ISOSCELES TRIANGLES IN $\mathbb{Q}^3$

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Abstract

In this article, we give a characterization of isosceles triangles having as their vertices points of the rational Euclidean space $\mathbb{Q}^3$. For positive integers d and r where $\sqrt{d}$ and $\sqrt{r}$ are both realized as distances in $\mathbb{Q}^3$ and $r = a^2 + b^2 + c^2$ for integers a, b, c, it is shown that the triangle with edges of length $\sqrt{d}$, $\sqrt{d}$, and $\sqrt{r}$ is embeddable in $\mathbb{Q}^3$ if and only if the Diophantine equation $x^2 + ry^2 - (4d - r)(a^2 + b^2)z^2 = 0$ has a non-trivial integer solution. This condition is used to give new proofs of a few recent results concerning $G(\mathbb{Q}^3, \sqrt{r})$, the Euclidean distance graph with vertex set $\mathbb{Q}^3$ where adjacent vertices are at distance $\sqrt{r}$ apart. Originally shown by E. J. Ionascu, we prove that $G(\mathbb{Q}^3, \sqrt{r})$ contains K_3 as a subgraph if and only if the square-free part of r is even but has no odd factor congruent to 2 (mod 3). We conclude by proving that each of those graphs $G(\mathbb{Q}^3, \sqrt{r})$ which contain K_3 as a subgraph have chromatic number 4.

1. Introduction

As is customary, the rings of real numbers, rational numbers, and integers are given by $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$, respectively. For any $S \subset \mathbb{R}$, S^n denotes the set of all n-tuples whose coordinate entries are elements of S. Throughout this paper, the distance between points of any S^n will be calculated using the standard Euclidean distance metric. For $a, b, c \in \mathbb{R}^+$, we will denote by $T(a, b, c)$ the triangle having edges of length a, b, and c, respectively. For any $S \subset \mathbb{R}^n$, define $T(a, b, c)$ to be *embeddable*, or alternately *representable*, in S if the triangle can be oriented such that its vertices are points of S.

We will phrase a large portion of our work in the language and notation of Euclidean distance graphs. For $d > 0$ and $S \subset \mathbb{R}^n$, let $G(S, d)$ denote the graph with vertex set S where any two vertices are adjacent if and only if they are a Euclidean distance d apart. Such graphs have been widely studied (see [7], [14], or [15]) with the foremost structural question being the determination of the chromatic number $\chi(S, d)$. In other words, $\chi(S, d)$ represents the minimum number of colors needed to color the vertices of $G(S, d)$ such that no two adjacent vertices receive the same color.

Central to our work will also be a few fundamentals from classical number theory. We expect the majority of readers to be familiar with these concepts, but in the event clarification is needed, one could consult virtually any textbook on elementary number theory, with our personal preference being [9]. For integers a, b, say that a is a *quadratic residue* of b if there exists an integer x such that $x^2 \equiv a \pmod{b}$. Often the word "quadratic" is dropped from print, which will be the case here as well. For an integer a and prime p, define the Legendre symbol $\left(\frac{a}{p}\right)$ as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } a \text{ is a multiple of } p \\ 1 & \text{if } a \text{ is a residue of } p \\ -1 & \text{if } a \text{ is not a residue of } p \end{cases}$$

An important multiplicative property of the Legendre symbol is that for integers a, b and prime p, $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$. Additional properties of the Legendre symbol relevant to our discussion are summarized by the law of quadratic reciprocity which is given with its common supplements below.

Theorem 1. *Let p, q be odd, positive primes. Then each of the following is true.*

(i) If at least one of p, q is congruent to $1 \pmod 4$, then $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$.

(ii) If both of p, q are congruent to $3 \pmod 4$, then $\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right)$.

(iii) $\left(\frac{-1}{p}\right) = 1$ if and only if $p \equiv 1 \pmod 4$.

(iv) $\left(\frac{2}{p}\right) = 1$ if and only if $p \equiv \pm 1 \pmod 8$.

The structure of the paper will be as follows. In Section 2, for positive integers r and d, we give a necessary and sufficient condition for $T(\sqrt{r}, \sqrt{d}, \sqrt{d})$ to be embeddable in $\mathbb{Q}^3$. Supposing that $\sqrt{r}$ and $\sqrt{d}$ are both distances actually realized between points of $\mathbb{Q}^3$ and $r = a^2 + b^2 + c^2$ for $a, b, c \in \mathbb{Z}$ with a, b not both equal to 0, it is shown that $T(\sqrt{r}, \sqrt{d}, \sqrt{d})$ is embeddable in $\mathbb{Q}^3$ if and only if the Diophantine equation $x^2 + ry^2 - (4d - r)(a^2 + b^2)z^2 = 0$ has a non-trivial solution. Furthermore, we show that the solvability of this equation is independent of which particular a, b, c are chosen to satisfy the above conditions. In Section 3, we use this characterization to prove a variant of a result of Ionascu [4] concerning the possible edge-lengths of equilateral triangles in $\mathbb{Q}^3$. We address Euclidean distance graphs $G(\mathbb{Q}^3, \sqrt{r})$ in Section 4, giving a new proof of the author's recent result [12] that if $G(\mathbb{Q}^3, \sqrt{r})$ contains a triangle, then $\chi(\mathbb{Q}^3, \sqrt{r}) = 4$.

