A NOTE ON FIBONACCI NUMBERS OF EVEN INDEX

Achille Frigeri
Dipartimento di Matematica, Politecnico di Milano, Milan, Italy
achille.frigeri@polimi.it

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Abstract

We introduce a representation of the integers based only on Fibonacci numbers of odd index. Then we give an elementary combinatorial proof of the fact that a positive integer n is a Fibonacci number of even index if and only if $\langle n\varphi\rangle + \frac{1}{n} > 1$.

1. Introduction

The Fibonacci numbers are recursively defined from $F_1 = F_2 = 1$ by $F_{n+2} = F_{n+1} + F_n$, for all $n \in \mathbb{N}$. We say that m is a Fibonacci number of even (resp. odd) index if there exists an even (resp. odd) number n such that $m = F_n$.

In the rest of the paper, we denote by φ the so called "golden section", i.e., $\varphi = \frac{1+\sqrt{5}}{2}$. For all $x \in \mathbb{R}$, we denote by $\langle x\rangle$ and by $\lfloor x\rfloor$ the fractional and the integer part of x, respectively. Clearly $\langle x\rangle = x - \lfloor x\rfloor$.

Some characterizations of the Fibonacci numbers of even and odd index appear in the literature. In [2], Herrmann showed a property related to the local minima/maxima of the sequence of fractional parts of multiples of the golden section. More precisely, ([2], Theorem 2.3), n is a Fibonacci number of even index if and only if $\langle n\varphi\rangle > \langle i\varphi\rangle$ for all $1 \leq i < n$, while it is a Fibonacci number of odd index if and only if $\langle n\varphi\rangle < \langle i\varphi\rangle$ for all $1 \leq i < n$. A completely different characterization is stated in [1]: n is a Fibonacci number if and only if $5n^2 + 4$ or $5n^2 - 4$ is a square and, more precisely, n is a Fibonacci number of even index if and only if $5n^2 + 4$ is a square (see [3]).

In this note, we introduce a representation of the integers based only on Fibonacci numbers of odd index, which differs from the well known Zeckendorf representation (see [7]). Then, we prove a characterization for the Fibonacci numbers of even index. From a result of Möbius (see [4]), we deduce the corresponding condition for the odd case.

2. Preliminaries

In [7], Zeckendorf proved that any positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. We observe that, under some assumptions, we can restrict such a sum only to Fibonacci numbers of odd index. Namely, every positive integer has a (not unique) ternary decomposition of the following form.

Proposition 1 (Fibonacci odd index ternary representation). *Let n be a positive integer, then it can be written in the form*

$$n = \sum_{h=0}^{s} a_h F_{2h+1} \tag{1}$$

for some $s \geq 0$ with $a_0 \in \{0,1\}$, $a_h \in \{0,1,2\}$ for $0 < h < s$ and $a_s \in \{1,2\}$. In this case, we write $n = (a_0 \ldots a_s)$. In the sequel, we will refer to such representation simply as "the representation of n".

Proof. The property easily follows by applying an algorithm of successive divisions starting from the greatest Fibonacci number of odd index less or equal than n. □

The procedure above gives a "normal representation", but in general such representation is not unique, e.g., $41 = (1222) = (01101)$ and $F_9 = (00001) = (1112)$. It is clear that, as we start from the greatest Fibonacci number of odd index less or equal than n, the normal representation for n is the longest representation of n. Nevertheless, there is an interesting class of integers for which we can prove the uniqueness.

Proposition 2. *Every Fibonacci number of even index has the unique representation $(1 \ldots 1)$.*

Proof. Let $n = F_{2m}$. Then n has the representation $(1 \ldots 1)$ of length m, which is the longest representation of n. Suppose $(a_0 \ldots a_h)$, with $h \leq m-1$, is a different representation of n. If we perform the element-wise difference between the two representations, we get the non-zero string $(b_1 \ldots b_{m-1}) = (a_0 \ldots a_h) - (1 \ldots 1)$ with $b_i \in \{-1, 0, 1\}$, for all $0 \leq i \leq m-1$. Then we have $\sum_{i=0}^{m-1} b_i F_{2i+1} = 0$. Now, let $A = \{i \in [0, m-1] \mid b_i = 1\}$ and let $B = \{i \in [0, m-1] \mid b_i = -1\}$. Obviously $A \neq B$. From the previous equation, we get $\sum_{i \in A} F_{2i+1} = \sum_{i \in B} F_{2i+1}$, which is absurd, as, from [7], every positive integer is represented uniquely as the sum of nonconsecutive Fibonacci numbers. □

The next two propositions recall some known properties of Fibonacci numbers.

