AN EQUATION INVOLVING ARITHMETIC FUNCTIONS AND RIESEL NUMBERS

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Received: 2/10/17, Revised: 6/4/18, Accepted: 10/18/18, Published: 10/26/18

Abstract

Let σ_a denote the generalized sum of divisors function. We prove that given any integers a, n, and s with $a \geq 0$, $n \geq 1$, and $s \geq 1$, there exist infinitely many pairs (k, N) such that $\sigma_a(N) = k \cdot 2^n$, where k is a Riesel number and N has exactly s distinct prime factors. In the case $a = 1$, we also show that additional conditions can be imposed to guarantee that N is square-free.

1. Introduction

In 1960, Sierpiński [4] published an article that proved the existence of infinitely many odd integers k such that $k \cdot 2^n + 1$ is composite for all positive integers n. Such numbers today are known as *Sierpiński numbers*. For Euler's totient function ϕ, it follows from the definition that if k is a Sierpiński number, then $k \cdot 2^n \neq \phi(q)$ for any prime q and any positive integer n. Very recently Gonzalez, Luca, and Huguet [2] published a proof of the following theorem.

Theorem 1. *For all integers $n \geq s \geq 2$ there exist infinitely many Sierpiński numbers k such that the equation*

$$2^n \cdot k = \phi(N)$$

holds with some positive integer N having exactly s distinct prime factors.

We note that if it is possible to loosen the restriction $n \geq s$ in Theorem 1, the best scenario would be the restriction $n \geq s-1$. To see this, let $q_1, q_2, \ldots, q_{s-1}$ be distinct odd primes, let $\ell_0, \ell_1, \ldots, \ell_{s-1}$ be positive integers, and let $N = 2^{\ell_0} q_1^{\ell_1} q_2^{\ell_2} \cdots q_{s-1}^{\ell_{s-1}}$. Then

$$\phi(N) = 2^{\ell_0 - 1} \prod_{i=1}^{s-1} q_i^{\ell_i - 1}(q_i - 1).$$

Since each q_i is odd, it follows that $\phi(N) \equiv 0 \pmod{2^{s-1}}$. Thus, if $2^n \cdot k = \phi(N)$ for some odd integer k, then n must be at least the number of distinct odd prime divisors of N.

Predating Sierpiński's article, in 1956 Riesel [3] published an article that proved the existence of infinitely many odd integers k such that $k \cdot 2^n - 1$ is composite for all positive integers n. Such numbers are known as *Riesel numbers*. Let σ_a be the generalized sums of divisors function defined by

$$\sigma_a(N) = \sum_{d|N} d^a.$$

It follows from the definition that if k is a Riesel number, then $k \cdot 2^n \neq \sigma_1(q)$ for any prime q and any positive integer n. In this article we will prove an analogue of Theorem 1 for the equation $2^n \cdot k = \sigma_a(N)$ where k is a Riesel number. In this new setting, the restriction $n \geq s$ in Theorem 1 is not required (see Theorem 6). We will also show that when $a = 1$ we can take N to be square-free (see Theorem 5).

2. Preliminary Results

An important concept to the proofs in this paper, originally due to Erdos [1], is that of a covering system of the integers. A *covering system* of the integers (or covering) is a finite collection of congruences such that every integer satisfies at least one of the congruences in the collection. Chen [5] provided the following definition regarding covering systems.

Definition 2. A covering system $\mathcal{C} = \{r_i \pmod{m_i}\}_{i=1}^t$ is called a $(2,1)$-*primitive covering* if $\mathcal{C}$ is a covering system and there exist distinct primes $p_1, p_2, \ldots, p_t$ such that p_i is a primitive prime divisor of $2^{m_i} - 1$ for $1 \leq i \leq t$.

Throughout this article we will present a $(2,1)$-primitive covering as a set of triples

$$\mathcal{C} = \{(r_i, m_i, p_i)\}_{i=1}^t$$

where the $r_i \pmod{m_i}$ are the congruences of the covering system, and the p_i are distinct primitive prime divisors of $2^{m_i} - 1$.

