ON THE ITERATES OF DIGIT MAPS

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Abstract
Given a base b, a "digit map" is a map $f : \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ of the form $f(\sum_{i=0}^{n} a_i b^i) = \sum_{i=0}^{n} f_*(a_i)$ where $\sum_i a_i b^i$ is the base b representation and $f_* : \{0, 1, \ldots, b-1\} \to \mathbb{Z}^{\geq 0}$ satisfies $f_*(0) = 0, f_*(1) = 1$, and $\gcd(f_*(b-1), b) = 1$. Suppose further there exists some m_*, $0 \leq m_* \leq b-1$, such that $f(m_*) - m_*$ is relatively prime to $f(b-1)$. We generalize recent results on so-called 'happy numbers' to general digit maps, showing that any periodic point, that is, under iterates of the digit map, can eventually be reached by arbitrarily long sequences of consecutive positive integers.

1. Introduction

Functions that act on digits of an integer, such as digit sums, are natural to study. In this paper, we look at functions that take in a positive integer and output the sum of its values on the digits of that integer. Precisely, for a fixed base b, we start with a function f_* that acts on the digits in base b, i.e., $f_* : \{0, 1, \ldots, b-1\} \to \mathbb{Z}^{\geq 0}$. Then, we extend f_* to a function f on the non-negative integers given by $f(\sum_{i=0}^{n} a_i b^i) = \sum_{i=0}^{n} f_*(a_i)$ where $0 \leq a_i \leq b-1$.

In Richard Guy's book "Unsolved Problems in Number Theory", Guy poses many questions regarding $(2, 10)$-happy numbers [2]. An (e, b)-*happy number* is a number that, under iterates of the digit map f induced by $f_*(m) = m^e$ (in base b), eventually reaches 1. In [1], Pan proved that there exist arbitrarily long sequences of consecutive (e, b)-happy numbers assuming that if a prime p divides $b-1$, then the integer $p-1$ does not divide $e-1$. As we will show, this is a special case of our main result.

A question appearing in Guy's book [2] is that of gaps in the happy number sequence. In this paper, we give an answer to this question by, in particular, showing that there are arbitrarily large gaps in the (e, b) happy number sequence, assuming that if a prime p divides $b-1$, then $p-1$ does not divide $e-1$.

We study more general digit maps f. It is well-known and easy to see that any positive integer eventually ends up in some finite cycle under digit maps, where here a finite cycle is some collection of positive integers $\{n_1, \ldots, n_k\}$ such that $f(n_i) = n_{i+1}$ for $1 \leq i \leq k-1$ and $f(n_k) = f(n_1)$. For example, the cycles generated by the $(2, 10)$-happy number digit map are $\{1\}$ and $\{4, 16, 37, 58, 89, 145, 42, 20\}$. A gap in the happy number sequence, therefore, corresponds to consecutive numbers that end up in the latter cycle. In this paper, a special case of what we prove is that indeed for any u in an (e, b)-happy number cycle, we can find arbitrarily long sequences of consecutive integers that end up in the same cycle as u. More interestingly, we show that for *any* digit map satisfying certain weak hypotheses, and any positive integer in a cycle, we can find arbitrarily long sequences of consecutive positive integers ending up in that cycle.

Fix a base b and a digit function f. We call a positive integer u in some cycle a *cycle number* and any positive integer ending up in that cycle a *u-integer*.

Theorem 1.1. *Fix a base b. Suppose f is a digit map such that $f(0) = 0, f(1) = 1, \gcd(f(b-1), b) = 1$, and there is some digit $0 \leq m_* \leq b-1$ such that $f(m_*) - m_*$ is relatively prime to $f(b-1)$. Then for any cycle number u and any positive integer n, there exist n consecutive u-integers.*

For example, working in base 10, if we construct the digit map $f_* : \{0, \ldots, 9\} \to \mathbb{Z}^{\geq 0}$ given by $f_*(0) = 0, f_*(1) = 1$ and $f_*(9) = 7$, then no matter where we send $2, 3, 4, 5, 6, 7,$ and 8, we are guaranteed that there will exist arbitrarily long sequences of consecutive positive integers such that when we apply f enough times, they will end up at 1, say. The result of Theorem 1.1 consumes the work of H. Pan [1], H. Grundman and E. A. Teeple in [3], and E. El-Sedy and S. Siksek in [4].