2. Isosceles Triangles in $\mathbb{Q}^3$

We begin with a warmup problem in the plane.

Theorem 2. *Let $r, d \in \mathbb{Q}^+$ where $\sqrt{r}$ and $\sqrt{d}$ are both realized as distances between points of $\mathbb{Q}^2$. An isosceles triangle $T(\sqrt{r}, \sqrt{d}, \sqrt{d})$ is embeddable in $\mathbb{Q}^2$ if and only if $\sqrt{r(4d-r)} \in \mathbb{Q}^+$.*

Proof. Let $T = T(\sqrt{r}, \sqrt{d}, \sqrt{d})$. As described in [13] (and probably elsewhere as well), for vectors $v_1, v_2 \in \mathbb{Q}^2$, there exists a distance-scaling bijection $\varphi : \mathbb{Q}^2 \to \mathbb{Q}^2$ such that $\varphi(v_1) = v_2$. In other words, for any points $p_1, p_2 \in \mathbb{Q}^2$, $|\varphi(p_1) - \varphi(p_2)| = \frac{|v_2|}{|v_1|}|p_1 - p_2|$. With this in mind, we have that T is embeddable in $\mathbb{Q}^2$ if and only if $T' = T(1, \sqrt{\frac{d}{r}}, \sqrt{\frac{d}{r}})$ is embeddable in $\mathbb{Q}^2$. Furthermore, we may assume that the side of T' of length 1 lies on the x-axis, say with endpoints $(0,0)$ and $(1,0)$. It follows that T' is embeddable in $\mathbb{Q}^2$ if and only if there exists a point $(\frac{1}{2}, y) \in \mathbb{Q}^2$ such that $(\frac{1}{2})^2 + y^2 = \frac{d}{r}$, or stated more simply, if and only if $\sqrt{r(4d-r)} \in \mathbb{Q}^+$. $\square$

In $\mathbb{Q}^3$, however, the situation is murkier. We do not have access to a 3-dimensional analogue of the transformation φ described in the proof of Theorem 2, and instead, we must make do with the following two lemmas. Lemma 1 is immediate as multiplication by any positive rational number is a distance-scaling bijection from any $\mathbb{Q}^n$ to $\mathbb{Q}^n$. Lemma 2 follows from Rodrigues' rotation formula and is given with proof in [12].

Lemma 1. *For any $q \in \mathbb{Q}^+$, there exists a bijection $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ that scales distance by a factor of q.*

Lemma 2. *For vectors $v_1, v_2 \in \mathbb{Q}^3$ with $|v_1| = |v_2|$, there exists a distance-preserving bijection $\phi : \mathbb{Q}^3 \to \mathbb{Q}^3$ such that $\phi(v_1) = v_2$.*

In attempting to determine if a triangle $T(\sqrt{r}, \sqrt{d}, \sqrt{d})$ is embeddable in $\mathbb{Q}^3$, without loss of generality we may assume by Lemma 1 that r, d are both integers. Now suppose $r = a^2 + b^2 + c^2$ for $a, b, c \in \mathbb{Z}$. The question of whether or not $T(\sqrt{r}, \sqrt{d}, \sqrt{d})$ is embeddable in $\mathbb{Q}^3$ is equivalent to the existence of a point $(x, y, z) \in \mathbb{Q}^3$ simultaneously at distance $\sqrt{d}$ from the origin and the point (a, b, c). And in light of Lemma 2, although r may have many representations as a sum of three squares, it does not matter how the a, b, c are chosen.

Consider the isosceles triangle T with vertices $(0,0,0)$, (a,b,c), and (x,y,z). We may extend T to a right triangle T' having vertices $(0,0,0)$, (a,b,c), and $(2x,2y,2z)$ as in Figure 1 below, and we have that T is embeddable in $\mathbb{Q}^3$ if and only if T' is embeddable in $\mathbb{Q}^3$. The question of whether T' is embeddable in $\mathbb{Q}^3$ is equivalent to whether there exist orthogonal $\mathbb{Q}^3$ vectors v_1 and v_2 such that $v_1 = \langle a, b, c \rangle$ and $v_2 = \langle \alpha, \beta, \gamma \rangle$ where $\alpha^2 + \beta^2 + \gamma^2 = 4d - r$. This brings us to the main theorem of this section.

