)$.

Using (3) we compute

$$S(30) = 12,\ S(33) = 8,\ S(35) = 8,\ S(34) = 9,\ S(38) = 10, \quad (4)$$
$$S(40) = 12,\ S(42) = 16,\ S(44) = 12,\ S(45) = 12,\ S(56) = 16.$$

3. Newforms in $S_2(\Gamma_0(N))$ for $N = 30, 33, 34, 35, 38, 40, 42, 44, 45, 56$ as Linear Combinations of Eta Quotients and Eisenstein Series

Theorem 2. *Let $N \in \{30, 33, 34, 35, 38, 40, 42, 44, 45, 56\}$. In* Table 1 *below we express all the newforms $F_N(z)$ in $S_2(\Gamma_0(N))$ as linear combinations of eta quotients and Eisenstein series.*

Level	Newform	Eta quotients and Eisenstein series
30	$F_{30}(z) =$	$6\dfrac{\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z)}{\eta(2z)\eta(5z)\eta(6z)\eta(15z)} + 2L_2(z) + L_3(z)$ $+\dfrac{1}{5}L_5(z) - 2L_6(z) - \dfrac{2}{5}L_{10}(z) - \dfrac{1}{5}L_{15}(z) + \dfrac{2}{5}L_{30}(z)$
33	$F_{33}(z) =$	$-10\dfrac{\eta^3(3z)\eta^3(33z)}{\eta(z)\eta(11z)} - 6\eta^2(3z)\eta^2(33z)$ $-2\eta(z)\eta(3z)\eta(11z)\eta(33z) + \dfrac{1}{3}L_3(z) + L_{11}(z) - \dfrac{1}{3}L_{33}(z)$
34	$F_{34}(z) =$	$6\dfrac{\eta^4(4z)\eta^4(68z)}{\eta^2(2z)\eta^2(34z)} + \dfrac{3}{8}\dfrac{\eta^{10}(2z)\eta^{10}(34z)}{\eta^4(z)\eta^4(4z)\eta^4(17z)\eta^4(68z)}$ $-\dfrac{1}{2}L_2(z) + \dfrac{1}{2}L_{17}(z) - \dfrac{1}{2}L_{34}(z)$
35	$F_{35}(z) =$	$3\dfrac{\eta^3(5z)\eta^3(7z)}{\eta(z)\eta(35z)} - \dfrac{\eta^3(z)\eta^3(35z)}{\eta(5z)\eta(7z)}$ $+\dfrac{4}{5}L_5(z) - \dfrac{5}{7}L_7(z) - \dfrac{73}{35}L_{35}(z)$
38	$F_{38A}(z) =$	$\dfrac{3}{7}\dfrac{\eta^3(z)\eta^3(38z)}{\eta(2z)\eta(19z)} - \dfrac{18}{7}\dfrac{\eta^4(2z)\eta^4(38z)}{\eta^2(z)\eta^2(19z)} - \dfrac{3}{7}\dfrac{\eta^3(2z)\eta^3(19z)}{\eta(z)\eta(38z)}$