Proposition 3 ([5], identity 28; [6], p. 52). *Let n be a positive integer, then:*

$$F_n \varphi = F_{n+1} + (-1)^{n+1} (\varphi - 1)^n, \tag{2}$$

and

$$F_n \varphi + F_{n-1} = \varphi^n. \tag{3}$$

Proposition 4. *Let n be a positive integer, then*

$$\lfloor F_n\varphi \rfloor = \begin{cases} F_{n+1}, & \text{if } n \text{ is odd;} \\ F_{n+1} - 1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Identity 103b of [6] states that $F_{n+1} - F_n\varphi = \frac{(-1)^n}{F_{n-1}+F_n\varphi}$. From the identity (3) and observing that $\frac{1}{\varphi} = \varphi - 1$, we have $F_n\varphi = F_{n+1} - (1-\varphi)^n$, and then the thesis. □

From the previous propositions immediately follows the next.

Proposition 5 ([2], Lemma 2.1). *Let n be a positive integer, then*

$$\langle F_n\varphi \rangle = \begin{cases} (\varphi - 1)^n, & \text{if } n \text{ is odd;} \\ 1 - (\varphi - 1)^n, & \text{if } n \text{ is even.} \end{cases} \tag{4}$$

A direct consequence of the previous is the following inequality.

Proposition 6. *Let r, s integers such that $1 \le r \le s$. Then*

$$\sum_{h=r}^{s} \langle F_{2h+1}\varphi \rangle < (\varphi - 1)^{2r}. \tag{5}$$

Proof. From the equality (4), and observing that $(\varphi - 1)^2 - 1 = 1 - \varphi$, we have:

$$\begin{aligned} \sum_{h=r}^{s} \langle F_{2h+1}\varphi \rangle &= \sum_{h=r}^{s} (\varphi - 1)^{2h+1} = (\varphi - 1)^{2r+1} \sum_{h=0}^{s-r} (\varphi - 1)^{2h} \\ &= (\varphi - 1)^{2r+1} \frac{((\varphi - 1)^2)^{s-r+1} - 1}{(\varphi - 1)^2 - 1} = (\varphi - 1)^{2r+1} \frac{1 - (\varphi - 1)^{2s-2r+2}}{\varphi - 1} \\ &= (\varphi - 1)^{2r} \langle F_{2(s-r+1)}\varphi \rangle < (\varphi - 1)^{2r}. \end{aligned}$$

□

3. Main Result

Before proving the main result, we need to observe that every representation $(a_0 \dots a_s)$ of a positive integer n belongs to one of the following six types:

(1) $a_0 = 0$, $a_i \in \{0, 1, 2\}$ for all $1 \le i < s$, and $a_s \in \{1, 2\}$;

(2) $a_i = 1$ for all $0 \le i \le s$;

(3) $a_i = 1$ for all $0 \le i < s - 1$, $a_{s-1} = 0$, and $a_s = 1$;

(4) $a_i = 1$ for all $0 \le i < s - 1$, $a_{s-1} = 0$, and $a_s = 2$;

(5) there is r with $1 \leq r < s-1$ such that $a_i = 1$ for all $0 \leq i < r$, $a_r = 0$, $a_j \in \{0,1,2\}$ for all $r < j < s$, and $a_s \in \{1,2\}$;

(6) there is r with $1 \leq r \leq s$ such that $a_i = 1$ for all $0 \leq i < r$, $a_r = 2$, $a_j \in \{0,1,2\}$ for all $r < j < s$, and $a_s \in \{1,2\}$.

For example, the representations $(1212\ldots12)$ and $(122\ldots2)$ are of type (6), while (111012) is of type (5). We can now prove the main result by analyzing the six types above, recalling that, from Proposition 2, n is a Fibonacci number of even index if and only if its representation is of type (2).

Theorem 1. *Let n be a positive integer. Then n is a Fibonacci number of even index if and only if $\langle n\varphi\rangle + \frac{1}{n} > 1$.*

Proof. Let $(a_0 \ldots a_s)$ be a representation of n. We analyze the six cases above and prove that only in case (2) we have $\langle n\varphi\rangle + \frac{1}{n} > 1$.

(1) Let $a_0 = 0$. Then

$$\langle n\varphi\rangle = \left\langle \sum_{h=1}^{s} a_h F_{2h+1}\varphi \right\rangle \leq \sum_{h=1}^{s} a_h \langle F_{2h+1}\varphi\rangle \leq 2\sum_{h=1}^{s} \langle F_{2h+1}\varphi\rangle,$$

and from the inequality (5) it follows that

$$\langle n\varphi\rangle < 2(\varphi-1)^2 < \frac{4}{5}.$$

If $n \geq 5$, then $\langle n\varphi\rangle + \frac{1}{n} < 1$. By direct calculation, the same inequality also holds for $n = 2$ and $n = 4$.