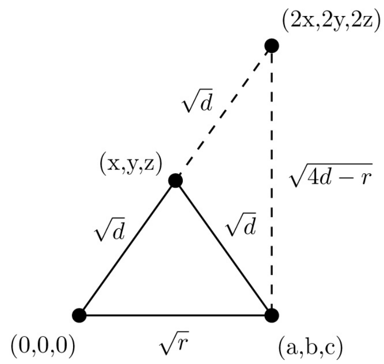

Figure 1:

Theorem 3. *Let $r, d \in \mathbb{Z}^+$ where $\sqrt{r}$ and $\sqrt{d}$ are both realized as distances in $\mathbb{Q}^3$. Let $r = a^2 + b^2 + c^2$ for $a, b, c \in \mathbb{Z}$ with a, b not both equal to zero. Then the triangle $T(\sqrt{r}, \sqrt{d}, \sqrt{d})$ is embeddable in $\mathbb{Q}^3$ if and only if the Diophantine equation $x^2 + ry^2 - (4d - r)(a^2 + b^2)z^2 = 0$ has a non-trivial integer solution.*

Proof. Let $T = T(\sqrt{r}, \sqrt{d}, \sqrt{d})$, and assume $a \neq 0$. With regards to the comments above, T is embeddable in $\mathbb{Q}^3$ if and only if there exist $\alpha, \beta, \gamma \in \mathbb{Q}$ such that $\alpha^2 + \beta^2 + \gamma^2 = 4d - r$ and $a\alpha + b\beta + c\gamma = 0$.

We first consider the case of $c = 0$. Hence $a\alpha + b\beta = 0$. If $b = 0$, we have that $a\alpha = 0$ implies $\alpha = 0$, which is equivalent to $4d - r$ being representable as a sum of two squares. Now considering the equation $x^2 + ry^2 - (4d - r)(a^2 + b^2)z^2 = 0$, we simplify to $x^2 + a^2y^2 - (4d - r)(a^2)z^2 = 0$ which, through "absorbing" the a^2 coefficient, is solvable if and only if $x^2 + y^2 - (4d - r)z^2 = 0$ is solvable. However, the last equation can be rewritten further as $(\frac{x}{z})^2 + (\frac{y}{z})^2 = 4d - r$ which is solvable if and only if $4d - r$ can be expressed as a sum of two squares. If $b \neq 0$, we write $\beta = \frac{a\alpha}{-b}$ and substitute to get $\alpha^2 + (\frac{a\alpha}{-b})^2 + \gamma^2 = 4d - r$. This simplifies to $\alpha^2(1 + \frac{a^2}{b^2}) + \gamma^2 = 4d - r$ where we may then write $\alpha = \frac{xb}{z}$ and $\gamma = \frac{y}{z}$, and then substitute again to get $(\frac{x}{z})^2(a^2 + b^2) + (\frac{y}{z})^2 = 4d - r$. Multiplying both sides by $z^2(a^2 + b^2)$ and then absorbing the $(a^2 + b^2)$ coefficient into x^2 leaves us with $x^2 + ry^2 - (4d - r)(a^2 + b^2)z^2 = 0$.

We now assume $c \neq 0$ and again consider the equation $\alpha^2 + \beta^2 + \gamma^2 = 4d - r$. Write $\gamma = \frac{a\alpha + b\beta}{-c}$ and substitute to obtain

$$\alpha^2 + \beta^2 + \left(\frac{a\alpha + b\beta}{-c}\right)^2 = 4d - r \tag{1}$$

which simplifies to Equation 2 below.

$$(a^2 + c^2)\alpha^2 + (b^2 + c^2)\beta^2 + 2ab\alpha\beta = (4d - r)c^2 \tag{2}$$

We now perform a linear transformation to eliminate the $\alpha\beta$ term. Let $\alpha = y_0\left(\frac{a}{a^2+b^2}\right) - x_0\left(\frac{b}{a^2+b^2}\right)$ and $\beta = y_0\left(\frac{b}{a^2+b^2}\right) + x_0\left(\frac{a}{a^2+b^2}\right)$. After making this substitution and executing the somewhat arduous process of simplifying, we are eventually left with Equation 3.

$$c^2{x_0}^2 + r{y_0}^2 = (4d-r)c^2(a^2+b^2) \tag{3}$$

Note that Equation 2 is solvable in rationals α, β if and only if Equation 3 is solvable in rationals x_0, y_0. Now moving to homogeneous coordinates by letting $x_0 = \frac{x}{cz}$ and $y_0 = \frac{y}{cz}$, we arrive at Equation 4.

$$x^2 + ry^2 - (4d-r)(a^2+b^2)z^2 = 0 \tag{4}$$

This completes the proof of the theorem. □

The advantage (at least as it appears to us) of Theorem 3 is that the potential solvability of homogeneous quadratic Diophantine equations in three variables (like Equation 4) is addressed by a classical and well-known theorem of Legendre which we give as Theorem 4. For a proof, see [9].