Level	Newform	Eta quotients and Eisenstein series
		$-\frac{9}{28}\frac{\eta^4(z)\eta^4(19z)}{\eta^2(2z)\eta^2(38z)} - \frac{1}{7}L_2(z) + \frac{1}{7}L_{19}(z) + \frac{1}{7}L_{38}(z)$
38	$F_{38B}(z) =$	$\frac{\eta^3(2z)\eta^3(19z)}{\eta(z)\eta(38z)} + \frac{\eta^3(z)\eta^3(38z)}{\eta(2z)\eta(19z)}$
40	$F_{40}(z) =$	$-4\frac{\eta^2(z)\eta^2(5z)\eta^2(8z)\eta^2(40z)}{\eta(2z)\eta(4z)\eta(10z)\eta(20z)} + \frac{3}{2}L_2(z) + \frac{3}{2}L_4(z) + L_5(z)$ $-L_8(z) - \frac{3}{2}L_{10}(z) - \frac{3}{2}L_{20}(z) + L_{40}(z)$
42	$F_{42}(z) =$	$-8\frac{\eta(2z)\eta(3z)\eta^2(7z)\eta^2(42z)}{\eta(z)\eta(6z)} - 8\frac{\eta(z)\eta(6z)\eta^2(14z)\eta^2(21z)}{\eta(2z)\eta(3z)}$ $+L_2(z) - L_3(z) + L_6(z) + \frac{1}{7}L_7(z) - \frac{1}{7}L_{14}(z)$ $+\frac{1}{7}L_{21}(z) - \frac{1}{7}L_{42}(z)$
44	$F_{44}(z) =$	$3\frac{\eta^3(4z)\eta^3(11z)}{\eta(z)\eta(44z)} - 3\frac{\eta^3(z)\eta^3(44z)}{\eta(4z)\eta(11z)} - 2L_4(z)$ $+2L_{11}(z) - 2L_{44}(z)$
45	$F_{45}(z) =$	$2\frac{\eta(3z)\eta^2(5z)\eta^2(9z)\eta(15z)}{\eta(z)\eta(45z)} - 2\frac{\eta^2(z)\eta(3z)\eta(15z)\eta^2(45z)}{\eta(5z)\eta(9z)}$ $+L_5(z) - \frac{2}{3}L_9(z) - \frac{2}{5}L_{15}(z) - \frac{14}{15}L_{45}(z)$
56	$F_{56A}(z) =$	$-4\frac{\eta(4z)\eta^3(14z)\eta(28z)}{\eta(2z)} - 2\frac{\eta^{10}(4z)\eta^4(56z)}{\eta^4(2z)\eta^4(8z)\eta^2(28z)}$ $+\frac{3}{2}L_2(z) - \frac{1}{2}L_4(z) + \frac{3}{7}L_7(z) - \frac{9}{14}L_{14}(z) + \frac{3}{14}L_{28}(z)$
56	$F_{56B}(z) =$	$-4\frac{\eta^3(4z)\eta(8z)\eta^2(14z)\eta(56z)}{\eta^2(2z)\eta(28z)} + 2\frac{\eta(2z)\eta^2(4z)\eta^6(28z)}{\eta(8z)\eta^3(14z)\eta(56z)}$

Level	Newform	Eta quotients and Eisenstein series
		$+\frac{3}{2}L_2(z) - \frac{1}{2}L_4(z) - L_7(z) + \frac{3}{2}L_{14}(z) - \frac{1}{2}L_{28}(z)$

Table 1: Newforms in $S_2(\Gamma_0(N))$ for $N = 30, 33, 34, 35, 38, 40, 42, 44, 45, 56$

Proof. In [3, Table 3] each newform in $S_2(\Gamma_0(N))$ for $N < 1000$ has been given by listing its Fourier coefficients for primes up to 100. Using the results from [3, p. 25] together with [3, Table 3] we determine the first $S(N) + 1$ terms of the Fourier series expansions of all the newforms in $S_2(\Gamma_0(N))$ for $N = 30$, 33, 34 35, 38, 40, 42, 44, 45, 56. We give them in Table 2 below.