(2) Let $a_0 = \ldots = a_s = 1$. In this case $n = F_{2m}$ for some m. From the identity (4), we have $\langle n\varphi\rangle = \langle F_{2m}\varphi\rangle = 1 - (\varphi-1)^{2m}$, while from the identity (3) we have $F_{2m} < F_{2m}\varphi + F_{2m-1} = \varphi^{2m}$. Then

$$\langle n\varphi\rangle + \frac{1}{n} > 1 - (\varphi-1)^{2m} + \frac{1}{\varphi^{2m}} = 1 - (\varphi-1)^{2m} + (\varphi-1)^{2m} = 1.$$

(3) Let $a_0 = \ldots = a_{s-2} = 1$, $a_{s-1} = 0$ and $a_s = 1$. Then, from the identity (5), we have

$$\begin{aligned}\langle n\varphi\rangle &= \langle (F_{2s-2} + F_{2s+1})\,\varphi\rangle \leq \langle F_{2s-2}\varphi\rangle + \langle F_{2s+1}\varphi\rangle \\ &< 1 - (\varphi-1)^{2s-2} + (\varphi-1)^{2s+1}.\end{aligned}$$

On the other hand, from the identity (4) we have

$$n = \sum_{h=0}^{s-2} F_{2h+1} + F_{2s+1} = F_{2s-2} + F_{2s+1} > F_{2s-2} + F_{2s-1}\varphi = \varphi^{2s-1},$$

i.e., $\frac{1}{n} < \left(\frac{1}{\varphi}\right)^{2s-1} = (\varphi-1)^{2s-1}$. Then,

$$\begin{aligned}\langle n\varphi\rangle + \frac{1}{n} &< 1-(\varphi-1)^{2s-2}+(\varphi-1)^{2s+1}+(\varphi-1)^{2s-1}\\ &= 1-(\varphi-1)^{2s+2} < 1,\end{aligned}$$

where in the last step we used the identity $1-(\varphi-1)^3-(\varphi-1)=(\varphi-1)^4$.

(4) Let $a_0 = \ldots = a_{s-2} = 1$, $a_{s-1} = 0$ and $a_s = 2$. We proceed analogously to the previous case. From the identity (5), we have

$$\begin{aligned}\langle n\varphi\rangle &= \langle (F_{2s-2}+2F_{2s+1})\,\varphi\rangle \le \langle F_{2s-2}\varphi\rangle + 2\,\langle F_{2s+1}\varphi\rangle\\ &< 1-(\varphi-1)^{2s-2}+2(\varphi-1)^{2s+1}.\end{aligned}$$

From the identity (4), we have

$$n=\sum_{h=0}^{s-2}F_{2h+1}+2F_{2s+1}=F_{2s-2}+F_{2s-1}+F_{2s+2}>F_{2s-1}+F_{2s}\varphi=\varphi^{2s},$$

i.e., $\frac{1}{n} < \left(\frac{1}{\varphi}\right)^{2s} = (\varphi-1)^{2s}$. Then

$$\begin{aligned}\langle n\varphi\rangle + \frac{1}{n} &< 1-(\varphi-1)^{2s-2}+2(\varphi-1)^{2s+1}+(\varphi-1)^{2s}\\ &= 1-(\varphi-1)^{2s+2} < 1,\end{aligned}$$

where in the last step we used the identity $1-2(\varphi-1)^3-(\varphi-1)^2=(\varphi-1)^4$.

(5) Let $a_0 = \ldots = a_{r-1} = 1$, $a_r = 0$ for some $1 \le r < s-1$, $a_j \in \{0,1,2\}$ for $r < j < s$, and $a_s \in \{1,2\}$. Then

$$n=\sum_{h=0}^{r-1}F_{2h+1}+\sum_{h=r+1}^{s}a_hF_{2h+1}=F_{2r}+\sum_{h=r+1}^{s}a_hF_{2h+1}. \tag{6}$$

From the previous equation, we get

$$\begin{aligned}\langle n\varphi\rangle &= \left\langle \left(F_{2r}+\sum_{h=r+1}^{s}a_hF_{2h+1}\right)\varphi\right\rangle = \left\langle F_{2r}\varphi+\sum_{h=r+1}^{s}a_hF_{2h+1}\varphi\right\rangle\\ &\le \langle F_{2r}\varphi\rangle + 2\sum_{h=r+1}^{s}\langle F_{2h+1}\varphi\rangle.\end{aligned}$$

From the identity (4) and the inequality (5), we get

$$\langle n\varphi\rangle < 1-(\varphi-1)^{2r}+2(\varphi-1)^{2r+2}.$$

Coming back to the equation (6), remembering that in this case $s \geq r+2$ and having in mind the identity (3), we have

$$n \geq F_{2r} + F_{2r+5} = F_{2r} + F_{2r+2} + 2F_{2r+3} > F_{2r+2} + F_{2r+3}\varphi = \varphi^{2r+3},$$

i.e., $\frac{1}{n} < (\varphi-1)^{2r+3}$. Then

$$\langle n\varphi \rangle + \frac{1}{n} < 1 - (\varphi-1)^{2r} + 2(\varphi-1)^{2r+2} + (\varphi-1)^{2r+3} = 1,$$

where in the last step we used the identity $2(\varphi-1)^2 + (\varphi-1)^3 = 1$.