Though Chen's definition came long after Sierpiński and Riesel's articles, $(2,1)$-primitive covering systems were instrumental in proving their results. In fact, Chen showed that if $\mathcal{C}$ is a $(2,1)$-primitive covering such that every integer satisfies at least M congruences of $\mathcal{C}$, then there exist infinitely many Sierpiński numbers k such that $k \cdot 2^n + 1$ has at least $M+1$ distinct prime factors for all positive integers n. Though not mentioned in his paper, Chen's result can easily be adjusted to prove the analogous result on Riesel numbers. To illustrate how $(2,1)$-primitive coverings will be used in this article, we end this section with a proof of Riesel's result.

Theorem 3. *There exist infinitely many odd integers k such that $k \cdot 2^n - 1$ is composite for all positive integers n.*

Proof. Let $\mathcal{C} = \{(r_i, m_i, p_i)\}_{i=1}^t$ be a $(2,1)$-primitive covering. Notice that since p_i is a prime divisor of $2^{m_i} - 1$, if $n \equiv r_i \pmod{m_i}$, then $2^n \equiv 2^{r_i} \pmod{p_i}$. Thus, for each $1 \leq i \leq t$, choosing k so that $k \cdot 2^{r_i} - 1 \equiv 0 \pmod{p_i}$ guarantees that $k \cdot 2^n - 1$ is divisible by p_i. Since k needs to be odd, we add the condition that $k \equiv 1 \pmod 2$. We note that infinitely many such k can be found using the Chinese remainder theorem since the definition of $(2,1)$-primitive covering ensures that each p_i is odd and distinct. Choosing k large enough ensures that for any n, $k \cdot 2^n - 1 \neq p_i$ for any $1 \leq i \leq t$. Since every positive integer n satisfies some congruence in $\mathcal{C}$, our choice of k guarantees that $k \cdot 2^n - 1$ is composite for all positive integers n.

We now illustrate the execution of this technique using the $(2,1)$-primitive covering

$$\mathcal{C} = \{(0,2,3), (1,4,5), (1,3,7), (11,12,13), (15,18,19), (27,36,37), (3,9,73)\}.$$

Notice that

$$\begin{aligned}
n \equiv 0 \pmod{2}, &\qquad k \equiv 1 \pmod{3} \text{ implies } k \cdot 2^n - 1 \equiv 0 \pmod{3};\\
n \equiv 1 \pmod{4}, &\qquad k \equiv 3 \pmod{5} \text{ implies } k \cdot 2^n - 1 \equiv 0 \pmod{5};\\
n \equiv 1 \pmod{3}, &\qquad k \equiv 4 \pmod{7} \text{ implies } k \cdot 2^n - 1 \equiv 0 \pmod{7};\\
n \equiv 11 \pmod{12}, &\qquad k \equiv 2 \pmod{13} \text{ implies } k \cdot 2^n - 1 \equiv 0 \pmod{13};\\
n \equiv 15 \pmod{18}, &\qquad k \equiv 8 \pmod{19} \text{ implies } k \cdot 2^n - 1 \equiv 0 \pmod{19};\\
n \equiv 27 \pmod{36}, &\qquad k \equiv 31 \pmod{37} \text{ implies } k \cdot 2^n - 1 \equiv 0 \pmod{37};\\
n \equiv 3 \pmod{9}, &\qquad k \equiv 64 \pmod{73} \text{ implies } k \cdot 2^n - 1 \equiv 0 \pmod{73}.
\end{aligned}$$

Thus, any $k \equiv 140022313 \pmod{2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73}$ will be a Riesel number. $\square$

Definition 4. If a Riesel number k arises, as in the proof of Theorem 3, from the $(2,1)$-primitive covering $\mathcal{C}$, then we call k a $\mathcal{C}$-Riesel number.