N	First $S(N) + 1$ terms of the newforms in $S_2(\Gamma_0(N))$
30	$q - q^2 + q^3 + q^4 - q^5 - q^6 - 4q^7 - q^8 + q^9 + q^{10} + q^{12} + O(q^{13})$,
33	$q + q^2 - q^3 - q^4 - 2q^5 - q^6 + 4q^7 - 3q^8 + O(q^9)$,
34	$q + q^2 - 2q^3 + q^4 - 2q^6 - 4q^7 + q^8 + O(q^9)$
35	$q + q^3 - 2q^4 - q^5 + q^7 + O(q^9)$,
$38A$	$q - q^2 + q^3 + q^4 - q^6 - q^7 - q^8 - 2q^9 + O(q^{11})$,
$38B$	$q + q^2 - q^3 + q^4 - 4q^5 - q^6 + 3q^7 + q^8 - 2q^9 - 4q^{10} + O(q^{11})$,
40	$q + q^5 - 4q^7 - 3q^9 + 4q^{11} + O(q^{13})$,
42	$q + q^2 - q^3 + q^4 - 2q^5 - q^6 - q^7 + q^8 + q^9 - 2q^{10} - 4q^{11} - q^{12}$
	$+6q^{13} - q^{14} + 2q^{15} + q^{16} + O(q^{17})$,
44	$q + q^3 - 3q^5 + 2q^7 - 2q^9 - q^{11} + O(q^{13})$,
45	$q + q^2 - q^4 - q^5 - 3q^8 - q^{10} + 4q^{11} + O(q^{13})$,
$56A$	$q + 2q^5 - q^7 - 3q^9 - 4q^{11} + 2q^{13} + O(q^{17})$,
$56B$	$q + 2q^2 - 4q^4 + q^6 + q^8 - 8q^{14} - 2q^{16} + O(q^{17})$.

Table 2: First $S(N) + 1$ terms of the Fourier series expansions of the newforms in $S_2(\Gamma_0(N))$

Let us consider the function $F_{30}(z)$ from Table 1. By (1) and (2), we have $F_{30}(z) \in M_2(\Gamma_0(30))$. Using MAPLE we determine the first 13 terms of the Fourier series expansion of $F_{30}(z)$ as

$$F_{30}(z) = q - q^2 + q^3 + q^4 - q^5 - q^6 - 4q^7 - q^8 + q^9 + q^{10} + q^{12} + O(q^{13}),$$

which are the same as the first 13 terms of the newform in $S_2(\Gamma_0(30))$ in Table 2. Thus, by Theorem 1, $F_{30}(z)$ must be equal to the newform in $S_2(\Gamma_0(30))$ in Table 2. The remaining cases can be proven similarly. Note that there are two different newforms in $S_2(\Gamma_0(38))$ and $S_2(\Gamma_0(56))$, we follow the notation in [3, Table 3] and label them as A and B in our tables. ☐

4. Main Results

We first note that if E is an elliptic curve over $\mathbb{Q}$ with conductor N, then by modularity theorem there exists a newform $f \in S_2(\Gamma_0(N))$ such that

$$[p]f_E(z) = p + 1 - |E(\mathbb{Z}_p)| \text{ for } p \nmid N,$$

see [5, p. 120], [4, Theorem 8.8.1]. We deduce Theorem 3 from [3, Table 1].

Theorem 3. Table 3 *below is a list of elliptic curves, more specifically strong Weil curves, corresponding to the newforms given in* Table 1.

Newform	Strong Weil curve	a_1	a_2	a_3	a_4	a_6
$F_{30}(z)$	E_{30A}	1	0	1	1	2
$F_{33}(z)$	E_{33A}	1	1	0	-11	0
$F_{34}(z)$	E_{34A}	1	0	0	-3	1
$F_{35}(z)$	E_{35A}	0	1	1	9	1
$F_{38A}(z)$	E_{38A}	1	0	1	9	90
$F_{38B}(z)$	E_{38B}	1	1	1	0	1
$F_{40}(z)$	E_{40A}	0	0	0	-7	-6
$F_{42}(z)$	E_{42A}	1	1	1	-4	5
$F_{44}(z)$	E_{44A}	0	1	0	3	-1
$F_{45}(z)$	E_{45A}	1	-1	0	0	-5
$F_{56A}(z)$	E_{56A}	0	0	0	1	2
$F_{56B}(z)$	E_{56B}	0	-1	0	0	-4

Table 3: $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$

We are now ready to present our main results in Theorem 4 below. We use Theorems 2 and 3 to give generating functions for the group orders $|E_N(\mathbb{Z}_p)|$ of elliptic curves in Table 3. Among many other possibilities, we have chosen the linear combinations of eta quotients and Eisenstein series in Table 1 in a way that we can deduce the congruences for these group orders in Corollary 1.