Note that the assumption in the theorem statement about the existence of m_* cannot be removed. Indeed, in base 10, consider the map $f_*(j) = j$ for $0 \leq j \leq 9$. The cycles are $\{1\}, \{2\}, \ldots, \{9\}$, and the obtained $f : \mathbb{N} \to \mathbb{N}$ preserves the residue mod 3 of the input. So, for example, we cannot have two consecutive 1-integers.

2. Proof of Theorem 1.1

We use a few short results in Pan's proof of Theorem 1.1, and introduce new techniques and results in Lemma 2.3 and Corollary 2.2. Specifically, the proofs of Lemma 2.1, Corollary 2.1, and Lemma 2.2 are basically identical to the proofs given by Pan; we just fit them to our notation.

Lemma 2.1. *Let x and m be arbitrary positive integers. Then for each $r \geq 1$, there exists a positive integer l such that*

$$f^r(l + y) = f^r(l) + f^r(y) = x + f^r(y)$$

for each $1 \le y \le m$*.*

Proof. We use induction on r. When $r = 1$, choose a positive integer s such that $b^s > m$ and let

$$l_1 = \sum_{j=0}^{x-1} b^{s+j}.$$

Clearly for any $1 \le y \le m$,

$$f(l_1 + y) = f(l_1) + f(y) = x + f(y).$$

Now assume $r > 1$ and the assertion of Lemma 2.1 holds for the smaller values of r. Note there exists an m' such that $f(y) \le m'$ for $1 \le y \le m$. Therefore, by induction hypothesis, there exists an l_{r-1} such that

$$f^{r-1}(l_{r-1} + f(y)) = f^{r-1}(l_{r-1}) + f^{r-1}(f(y)) = x + f^r(y)$$

for $1 \le y \le m$. Let

$$l_r = \sum_{j=0}^{l_{r-1}-1} b^{s+j}$$

where s satisfies $b^s > m$. Then,

$$f^r(l_r) = f^{r-1}(f(l_r)) = f^{r-1}(l_{r-1}) = x$$

and for each $1 \le y \le m$,

$$\begin{aligned} f^r(l_r + y) &= f^{r-1}(f(l_r + y)) = f^{r-1}(f(l_r) + f(y)) \\ &= f^{r-1}(l_{r-1} + f(y)) = f^{r-1}(l_{r-1}) + f^r(y) = f^r(l_r) + f^r(y). \end{aligned}$$

□

Definition 2.1. Let $D = D(f_*, b)$ be the set of all positive integers that are in some cycle, that is $n \in D$ if and only if $f^r(n) = n$ for some $r \ge 1$. It is easy to see that D is finite.

Definition 2.2. Take some $u \in D$. Then we say a positive integer n is a *u-integer* if $f^r(n) = u$ for some $r \ge 1$. Further, we say two positive integers m, n are *concurrently u-integers* if for some $r \ge 1$, $f^r(m) = f^r(n) = u$.

Note that two u-integers m, n are not concurrently u-integers only if u belongs to a cycle of length greater than 1 in D and m, n are at different places in the cycle at a certain time. Note "concurrently u-integers" is a transitive relation. Now fix u and we will prove that there are arbitrarily long sequences of consecutive u-integers. First, we make a reduction.

Corollary 2.1. *Assume that there exists $h \in \mathbb{Z}^+$ such that $h+x$ is a u-integer for all $x \in D$. Then for arbitrary $m \in \mathbb{Z}^+$, there exists $l \in \mathbb{Z}^+$ such that $l+1, l+2, \ldots, l+m$ are u-integers.*

Proof. By the definition of D, there exists $r \in \mathbb{Z}^+$ such that $f^r(y) \in D$ for all $1 \leq y \leq m$. By Lemma 2.1, there exists $l \in \mathbb{Z}^+$ so that

$$f^r(l+y) = h + f^r(y)$$

for $1 \leq y \leq m$. Since $f^r(l+y)$ is then a u-integer, $l+y$ is as well, for $1 \leq y \leq m$. $\square$

Lemma 2.2. *Assume that for each $x \in D$ there exists $h_x \in \mathbb{Z}^+$ such that $h_x + u$ and $h_x + x$ are concurrently u-integers. Then there exists $h \in \mathbb{Z}^+$ such that $h + x$ is a u-integer for each $x \in D$.*

Proof. We shall prove that, under the assumption of Lemma 2.2, for each subset X of D containing u, there exists $h_X \in \mathbb{Z}^+$ such that $h_X + x$ is a u-integer for each $x \in X$.