Theorem 4. *Let a, b, c be non-zero integers, not each positive or each negative, and suppose that abc is square-free. Then the equation*

$$ax^2 + by^2 + cz^2 = 0$$

has a non-trivial integer solution (x, y, z) if and only if each of the following are satisfied:

(i) $-ab$ is a quadratic residue of c

(ii) $-ac$ is a quadratic residue of b

(iii) $-bc$ is a quadratic residue of a.

As an example of the interplay between Theorems 3 and 4, consider the triangle $T(\sqrt{6}, \sqrt{10}, \sqrt{10})$. Writing $6 = 1^2 + 1^2 + 2^2$, Equation 4 becomes $x^2 + 6y^2 - (34)(2)z^2 = 0$. Absorbing squared terms into z, we have that the previous equation is solvable if and only if $x^2 + 6y^2 - 17z^2 = 0$ is solvable. Now applying Theorem 4 along with Theorem 1, we have that this equation is in fact not solvable as the Legendre symbol $(\frac{2}{3}) = -1$ and thus 17 is not a residue of 6. So $T(\sqrt{6}, \sqrt{10}, \sqrt{10})$ is not embeddable in $\mathbb{Q}^3$.

For another cavalierly-generated example, consider $T(\sqrt{134}, \sqrt{110}, \sqrt{110})$. Writing $134 = 2^2 + 3^2 + 11^2$, Equation 4 becomes $x^2 + 134y^2 - (306)(13)z^2 = 0$ which is solvable if and only if $2x^2 + 67y^2 - (13)(17)z^2 = 0$ is solvable. Manipulation of Legendre symbols in conjunction with Theorem 1 shows that each of the expressions

$\left(\frac{2}{67}\right)\left(\frac{13}{67}\right)\left(\frac{17}{67}\right)$, $\left(\frac{-1}{13}\right)\left(\frac{2}{13}\right)\left(\frac{67}{13}\right)$, and $\left(\frac{-1}{17}\right)\left(\frac{2}{17}\right)\left(\frac{67}{17}\right)$ are equal to 1. By Theorem 3, $T(\sqrt{134}, \sqrt{110}, \sqrt{110})$ is embeddable in $\mathbb{Q}^3$. And for those readers who remain skeptical, we note that the points $(0,0,0)$, $(9,5,-2)$, and $(2,11,3)$ do indeed constitute the vertices of such a triangle.

We close this section by remarking that Lemma 2 guarantees that for given $r, d \in \mathbb{Z}^+$, either all pairs of $\mathbb{Q}^3$ points distance $\sqrt{r}$ apart form two vertices of the isosceles triangle $T(\sqrt{r}, \sqrt{d}, \sqrt{d})$ or no pairs of $\mathbb{Q}^3$ points distance $\sqrt{r}$ apart form two vertices of $T(\sqrt{r}, \sqrt{d}, \sqrt{d})$. However, when letting $r = a^2 + b^2 + c^2$, it is not immediately obvious that the solvability of Equation 4 is independent of the selection of a and b. We allay those concerns with the following observation. Since $r - (a^2 + b^2) = c^2$, $-(a^2 + b^2)$ is always a residue of r and r is always a residue of $(a^2 + b^2)$. Furthermore, by a well-known characterization of integers that are representable as a sum of two squares (again, see [9] for elaboration), we have that the square-free part of $(a^2 + b^2)$ can contain no prime factors congruent to 3 (mod 4). By Theorem 1, $-r$ is a residue of the square-free part of $(a^2 + b^2)$ as well.

3. Equilateral Triangles in $\mathbb{Q}^3$

In a series of papers (see [4] and [5]), Ionascu develops a characterization of equilateral triangles whose vertices are points of $\mathbb{Z}^3$. Appearing in [4] is the following result which in the author's eyes has proven particularly useful to the study of Euclidean distance graphs $G(\mathbb{Q}^3, d)$.

Theorem 5. *Let $r \in \mathbb{Z}^+$. An equilateral triangle of side length $\sqrt{r}$ is embeddable in $\mathbb{Z}^3$ if and only if the square-free part of r is even but contains no odd factor congruent to* 2 (mod 3).