Theorem 4. *Consider the elliptic curves listed in Table 3. We have*

$$|E_{30A}(\mathbb{Z}_p)| = -6[p]\left(\frac{\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z)}{\eta(2z)\eta(5z)\eta(6z)\eta(15z)}\right) \text{ for all } p \nmid 30,$$

$$|E_{33A}(\mathbb{Z}_p)| = 2[p]\left(5\frac{\eta^3(3z)\eta^3(33z)}{\eta(z)\eta(11z)} + \eta(z)\eta(3z)\eta(11z)\eta(33z)\right) \text{ for all } p \nmid 33,$$

$$|E_{34A}(\mathbb{Z}_p)| = \frac{3(p+1)}{2} - 3[p]\Big(2\frac{\eta^4(4z)\eta^4(68z)}{\eta^2(2z)\eta^2(34z)} + \frac{1}{8}\frac{\eta^{10}(2z)\eta^{10}(34z)}{\eta^4(z)\eta^4(4z)\eta^4(17z)\eta^4(68z)}\Big) \text{ for all } p \nmid 34,$$

$$|E_{35A}(\mathbb{Z}_p)| = 3(p+1) - [p]\left(3\frac{\eta^3(5z)\eta^3(7z)}{\eta(z)\eta(35z)} - \frac{\eta^3(z)\eta^3(35z)}{\eta(5z)\eta(7z)}\right) \text{ for all } p \nmid 35,$$

$$|E_{38A}(\mathbb{Z}_p)| = \frac{6}{7}(p+1) - \frac{3}{7}[p]\left(\frac{\eta^3(z)\eta^3(38z)}{\eta(2z)\eta(19z)} - 6\frac{\eta^4(2z)\eta^4(38z)}{\eta^2(z)\eta^2(19z)} - \frac{\eta^3(2z)\eta^3(19z)}{\eta(z)\eta(38z)} - \frac{3}{4}\frac{\eta^4(z)\eta^4(19z)}{\eta^2(2z)\eta^2(38z)}\right) \text{ for all } p \nmid 38,$$

$$|E_{38B}(\mathbb{Z}_p)| = (p+1) - [p]\left(\frac{\eta^3(2z)\eta^3(19z)}{\eta(z)\eta(38z)} + \frac{\eta^3(z)\eta^3(38z)}{\eta(2z)\eta(19z)}\right) \text{ for all } p \nmid 38,$$

$$|E_{40A}(\mathbb{Z}_p)| = 4[p]\left(\frac{\eta^2(z)\eta^2(5z)\eta^2(8z)\eta^2(40z)}{\eta(2z)\eta(4z)\eta(10z)\eta(20z)}\right) \text{ for all } p \nmid 40,$$

$$|E_{42A}(\mathbb{Z}_p)| = 8[p]\left(\frac{\eta(2z)\eta(3z)\eta^2(7z)\eta^2(42z)}{\eta(z)\eta(6z)} + \frac{\eta(z)\eta(6z)\eta^2(14z)\eta^2(21z)}{\eta(2z)\eta(3z)}\right) \text{ for all } p \nmid 42.$$

$$|E_{44A}(\mathbb{Z}_p)| = 3(p+1) - 3[p]\left(\frac{\eta^3(4z)\eta^3(11z)}{\eta(z)\eta(44z)} - \frac{\eta^3(z)\eta^3(44z)}{\eta(4z)\eta(11z)}\right) \text{ for all } p \nmid 44,$$

$$|E_{45A}(\mathbb{Z}_p)| = 2(p+1) - 2[p]\left(\frac{\eta(3z)\eta^2(5z)\eta^2(9z)\eta(15z)}{\eta(z)\eta(45z)} - \frac{\eta^2(z)\eta(3z)\eta(15z)\eta^2(45z)}{\eta(5z)\eta(9z)}\right) \text{ for all } p \nmid 45,$$