The cases $|X| = 1$ and $|X| = 2$ are clear. Assume $|X| > 2$ and that the assertion holds for every smaller value of $|X|$. Take some $x \in X$, with $x \neq u$. Then h_x+u and h_x+x are concurrently u-integers, so take $r \in \mathbb{Z}^+$ large enough so that $f^r(h_x+u) = f^r(h_x+x) = u$ and $f^r(h_x+y) \in D$ for all $y \in X$. Let $X^* = \{f^r(h_x+y) | y \in X\}$. Then, X^* is clearly a subset of D containing u with $|X^*| < |X|$. Therefore, by induction, there exists $h_{X^*} \in \mathbb{Z}^+$ such that $h_{X^*} + f^r(h_x+y)$ is a u-integer for each $y \in X$. By Lemma 2.1, there exists $l \in \mathbb{Z}^+$ satisfying

$$f^r(l + h_x + y) = h_{X^*} + f^r(h_x + y)$$

for every $y \in X$. Thus, $h_X := l + h_x$ works. $\square$

We now proceed to prove the hypothesis of Lemma 2.2. Note that it suffices to show that for any fixed difference d, we can find two concurrent u-integers with difference d. This is the statement of Corollary 2.2.

Lemma 2.3. *Let h be a u-integer. Then for every integer a, there exists a u-integer l such that $l \equiv a \pmod{f(b-1)}$, and such that l and h are concurrently u-integers.*

Proof. Let l_1 be a u-integer such that $l_1 > f(b-1)f(m_*)$. We show we can take l_2 such that $l_2 \equiv a \pmod{f(b-1)}$ and $f(l_2) = l_1$.

Since $f(m_*) - m_*$ is relatively prime to $f(b-1)$, we can solve

$$l_1 - r(f(m_*) - m_*) \equiv a \pmod{f(b-1)}$$

with $0 \leq r < f(b-1)$. Note there are infinitely many $j \geq 1$ with $b^j \equiv 1 \pmod{f(b-1)}$. Let $j_1 < j_2 < \cdots < j_{l_1 - rf(m_*)} < t_1 < t_2 < \cdots < t_r$ satisfy $b^{j_i} \equiv b^{t_i} \equiv 1 \pmod{f(b-1)}$ and $b^{t_{j+1}} > m_* b^{t_j}$. Now let

$$l_2 = \sum_{n=1}^{l_1 - rf(m_*)} b^{j_n} + m_* b^{t_1} + \cdots + m_* b^{t_r}.$$

Note by our choice of l_1, the indices of the j's and of the sum are valid. So,

$$f(l_2) = l_1 - rf(m_*) + rf(m_*) = l_1$$

and

$$l_2 \equiv l_1 - rf(m_*) + rm_* \equiv a \pmod{f(b-1)},$$

as desired.

Now we generate $l_3, l_4, \ldots$ inductively by choosing l_{n+1} so that $l_{n+1} \equiv a \pmod{f(b-1)}$ and $f(l_{n+1}) = l_n$. Note that since the cycle that u is in is finite, it must be that one of the l_n's is concurrently a u-integer with h. □

Corollary 2.2. *For each $x \in \mathbb{Z}^+$, there is a u-integer l such that l and $l + x$ are concurrently u-integers.*

Proof. Fix $x \in \mathbb{Z}^+$. Take $s \in \mathbb{Z}^+$ such that $b^s > x$. Let $x_1 = b^s - x$. Take a u-integer h' such that

$$h' \equiv f(x_1) \pmod{f(b-1)}.$$

Let V be the cycle set that u is in. By Lemma 2.3, for each $v' \in V$, there exists $l_{v'}$ such that $l_{v'} \equiv 1 \pmod{f(b-1)}$, and $l_{v'}$ and v' are concurrently u-integers. Fixing an $l_{v'}$ for each $v' \in V$, let $M = \max_{v' \in V} l_{v'}$.