Remark 1. *Using the proof of Theorem 3, it follows that if κ is a $\mathcal{C}$-Riesel number, then any integer k satisfying $k \equiv \kappa \pmod{2 \cdot p_1 \cdot p_2 \cdots p_t}$ is also a $\mathcal{C}$-Riesel number.*

3. Main Results

In this section we investigate the equation $\sigma_a(N) = k \cdot 2^n$. Suppose k is an odd integer and let $q_1, q_2, \ldots, q_s$ be distinct odd primes. If $N = q_1 \cdot q_2 \cdots q_s$, then $\sigma_a(N) = \prod_{i=1}^{s}(q_i^a + 1)$. Since each q_i is odd, it follows that $\sigma_a(N) \equiv 0 \pmod{2^s}$. Thus, if k is odd, N is square-free, and $\sigma_a(N) = k \cdot 2^n$, then n must be at least the number of distinct prime divisors of N. Hence, the restriction $n \geq s$ is necessary in the following theorem.

Theorem 5. *Let $\mathcal{C} = \{(r_j, m_j, p_j)\}_{j=1}^{t}$ with $3 \leq p_1 < p_2 < \cdots < p_t$ be a $(2,1)$-primitive covering and let κ be a $\mathcal{C}$-Riesel number. For integers $n \geq s \geq 2$, if one of the conditions*

(i) $p_1 \geq 5$,

(ii) $p_1 = 3$, $\kappa \cdot 2^n \equiv 1 \pmod 3$ *and s is even, or*

(iii) $p_1 = 3$, $\kappa \cdot 2^n \equiv 2 \pmod 3$ *and s is odd,*

holds, then there exist infinitely many pairs (k, N), where k is a $\mathcal{C}$-Riesel number and N is a square-free odd positive integer with s distinct prime divisors, such that $\sigma_1(N) = k \cdot 2^n$.

Proof. For each j with $1 \leq j \leq t$, if $p_j \geq 5$ we let $q_1, q_2, \ldots, q_s$ be primes satisfying

$$\prod_{i=1}^{s}(q_i + 1) \equiv \kappa \cdot 2^n \pmod{p_j}$$

in the following way.

Suppose that s is even. If p_j does not divide $\kappa \cdot 2^n + 1$, then we choose $q_1 \equiv -(\kappa \cdot 2^n + 1) \pmod{p_j}$ and for $2 \leq i \leq s$ we choose $q_i \equiv -2 \pmod{p_j}$. If p_j does divide $\kappa \cdot 2^n + 1$, then it does not divide $\kappa \cdot 2^{n-1} + 1$. In this case we choose $q_1 \equiv -(\kappa \cdot 2^{n-1} + 1) \pmod{p_j}$, $q_2 \equiv -3 \pmod{p_j}$, and for $3 \leq i \leq s$ we choose $q_i \equiv -2 \pmod{p_j}$.

Now suppose that s is odd. If p_j does not divide $\kappa \cdot 2^n - 1$, then we choose $q_1 \equiv \kappa \cdot 2^n - 1 \pmod{p_j}$ and for $2 \leq i \leq s$ we choose $q_i \equiv -2 \pmod{p_j}$. If p_j does divide $\kappa \cdot 2^n - 1$, then it does not divide $\kappa \cdot 2^{n-1} - 1$. In this case we choose $q_1 \equiv \kappa \cdot 2^{n-1} - 1 \pmod{p_j}$, $q_2 \equiv -3 \pmod{p_j}$, and for $3 \leq i \leq s$ we choose $q_i \equiv -2 \pmod{p_j}$.