Ionascu proves Theorem 5 by first establishing a few propositions concerning binary quadratic forms. However, in this section, we aim to prove a weaker version of his result using only Theorems 3 and 4 along with quadratic reciprocity. Note also that the embeddability of an equilateral triangle $T(\sqrt{r}, \sqrt{r}, \sqrt{r})$ in $\mathbb{Q}^3$ is equivalent to the existence of the complete graph K_3 as a subgraph of $G(\mathbb{Q}^3, \sqrt{r})$.

Theorem 6. *Let $r \in \mathbb{Z}^+$. The distance graph $G(\mathbb{Q}^3, \sqrt{r})$ contains K_3 as a subgraph if and only if the square-free part of r is even but contains no odd factor congruent to* 2 (mod 3).

Proof. Regarding Lemma 1, we may without loss of generality assume r to be square-free. Writing $r = a^2 + b^2 + c^2$ for integers a, b, c with a non-zero, and setting $d = r$ in Equation 4, we arrive at Equation 5 below.

$$rx^2 + y^2 - 3(a^2 + b^2)z^2 = 0 \tag{5}$$

Let w be the square-free part of (a^2+b^2) and note that we may select the a, b so that w is odd. Furthermore, note that 3 does not divide w as w is a square-free sum of two squares. Let $g = \gcd(r, w)$, and set $r_0 = \frac{r}{g}$ and $w_0 = \frac{w}{g}$. Equation 6 now fits the required form of Theorem 4 and is solvable if and only if Equation 5 is solvable.

$$r_0x^2 + gy^2 - 3w_0z^2 = 0 \tag{6}$$

For sufficiency, we must verify three things – i) $3gw_0$ is a residue of r_0, ii) $-gr_0$ is a residue of $3w_0$, and iii) $3w_0r_0$ is a residue of g. We first consider the case of r not divisible by 3.

For i), we have $3gw_0 = 3w$. Since $r - (a^2+b^2) = c^2$, and $-(a^2+b^2)$ equals $-w$ times a square, we have that $-w$ is a residue of r (and thus of r_0). We now need to show that -3 is a residue of r_0 as well. Consider an odd prime p dividing r_0 where by hypothesis $p \equiv 1 \pmod 3$. If $p \equiv 1 \pmod 4$, we have $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = (1)\left(\frac{p}{3}\right) = 1$. If $p \equiv 3 \pmod 4$, we have $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = (-1)(-1)\left(\frac{p}{3}\right) = 1$.

For ii), we have $-gr_0 = -r$. Since $r - (a^2+b^2) = c^2$ and every prime factor of w_0 is congruent to $1 \pmod 4$, we have that $-r$ is a residue of w_0. Since $r \equiv 2 \pmod 3$, $-r \equiv 1 \pmod 3$, and we are left with $-r$ being a residue of 3.

For iii), let p be a prime dividing g. Since $p|r$ and $p|w$, we have that $p \equiv 1 \pmod{12}$. Thus $\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{1}{3}\right) = 1$. Again considering $r - (a^2+b^2) = c^2$, we rewrite this expression as $r_0g - w_0d^2g = c^2$ for some integer d. This implies $c \equiv 0 \pmod g$, so write $c = c_0g$ and then simplify to $r_0 - w_0d^2 = g{c_0}^2$. This gives us $r_0 \equiv w_0d^2 \pmod g$, and it follows that r_0, w_0 are either both residues of g or both non-residues of g. In either case, we have that $\left(\frac{r_0w_0}{p}\right) = 1$. We are left with $3w_0r_0$ being a residue of g.

In the case of r being divisible by 3, write $r_0 = 3r_1$. Equation 6 is solvable if and only if the following Equation 7 is solvable.

$$r_1x^2 + 3gy^2 - w_0z^2 = 0 \tag{7}$$

Regarding Theorem 4, to show Equation 7 is solvable, the only thing we need apart from the work already done above is to establish that r_1w_0 is a residue of 3. As $r_1 \equiv 2 \pmod 3$, we have that $\left(\frac{r_1}{3}\right) = -1$. As $r = a^2+b^2+c^2$ is congruent to $0 \pmod 3$ but not $0 \pmod 9$, it follows that each of $a, b, c \not\equiv 0 \pmod 3$. Thus $(a^2+b^2) \equiv 2 \pmod 3$ and $wd^2 \equiv 2 \pmod 3$ implies $w \equiv 2 \pmod 3$. Since each factor of g is congruent to $1 \pmod 3$, we are left with $w_0 \equiv 2 \pmod 3$ as well. So $\left(\frac{w_0}{3}\right) = -1$ and we have shown that the product r_1w_0 is a residue of 3. This concludes the sufficiency part of the argument.