Since the proof of Lemma 2.3 guarantees infinitely many u-integers in a given residue, we may (and do) fix $h > f(x_1) + M$ to be a u-integer with $h \equiv f(x_1) \pmod{f(b-1)}$. Let v be in the cycle of u so that h and v are concurrently u-integers. Now take the u-integer $N = l_v$ so that $N \equiv 1 \pmod{f(b-1)}$, and N and v are concurrently u-integers. Take a positive integer t so that $b^t > b^{s + \lfloor \frac{h}{f(b-1)} \rfloor + 1}$. Let $x_2 = x_1 + b^t \sum_{j=1}^{N-1} b^j$. Note $f(x_2) = f(x_1) + (N-1)$ since $b^t > b^s > x_1$. Thus,

$$f(x_2) \equiv f(x_1) \equiv h \pmod{f(b-1)}.$$

Also note, $f(x_2) = f(x_1) + (N-1) \leq f(x_1) + M - 1 < h$. Write $h = f(b-1)k + f(x_2)$ and note that we have $k > 0$. Also note $k \leq \lfloor \frac{h}{f(b-1)} \rfloor + 1 < t - s$. Let

$$l = x_2 + \sum_{j=0}^{k-1} (b-1) b^{s+j}.$$

Then,

$$\begin{aligned} f(l) &= f\left(x_1 + b^t \sum_{j=1}^{N-1} b^j + b^s \sum_{j=0}^{k-1} (b-1)b^j\right) \\ &= f\left(x_1 + b^s[b^{t-s} \sum_{j=1}^{N-1} b^j + \sum_{j=0}^{k-1} (b-1)b^j]\right) \\ &= f(x_1) + f(b^{t-s} \sum_{j=1}^{N-1} b^j + \sum_{j=0}^{k-1} (b-1)b^j), \end{aligned}$$

and since $\sum_{j=0}^{k-1}(b-1)b^j = b^k - 1 < b^{t-s}$,

$$f(l) = f(x_1) + (N-1) + kf(b-1) = f(x_2) + kf(b-1) = h.$$

Further,

$$f(l+x) = f\left(b^s + \sum_{j=0}^{k-1}(b-1)b^{s+j} + b^t \sum_{j=1}^{N-1} b^j\right) = f\left(b^{s+k} + b^t \sum_{j=1}^{N-1} b^j\right)$$

which is equal to N. Since h and N are concurrently u-integers, it follows that l and $l+x$ are concurrently u-integers, as desired. □

Theorem 1.1 now follows from Corollary 2.1, Lemma 2.2, and Corollary 2.2.

We finish by showing that, in the generalized happy number setup, the assumption in Theorem 1.1 is satisfied (clearly $f(0) = 0, f(1) = 1$, and $\gcd(b, f(b-1)) = 1$). Recall we assume that if a prime p divides $b-1$, then $p-1$ does not divide $e-1$. Write $b - 1 = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ for distinct primes p_i. For all i, let g_i satisfy $(\frac{b-1}{p_i^{\alpha_i}} g_i)^e \not\equiv \frac{b-1}{p_i^{\alpha_i}} g_i \pmod{p_i}$ — any generator of $\mathbb{Z}_{p_i}^\times$ times $(\frac{b-1}{p_i^{\alpha_i}})^{-1} \pmod{p_i}$ will do. Then let $g = \sum_i \frac{b-1}{p_i^{\alpha_i}} g_i$ so that $g \equiv \frac{b-1}{p_i^{\alpha_i}} g_i \pmod{p_i}$, and thus for all i, $g^e \equiv (\frac{b-1}{p_i^{\alpha_i}})^e g_i^e \pmod{p_i}$ which is indeed not congruent to $g \pmod{p_i}$ by our choice of g_i. Now just reduce $g \pmod{b-1}$ so that we get a digit g that satisfies $f(g) - g \not\equiv 0 \pmod{p_i}$ for every $p_i \mid b-1$. Hence $\gcd(f(g) - g, f(b-1)) = \gcd(f(g) - g, (b-1)^e) = 1$, as desired.

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