If $p_1 = 3$, then also let $q_i \equiv 1 \pmod 3$ for $1 \leq i \leq s$. Further, let

$$q_1 \equiv 2^{n-s+1} - 1 \pmod{2^{n-s+2}}, \tag{1}$$

$$q_i \equiv 1 \pmod 4 \quad \text{for } 2 \leq i \leq s. \tag{2}$$

Since the p_j are distinct odd primes, the Chinese remainder theorem allows us to find positive integers q_i satisfying each of the required conditions, and by Dirichlet's theorem on primes in an arithmetic progression, we may require additionally that each q_i be prime. Now let $N = q_1 q_2 \cdots q_s$. Notice that for each $1 \leq j \leq t$,

$$\sigma_1(N) = \prod_{i=1}^{s}(q_i + 1) \equiv \kappa \cdot 2^n \pmod{p_j}.$$

Furthermore, (1) and (2) guarantee that $k = \sigma_1(N)/2^n$ is an odd integer. Thus, k is an integer satisfying $k \equiv \kappa \pmod{2 \cdot p_1 \cdot p_2 \cdots p_t}$. By Remark 1, k is a $\mathcal{C}$-Riesel number and $\sigma_1(N) = k \cdot 2^n$, as desired. □

Let $q_1, q_2, \ldots, q_s$ be distinct odd primes and let $M = q_1 \cdot q_2 \cdots q_s$. If $q_i \equiv -2 \pmod{p}$ for each i and some prime p, then $\sigma_1(q_i) \equiv -1 \pmod{p}$ since $\sigma_1(q_i) = q_i + 1$. Thus, $\sigma_1(M) \equiv (-1)^s \pmod{p}$. This observation plays a crucial role in the proof of Theorem 5 and explains the need for (ii) and (iii) in the statement of the theorem. In the concluding remarks of this article we will present a totient function analogue of Theorem 5. We note here that if $q_i \equiv 2 \pmod{p}$ for each i and some prime p, then $\phi(q_i) \equiv 1 \pmod{p}$ since $\phi(q_i) = q_i - 1$. Therefore, $\phi(M) \equiv 1 \pmod{p}$. This observation will allow us to ignore the parity of s in the totient function results at the end of this article.

Example 1. Suppose $n = s = 2$. In the proof of Theorem 3 we proved that $\kappa = 140022313$ is a $\mathcal{C}$-Riesel number for the $(2, 1)$-primitive covering

$$\mathcal{C} = \{(0, 2, 3), (1, 4, 5), (1, 3, 7), (11, 12, 13), (15, 18, 19), (27, 36, 37), (3, 9, 73)\}.$$

Following the proof of Theorem 5 we select $q_1 = 140415097$ and $q_2 = 2311664353$. Let $N = q_1 \cdot q_2 = 324592574357937241$ and let $k = 81148144202504173$. Then k is a $\mathcal{C}$-Riesel number, and $\sigma_1(N) = k \cdot 2^n$. Table 1 provides more examples for small values of n and s using the covering system $\mathcal{C}$.

n	s	q_i	k
2	2	140415097 231164353	81148144202504173
4	2	5044888231 2311664353	728880518489280133
6	2	22067227807 763181053	263145159021367963
8	2	152449857151 2311664353	1376612892776056093
3	3	619269913 189040213 2311664353	33827427401861235921698623
5	3	3023664967 798574957 2311664353	174431193937684642949939743
7	3	3512480863 110079253 2311664353	6982874731886299575240433
4	4	701761261 2311664353 5674085233 8476102633	4876253181299890612056417864929223643
6	4	6936333847 763181053 2311664353 5674085233	1084921275064059379558490666346206533
5	5	81546697 798574957 2311664353 5674085233 8476102633	226250176939959213459503006978362438695668263
7	5	2952077383 110079253 2311664353 5674085233 8476102633	282253986164145432879309224321736867939712663

Table 1: Small values of n

Notice that if q is an odd prime and ℓ is even, then $\sigma_a(q^\ell) = 1+q^a+q^{2a}+\cdots+q^{\ell a}$ is odd. Thus, if we remove the square-free condition on N in Theorem 5, the restriction

$n \geq s \geq 2$ may no longer be necessary. This observation leads us to the following theorem.