(6) Let $a_0 = \ldots = a_{r-1} = 1$, $a_r = 2$ for some $1 \leq r \leq s$, $a_j \in \{0,1,2\}$ for $r < j < s$, and $a_s \in \{1,2\}$. Proceeding as in the previous case, we have

$$n = \sum_{h=0}^{r-1} F_{2h+1} + 2F_{2r+1} + \sum_{h=r+1}^{s} a_h F_{2h+1} = F_{2r+3} + \sum_{h=r+1}^{s} a_h F_{2h+1}.$$

Then

$$\begin{aligned}\langle n\varphi \rangle &= \left\langle \left(F_{2r+3} + \sum_{h=r+1}^{s} a_h F_{2h+1} \right) \varphi \right\rangle = \left\langle F_{2r+3}\varphi + \sum_{h=r+1}^{s} a_h F_{2h+1}\varphi \right\rangle \\ &\leq \langle F_{2r+3}\varphi \rangle + 2 \sum_{h=r+1}^{s} \langle F_{2h+1}\varphi \rangle .\end{aligned}$$

From the identity (4) and the inequality (5), we get

$$\langle n\varphi \rangle < (\varphi-1)^{2r+3} + 2(\varphi-1)^{2r+2} \leq (\varphi-1)^5 + 2(\varphi-1)^4 < \frac{2}{5}.$$

Then, as $n > 1$, we have $\langle n\varphi \rangle + \frac{1}{n} < 1$. □

From the following result of Möbius [4], we easily obtain the characterization for the Fibonacci numbers of odd index.

Theorem 2 ([4]). *Let n be a positive integer. Then n is a Fibonacci number if and only if the real open interval $\left(n\varphi - \frac{1}{n}, n\varphi + \frac{1}{n}\right)$ contains exactly one integer.*

Corollary 1. *Let n be a positive integer. Then n is a Fibonacci number of odd index if and only if $\langle n\varphi \rangle - \frac{1}{n} < 0$.*

Proof. Let $\langle n\varphi \rangle - \frac{1}{n} < 0$, $m = \left\lceil n\varphi - \frac{1}{n} \right\rceil$ and suppose $n > 1$. From the hypothesis, $\langle n\varphi - \frac{1}{n} \rangle = \langle n\varphi \rangle - \frac{1}{n} + 1$ and then

$$n\varphi - \frac{1}{n} = \left\lfloor n\varphi - \frac{1}{n} \right\rfloor + \left\langle \varphi - \frac{1}{n} \right\rangle = m - 1 + \langle n\varphi \rangle - \frac{1}{n} + 1 = m + \langle n\varphi \rangle - \frac{1}{n} < m.$$

On the other hand,

$$n\varphi + \frac{1}{n} = n\varphi - \frac{1}{n} + \frac{2}{n} = m + \langle n\varphi \rangle + \frac{1}{n} > m,$$

and then the interval $\left(n\varphi - \frac{1}{n}, n\varphi + \frac{1}{n}\right)$ contains exactly one integer, so n is a Fibonacci number. Moreover

$$\langle n\varphi \rangle + \frac{1}{n} = \langle n\varphi \rangle - \frac{1}{n} + \frac{2}{n} < \frac{2}{n} < 1,$$

and, by Theorem 1, it does not have even index. Conversely, suppose $n = F_{2m+1}$. From the identity (4), $\langle F_{2m+1}\varphi \rangle = (\varphi - 1)^{2m+1}$, and from the well known Binet's formula (see [6], p. 52) we have $F_{2m+1} = \frac{\varphi^{2m+1} + (\varphi-1)^{2m+1}}{\sqrt{5}}$. Then, with some straightforward calculation, we get

$$\langle F_{2m+1}\varphi \rangle - \frac{1}{n} = (\varphi - 1)^{2m+1} - \frac{1}{F_{2m+1}} = (\varphi - 1)^{2m+1} - \frac{\sqrt{5}}{\varphi^{2m+1} + (\varphi - 1)^{2m+1}} < 0.$$

□

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