For necessity, we consider separately the possibility of r being odd and the possibility of r having some odd prime factor that is congruent to $2 \pmod 3$. In each case we will show that Equation 6 is not solvable.

First, suppose that $q \equiv 2 \pmod 3$ where q is an odd prime and $q|r$. For solvability, we must have $3gw_0 = 3w$ being a residue of of q. As $-w$ is a residue of r, it is in turn a residue of q, and we are left with requiring $\left(\frac{-3}{q}\right) = 1$. If $q \equiv 1 \pmod 4$, $\left(\frac{-3}{q}\right) = \left(\frac{-1}{q}\right)\left(\frac{3}{q}\right) = \left(\frac{q}{3}\right) = \left(\frac{2}{3}\right) = -1$. If $q \equiv 3 \pmod 4$, we have $\left(\frac{-3}{q}\right) = \left(\frac{-1}{q}\right)\left(\frac{3}{q}\right) = (-1)(-1)\left(\frac{q}{3}\right) = \left(\frac{2}{3}\right) = -1$. Thus Equation 6 or 7 (whichever is applicable) is not solvable.

Now assume that r is odd, and first consider the case of r not divisible by 3. Let p be any prime dividing r_0. For solvability of Equation 6, we need $3w$ to be a residue of p. By the argument in the preceding paragraph, we have that $p \equiv 1 \pmod 3$. However, for solvability of Equation 6, we also need $-r$ to be a residue of 3. This gives a contradiction as $\left(\frac{-r}{3}\right) = \left(\frac{2}{3}\right) = -1$. Considering the case of 3 dividing r, we shift our attention to Equation 7 and note that for solvability, we need r_1w_0 to be a residue of 3 and $3w$ to be a residue of r_1. By previous arguments, $3w$ being a residue of r_1 implies that -3 is a residue of r_1 and thus every prime factor of r_1 is congruent to $1 \pmod 3$. As r_1 is then congruent to $1 \pmod 3$, for r_1w_0 to be a residue of 3, we must have $w_0 \equiv 1 \pmod 3$. This cannot happen because of the previously mentioned reason of $r = a^2 + b^2 + c^2$ being congruent to $0 \pmod 3$ but not $0 \pmod 9$ results in $(a^2 + b^2) \equiv 2 \pmod 3$. It follows that $w_0 \equiv 2 \pmod 3$ as well. This completes the necessity portion of the argument. $\square$

4. Euclidean Distance Graphs in $\mathbb{Q}^3$

The "holy grail" of Euclidean distance graph coloring has always been the determination of $\chi(\mathbb{R}^2, 1)$, the fabled "chromatic number of the plane." Those pessimists among us may say that not only has it always been, it very likely will always be. It is known that $4 \leq \chi(\mathbb{R}^2, 1) \leq 7$ and frustratingly, no improvement has been made to these bounds in over sixty years of effort. Interestingly enough, both bounds can be obtained through elementary methods, with the simplest derivation of the lower bound given by Leo and Willy Moser in [11] where they observe that the graph in Figure 2 is a subgraph of $G(\mathbb{R}^2, 1)$. This graph is commonly referred to as the *Moser spindle* and is easily seen to have chromatic number 4.

For a mostly up-to-date reference on the status of coloring problems in the rational space $\mathbb{Q}^n$, see Johnson's compendium [7] on the subject. We will not go nearly as in-depth here, but still, we should relay a few facts of this case. Although Woodall was the first to consider coloring $\mathbb{Q}^n$, showing in a 1972 article [16] that $\chi(\mathbb{Q}^2, 1) = 2$, the most influential work was done by Benda and Perles in [1]. In their manuscript [1], they obtain a number of results on coloring $\mathbb{Q}^n$ for small values of n and also pose a few still open problems including the following:

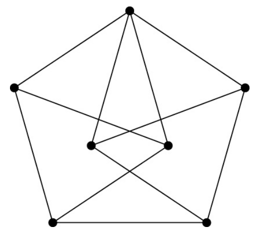

Figure 2:

"Does there exist $r \in \mathbb{Z}^+$ such that $G(\mathbb{Q}^3, \sqrt{r}) = 3$?"

Regarding Lemma 1, we may assume r to be square-free. It is shown in [6] that for odd r, the graph $G(\mathbb{Q}^3, \sqrt{r})$ can be properly 2-colored. For even r, it is shown in [2] that $3 \leq \chi(\mathbb{Q}^3, \sqrt{r})$ and in [8] that $\chi(\mathbb{Q}^3, \sqrt{r}) \leq 4$. So to completely resolve Benda and Perles' question, we are left with having to decide for each even square-free r whether $\chi(\mathbb{Q}^3, \sqrt{r}) = 3$ or $\chi(\mathbb{Q}^3, \sqrt{r}) = 4$. In [12], the author gives a partial answer with the following theorem given as a corollary to a few stronger results concerning colorings of the graph $G(\mathbb{Z}^3, \sqrt{r})$.