$$|E_{56A}(\mathbb{Z}_p)| = 2[p]\left(2\frac{\eta(4z)\eta^3(14z)\eta(28z)}{\eta(2z)} + \frac{\eta^{10}(4z)\eta^4(56z)}{\eta^4(2z)\eta^4(8z)\eta^2(28z)}\right) \text{ for all } p \nmid 56,$$

$$|E_{56B}(\mathbb{Z}_p)| = 2[p]\left(2\frac{\eta^3(4z)\eta(8z)\eta^2(14z)\eta(56z)}{\eta^2(2z)\eta(28z)} - \frac{\eta(2z)\eta^2(4z)\eta^6(28z)}{\eta(8z)\eta^3(14z)\eta(56z)}\right) \text{ for all } p \nmid 56.$$

Proof. We just prove the equalities for $|E_{30A}(\mathbb{Z}_p)|$ and $|E_{45A}(\mathbb{Z}_p)|$ as the remaining ones can be proven similarly. By Theorems 2, 3 and modularity theorem, for all $p \nmid 30$, we have

$$\begin{aligned}
|E_{30A}(\mathbb{Z}_p)| &= p + 1 - [p]F_{30A}(z) \\
&= p + 1 - [p]\Big(6\frac{\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z)}{\eta(2z)\eta(5z)\eta(6z)\eta(15z)} \\
&\qquad + 2L_2(z) + L_3(z) + \frac{1}{5}L_5(z) - 2L_6(z) - \frac{2}{5}L_{10}(z) - \frac{1}{5}L_{15}(z) + \frac{2}{5}L_{30}(z)\Big) \\
&= p + 1 - 6[p]\Big(\frac{\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z)}{\eta(2z)\eta(5z)\eta(6z)\eta(15z)}\Big) - \sigma(p)
\end{aligned}$$

$$= -6[p]\Big(\frac{\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z)}{\eta(2z)\eta(5z)\eta(6z)\eta(15z)}\Big),$$

which completes the proof of the first equality.

Again by Theorems 2, 3 and modularity theorem, for all $p \nmid 45$, we have

$$\begin{aligned}
|E_{45A}(\mathbb{Z}_p)| &= p + 1 - [p]F_{45}(z)\\
&= p + 1 - [p]\Big(2\frac{\eta(3z)\eta^2(5z)\eta^2(9z)\eta(15z)}{\eta(z)\eta(45z)} - 2\frac{\eta^2(z)\eta(3z)\eta(15z)\eta^2(45z)}{\eta(5z)\eta(9z)}\\
&\qquad + L_5(z) - \frac{2}{3}L_9(z) - \frac{2}{5}L_{15}(z) - \frac{14}{15}L_{45}(z)\Big)\\
&= p + 1 - 2[p]\Big(\frac{\eta(3z)\eta^2(5z)\eta^2(9z)\eta(15z)}{\eta(z)\eta(45z)} - \frac{\eta^2(z)\eta(3z)\eta(15z)\eta^2(45z)}{\eta(5z)\eta(9z)}\Big) + \sigma(p)\\
&= 2(p+1) - 2[p]\Big(\frac{\eta(3z)\eta^2(5z)\eta^2(9z)\eta(15z)}{\eta(z)\eta(45z)} - \frac{\eta^2(z)\eta(3z)\eta(15z)\eta^2(45z)}{\eta(5z)\eta(9z)}\Big),
\end{aligned}$$

which completes the proof of the last equality. □

The following congruences follow immediately from Theorem 4.