Theorem 6. *Let $\mathcal{C}$ be a $(2,1)$-primitive covering. Let a, n, and s be integers with $a \geq 0$, $n \geq 1$, and $s \geq 1$. There exist infinitely many pairs (k, N), where k is a $\mathcal{C}$-Riesel number and N is an odd positive integer with s distinct prime divisors, such that $\sigma_a(N) = k \cdot 2^n$.*

Proof. Let $\mathcal{C} = \{(r_i, m_i, p_i\}_{i=1}^{t}$ and let κ be a $\mathcal{C}$-Riesel number. For each $1 \leq j \leq t$, we let $q_1, q_2, \ldots, q_s$ be primes and $\ell_1, \ell_2, \ldots, \ell_s$ be positive integers satisfying

$$\prod_{i=1}^{s} \left(1 + q_i^a + q_i^{2a} + \cdots + q_i^{\ell_i a}\right) \equiv \kappa \cdot 2^n \pmod{p_j} \tag{3}$$

in the following way. For $1 \leq i \leq s$, let $q_i \equiv 1 \pmod{p_j}$ so that

$$1 + q_i^a + q_i^{2a} + \cdots + q_i^{\ell_i a} \equiv 1 + \ell_i \pmod{p_j}.$$

Now let $\ell_1 \equiv \kappa \cdot 2^n - 1 \pmod{p_j}$ and for $2 \leq i \leq s$ let $\ell_i \equiv 0 \pmod{p_j}$. These conditions then ensure that (3) holds. We now impose the following additional conditions:

$$q_i \equiv 1 \pmod{2^{n+1}} \quad \text{for } 1 \leq i \leq s, \tag{4}$$

$$\ell_1 \equiv 2^n - 1 \pmod{2^{n+1}}, \tag{5}$$

$$\ell_i \equiv 0 \pmod{2^{n+1}} \quad \text{when } 2 \leq i \leq s. \tag{6}$$

The Chinese remainder theorem guarantees the existence of q_i and ℓ_i satisfying the mentioned conditions, and Dirichlet's theorem on primes in an arithmetic progression further allows us to require that each q_i be prime. Now let $N = q_1^{\ell_1} q_2^{\ell_2} \cdots q_s^{\ell_s}$. Then

$$\sigma_a(N) = \prod_{i=1}^{s} \left(1 + q_i^a + q_i^{2a} + \cdots + q_i^{\ell_i a}\right).$$

Notice that (4), (5), and (6) ensure that

$$1 + q_1^a + q_1^{2a} + \cdots + q_1^{\ell_1 a} \equiv 2^n \pmod{2^{n+1}}$$

and for $2 \leq i \leq s$,

$$1 + q_i^a + q_i^{2a} + \cdots + q_i^{\ell_i a} \equiv 1 \pmod{2^{n+1}}$$

so that $k = \sigma_a(N)/2^n$ is an odd integer. Thus, k is an integer satisfying $k \equiv \kappa \pmod{2 \cdot p_1 \cdot p_2 \cdots p_t}$. By Remark 1, k is a $\mathcal{C}$-Riesel number and $\sigma_a(N) = k \cdot 2^n$, as desired. $\square$

Explicit examples of (N, k) for Theorem 6 do not lend themselves as easily as those given after Theorem 5. The construction laid out in the proof of Theorem 6 requires each ℓ_i to be significantly large. Specifically, for $2 \leq i \leq s$ we need $\ell_i \geq \prod_{j=1}^{t} p_j$. Thus, if we were to use the covering system $\mathcal{C}$ that was presented in the proof of Theorem 3, we would need each $\ell_i \geq 70050435$. This makes examples (N, k) far too large to give an explicit example in this article.

4. Concluding Remarks

Techniques similar to those used in the proofs in this article can be used to prove the following theorems, which we state without proof.

The following two theorems are the Sierpiński analogues of Theorems 5 and 6, respectively.

Theorem 7. *Let $\mathcal{C} = \{(r_j, m_j, p_j)\}_{j=1}^{t}$ with $3 \leq p_1 < p_2 < \cdots < p_t$ be a $(2,1)$-primitive covering and let κ be a $\mathcal{C}$-Sierpiński number. For integers $n \geq s \geq 2$, if one of the conditions*

(i) $p_1 \geq 5$,

(ii) $p_1 = 3$, $\kappa \cdot 2^n \equiv 1 \pmod 3$ *and s is even, or*

(iii) $p_1 = 3$, $\kappa \cdot 2^n \equiv 2 \pmod 3$ *and s is odd,*

holds, then there exist infinitely many pairs (k, N), where k is a $\mathcal{C}$-Sierpiński number and N is a square-free odd positive integer with s distinct prime divisors, such that $\sigma_1(N) = k \cdot 2^n$.