Theorem 7. *Let $r \in \mathbb{Z}^+$ where the square-free part of r is even but contains no odd factor congruent to* $2 \pmod 3$*. Then $\chi(\mathbb{Q}^3, \sqrt{r}) = 4$.*

The method of proof is straightforward. For each r it is shown that for some m, the graph in Figure 3 is a subgraph of $G(\mathbb{Q}^3, \sqrt{r})$. We will refer to this graph as a *generalized spindle*, and it too is easily seen to have chromatic number 4.

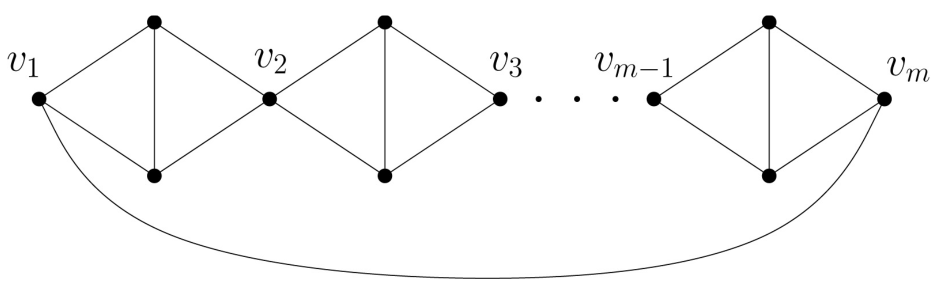

Figure 3:

As a 3-dimensional representation of the generalized spindle is highly non-rigid, particularly at the vertices $v_2, \ldots, v_{m-1}$, the mechanics of the proof of Theorem 7 just consist of showing that the graph can be drawn in $\mathbb{Q}^3$ with v_1 and v_m distance $\sqrt{r}$ apart. Now, this line of proof is satisfactory is establishing the theorem, but it is a little unsatisfying for the following reasons. First of all, not much is being said about the placement of the vertices $v_2, \ldots, v_{m-1}$. A representation of the generalized spindle embedded as a Euclidean distance graph in $\mathbb{Q}^3$ would not at all resemble the way the graph is drawn in Figure 3. So, what would it look like? Additionally, nothing is being said about what values of m can be used to embed the graph in $\mathbb{Q}^3$. With this in mind, we will use the remainder of this section to offer an alternate proof of Theorem 7, one that possesses a more "geometric flavor" than that outlined above.

An important player in this new proof of Theorem 7 will be the Diophantine equation $3x^2 - py^2 = 1$ where p is a positive prime congruent to $11 \pmod{12}$. Mollin gives Pell-type Diophantine equations of this form an extensive treatment in [10], where the following theorem appears.

Theorem 8. *Let $D > 2$ with D not a perfect square. Then $l(\sqrt{D})$ is even if and only if one of the following holds:*

(i) There exists a factorization $D = ab$ with $1 < a < b$ such that $ax^2 - by^2 = \pm 1$ is solvable.

(ii) There exists a factorization $D = ab$ with $1 \leq a < b$ such that $ax^2 - by^2 = \pm 2$ is solvable.

The parameter $l(\sqrt{D})$ denotes the length of the period in the continued fraction expansion of $\sqrt{D}$. A digression into the realm of continued fractions would take us far afield, but all that is really needed for our proof is observed in [3], where it is shown that $l(\sqrt{D})$ odd implies that D has no prime factor congruent to $3 \pmod{4}$.

Lemma 3. *Let p be a positive prime congruent to $11 \pmod{12}$. Then the Diophantine equation $3x^2 - py^2 = 1$ is solvable.*

Proof. Let $D = 3p$. Then $l(\sqrt{D})$ is even and Theorem 8 implies that one of the equations $3x^2 - py^2 = \pm 1$, $3x^2 - py^2 = \pm 2$, or $x^2 - 3py^2 = \pm 2$ is solvable. First, note that $3x^2 - py^2 = -1$ is not solvable, as considering both sides of the equation modulo 3, we see that a solution would require $y^2 \equiv 2 \pmod{3}$ which is impossible. Similarly, neither $3x^2 - py^2 = \pm 2$ nor $x^2 - 3py^2 = \pm 2$ is solvable as either of the congruences $3x^2 - py^2 \equiv 2 \pmod{4}$ or $x^2 - 3py^2 \equiv 2 \pmod{4}$ would contradict the fact that for any integer n, $n^2 \equiv 0$ or $1 \pmod{4}$. We are then left with the fact that $3x^2 - py^2 = 1$ must be solvable. $\square$

Note that in any integer solution x, y of $3x^2 - py^2 = 1$, we must have x even, for if x was odd, $3x^2 \equiv 3 \pmod{4}$ implies $py^2 \equiv 2 \pmod{4}$ which in turn implies that $y^2 \equiv 2 \pmod{4}$ which is impossible. So writing $x = 2x_0$, we substitute back into the original equation and arrive at the following corollary.