Corollary 1. *We have*

$$\begin{aligned}
|E_{30A}(\mathbb{Z}_p)| &\equiv 0 \pmod 6 \text{ for all } p \nmid 30,\\
|E_{33A}(\mathbb{Z}_p)| &\equiv 0 \pmod 2 \text{ for all } p \nmid 33,\\
|E_{34A}(\mathbb{Z}_p)| &\equiv 0 \pmod 3 \text{ for all } p \nmid 34,\\
|E_{38A}(\mathbb{Z}_p)| &\equiv 0 \pmod 3 \text{ for all } p \nmid 38,\\
|E_{40A}(\mathbb{Z}_p)| &\equiv 0 \pmod 4 \text{ for all } p \nmid 40,\\
|E_{42A}(\mathbb{Z}_p)| &\equiv 0 \pmod 8 \text{ for all } p \nmid 42,\\
|E_{44A}(\mathbb{Z}_p)| &\equiv 0 \pmod 3 \text{ for all } p \nmid 44,\\
|E_{45A}(\mathbb{Z}_p)| &\equiv 0 \pmod 2 \text{ for all } p \nmid 45,\\
|E_{56A}(\mathbb{Z}_p)| &\equiv 0 \pmod 2 \text{ for all } p \nmid 56,\\
|E_{56B}(\mathbb{Z}_p)| &\equiv 0 \pmod 2 \text{ for all } p \nmid 56.
\end{aligned}$$

5. Alternative Representations for Newforms in $S_2(\Gamma_0(N))$ for $N = 11$, 14, 15, 20, 24, 33, 40, 42 and Further Congruences

We use a computer algorithm to go through all possible linear combinations of eta quotients and Eisenstein series which correspond to newforms in $S_2(\Gamma_0(N))$ with $N = 30$, 33, 34,35, 38, 40, 42, 44, 45 and 56. In Table 1 we give the linear combinations which are suitable for deducing the congruences for the group orders in Corollary 1. In Table 4 below we give alternative representations for the newforms $F_N(z)$ in $S_2(\Gamma_0(N))$ for $N = 33, 40, 42$, which have fewer number of functions in the linear combinations. We note that the representation for $F'_{33}(q)$ is the same as that of the one in [10].

Level	Newform	Eta quotients and Eisenstein series
33	$F'_{33}(q) =$	$3\eta^2(3z)\eta^2(33z) + 3\eta(z)\eta(3z)\eta(11z)\eta(33z) + \eta^2(z)\eta^2(11z)$
40	$F'_{40}(q) =$	$2\eta^2(2z)\eta^2(10z) - \dfrac{\eta(2z)\eta^2(8z)\eta^5(20z)}{\eta(4z)\eta(10z)\eta^2(40z)}$
42	$F'_{42}(q) =$	$\dfrac{\eta^2(2z)\eta^2(3z)\eta^2(14z)\eta^2(21z)}{\eta(z)\eta(6z)\eta(7z)\eta(42z)} - \dfrac{\eta^2(z)\eta^2(6z)\eta^2(7z)\eta^2(42z)}{\eta(2z)\eta(3z)\eta(14z)\eta(21z)}$

Table 4: Alternative Representations for Newforms in $M_2(\Gamma_0(N))$, where $N = 33, 40, 42$.

In [8], Martin and Ono represented all the newforms in $S_2(\Gamma_0(N))$ for $N = 11$, 14, 15, 20, 24, 27, 32, 36, 48, 64, 80, 144 in terms of single eta quotients. In Table 5 below we give alternative representations for the newforms $F_N(z) \in S_2(\Gamma_0(N))$ for $N = 11$, 14, 15, 20, 24 by using arguments from this paper. We then deduce congruence relations similar to Corollary 1 for the group orders of the corresponding strong Weil curves with conductors 11, 14, 15, 20 and 24.