Theorem 8. *Let $\mathcal{C}$ be a $(2,1)$-primitive covering. Let a, n, and s be integers with $a \geq 0$, $n \geq 1$, and $s \geq 1$. There exist infinitely many pairs (k, N), where k is a $\mathcal{C}$-Sierpiński number and N is an odd positive integer with s distinct prime divisors, such that $\sigma_a(N) = k \cdot 2^n$.*

The next two theorems are totient function analogues of Theorems 5 and 6, respectively.

Theorem 9. *Let $\mathcal{C} = \{(r_j, m_j, p_j)\}_{j=1}^{t}$ with $3 \leq p_1 < p_2 < \cdots < p_t$ be a $(2,1)$-primitive covering and let κ be a $\mathcal{C}$-Riesel number. For integers $n \geq s \geq 2$, if one of the conditions*

(i) $p_1 \geq 5$ *or*

(ii) $p_1 = 3$ *and* $\kappa \cdot 2^n \equiv 1 \pmod 3$

holds, then there exist infinitely many pairs (k, N)*, where* k *is a* $\mathcal{C}$*-Riesel number and* N *is a square-free positive integer with* s *distinct prime divisors, such that* $\phi(N) = k \cdot 2^n$*.*

Theorem 10. *Let* $\mathcal{C}$ *be a* $(2, 1)$*-primitive covering. For integers* $n \geq s \geq 2$ *there exist infinitely many pairs* (k, N)*, where* k *is a* $\mathcal{C}$*-Riesel number and* N *is a positive integer with* s *distinct prime divisors, such that* $\phi(N) = k \cdot 2^n$*.*

The last two theorems are totient function analogues of Theorems 7 and 8, respectively.

Theorem 11. *Let* $\mathcal{C} = \{(r_j, m_j, p_j)\}_{j=1}^t$ *with* $3 \leq p_1 < p_2 < \cdots < p_t$ *be a* $(2, 1)$*-primitive covering and let* κ *be a* $\mathcal{C}$*-Sierpiński number. For integers* $n \geq s \geq 2$*, if one of the conditions*

(i) $p_1 \geq 5$ *or*

(ii) $p_1 = 3$ *and* $\kappa \cdot 2^n \equiv 1 \pmod 3$

holds, then there exist infinitely many pairs (k, N)*, where* k *is a* $\mathcal{C}$*-Sierpiński number and* N *is a square-free positive integer with* s *distinct prime divisors, such that* $\phi(N) = k \cdot 2^n$*.*

Theorem 12. *Let* $\mathcal{C}$ *be a* $(2, 1)$*-primitive covering. For integers* $n \geq s \geq 2$ *there exist infinitely many pairs* (k, N)*, where* k *is a* $\mathcal{C}$*-Sierpiński number and* N *is a positive integer with* s *distinct prime divisors, such that* $\phi(N) = k \cdot 2^n$*.*

References

[1] P. Erdős, On integers of the form $2^k + p$ and some related problems, *Summa Brasil. Math.* (1950), 113-123.

[2] M. González, F. Luca, V. Huguet, On Sierpiński numbers of the form $\phi(N)/2^n$, *Integers* **16** (2016), A89, 7pp.

[3] H. Riesel, Några stora primtal, *Elementa* **39** (1956), 258-260.

[4] W. Sierpiński, Sur un probleme concernant les nombres $k2^n+1$, *Elem. Math.* **15** (1960), 73-74.

[5] Y.G. Chen, On integers of the form $k - 2^n$ and $k2^n + 1$, *J. Number Theory* **89** (2001), 121-125.