Level	Newform	Eta quotients and Eisenstein series
11	$F_{11}(z) =$	$-5\dfrac{\eta^4(2z)\eta^4(22z)}{\eta^2(z)\eta^2(11z)} - 4\eta^2(2z)\eta^2(22z) + \dfrac{1}{2}L_2(z) + L_{11}(z) - \dfrac{1}{2}L_{22}(z)$
14	$F_{14}(z) =$	$\dfrac{6}{5}\dfrac{\eta^5(z)\eta^5(14z)}{\eta^3(2z)\eta^3(7z)} + \dfrac{13}{5}L_2(z) - \dfrac{13}{5}L_7(z) + L_{14}(z)$
15	$F_{15}(z) =$	$-4\dfrac{\eta^3(3z)\eta^3(15z)}{\eta(z)\eta(5z)} + \dfrac{1}{3}L_3(z) + L_5(z) - \dfrac{1}{3}L_{15}(z)$
20	$F_{20}(z) =$	$-6\eta_{20}[0,-2,4,0,-2,4](z)$ $+\frac{3}{2}L_2(z) - \frac{1}{2}L_4(z) - L_5(z) + \frac{3}{2}L_{10}(z) - \frac{1}{2}L_{20}(z)$

Level	Newform	Eta quotients and Eisenstein series
24	$F_{24}(z) =$	$-2\eta_{24}[4,-2,0,0,0,0,-2,4](z) + \frac{3}{2}L_2(z) + L_3(z)$ $+\frac{3}{2}L_4(z) - \frac{3}{2}L_6(z) - L_8(z) - \frac{3}{2}L_{12}(z) + L_{24}(z)$

Table 5: Alternative Representations for Newforms in $M_2(\Gamma_0(N))$, where $N = 11, 14, 15, 20, 24$.

Corresponding strong Weil curves with conductors 11, 14, 15, 20 and 24 are

$$\begin{aligned}
&E_{11A} : y^2 + y = x^3 - x^2 - 10x - 20,\\
&E_{14A} : y^2 + xy + y = x^3 + 4x - 6,\\
&E_{15A} : y^2 + xy + y = x^3 + x^2 - 10x - 10,\\
&E_{20A} : y^2 = x^3 + x^2 + 4x + 4,\\
&E_{24A} : y^2 = x^3 - x^2 - 4x + 4,
\end{aligned}$$

respectively, see [3, Table 1] and [8]. Similar to Theorem 4, we obtain

$$\begin{aligned}
|E_{11A}(\mathbb{Z}_p)| &= 5[p]\left(\frac{\eta^4(2z)\eta^4(22z)}{\eta^2(z)\eta^2(11z)}\right) \text{ for all } p \nmid 11,\\
|E_{14A}(\mathbb{Z}_p)| &= -\frac{6}{5}[p]\left(\frac{\eta^5(z)\eta^5(14z)}{\eta^3(2z)\eta^3(7z)}\right) \text{ for all } p \nmid 14,\\
|E_{15A}(\mathbb{Z}_p)| &= 4[p]\left(\frac{\eta^3(3z)\eta^3(15z)}{\eta(z)\eta(5z)}\right) \text{ for all } p \nmid 15,\\
|E_{20A}(\mathbb{Z}_p)| &= 6[p]\left(\frac{\eta^4(4z)\eta^4(20z)}{\eta^2(2z)\eta^2(10z)}\right) \text{ for all } p \nmid 20,\\
|E_{24A}(\mathbb{Z}_p)| &= 2[p]\left(\frac{\eta^4(z)\eta^4(24z)}{\eta^2(2z)\eta^2(12z)}\right) \text{ for all } p \nmid 24.
\end{aligned}$$

Thus, similar to Corollary 1, we deduce the congruence relations

$$\begin{aligned}
&|E_{11A}(\mathbb{Z}_p)| \equiv 0 \pmod 5 \text{ for all } p \nmid 11,\\
&|E_{14A}(\mathbb{Z}_p)| \equiv 0 \pmod 6 \text{ for all } p \nmid 14,\\
&|E_{15A}(\mathbb{Z}_p)| \equiv 0 \pmod 4 \text{ for all } p \nmid 15,\\
&|E_{20A}(\mathbb{Z}_p)| \equiv 0 \,(\text{mod } 6) \text{ for all } p \nmid 20,\\
&|E_{24A}(\mathbb{Z}_p)| \equiv 0 \,(\text{mod } 2) \text{ for all } p \nmid 24.
\end{aligned}$$

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