Corollary 1. *Let p be a positive prime congruent to* $11 \pmod{12}$. *Then the Diophantine equation* $12x^2 - py^2 = 1$ *is solvable.*

The final tool we will need in our alternate proof of Theorem 7 is Dirichlet's well-known theorem on primes in arithmetic progressions. It is given below.

Theorem 9. *Let* $a, b \in \mathbb{Z}^+$ *where* $\gcd(a, b) = 1$. *Then the arithmetic progression* $a, a + b, a + 2b, \ldots$ *contains infinitely many primes.*

We are now ready for a proof of Theorem 7. Let r be a square-free positive integer that is even but contains no odd factor congruent to $2 \pmod 3$. Our plan will be to use Theorem 3 to show that for some $n \in \mathbb{Z}$, the isosceles triangle $T(\sqrt{r}, n\sqrt{3r}, n\sqrt{3r})$ is embeddable in $\mathbb{Q}^3$. Such an embedding guarantees that $\chi(\mathbb{Q}^3, \sqrt{r}) \geq 4$ by the following rationale. First note that if T_1 and T_2 are equilateral triangles, each of side length $\sqrt{r}$, that lie in the same plane and share exactly one edge, their unshared vertices must be distance $\sqrt{3r}$ apart.

Suppose that $T = T(\sqrt{r}, n\sqrt{3r}, n\sqrt{3r})$ is in fact embeddable in $\mathbb{Q}^3$. Label the vertices of T as v, x_n, y_n where $|x_n - y_n| = \sqrt{r}$. There are rational points $x_1, \ldots, x_{n-1}$ that split the edge from v to x_n into n line segments, each of length $\sqrt{3r}$. As well, there are rational points $y_1, \ldots, y_{n-1}$ that split the edge from v to y_n into n line segments, each of length $\sqrt{3r}$. Now designate by $\mathcal{S}$ the set of all two-element subsets $\{q_1, q_2\} \subset \mathbb{Q}^3$ such that $|q_1 - q_2| = \sqrt{r}$. For each $i \in \{1, \ldots, n-1\}$ and pair of points x_i and x_{i+1}, Theorems 2 and 6 together guarantee that there exists some $Q \in \mathcal{S}$ such that the two points of Q are at distance $\sqrt{r}$ from each of x_i and x_{i+1}. A similar claim can be made about the pairs of points y_i and y_{i+1} for $i \in \{1, \ldots, n-1\}$ and the pairs of points v, x_1 and v, y_1. So in summary, if we can use Theorem 3 to show that T is embeddable in $\mathbb{Q}^3$ for some n, we have in effect shown that a generalized spindle with $m = 2n$ is a subgraph of $G(\mathbb{Q}^3, \sqrt{r})$ and thus $\chi(\mathbb{Q}^3, \sqrt{r}) = 4$. We do that with the following theorem.

Theorem 10. *Let r be a square-free, even positive integer that contains no odd factor congruent to* $2 \pmod 3$. *Then there exists* $n \in \mathbb{Z}^+$ *such that the triangle* $T(\sqrt{r}, n\sqrt{3r}, n\sqrt{3r})$ *is embeddable in* $\mathbb{Q}^3$.

Proof. To show that $T = T(\sqrt{r}, n\sqrt{3r}, n\sqrt{3r})$ is embeddable in $\mathbb{Q}^3$, Theorem 3 indicates that we must establish the solvability of the equation

$$x^2 + ry^2 - (12rn^2 - r)(a^2 + b^2)z^2 = 0 \tag{8}$$

where $r = a^2 + b^2 + c^2$ for $a, b, c \in \mathbb{Z}$. Equation 8 is solvable if and only if Equation 9 below is solvable.

$$rx^2 + y^2 - (12n^2 - 1)(a^2 + b^2)z^2 = 0 \tag{9}$$

Now suppose r has prime factorization $r = 2p_1 \cdots p_\alpha q_1 \cdots q_\beta$ where each $p_i \equiv 1 \pmod 4$ and each $q_j \equiv 3 \pmod 4$. By the Chinese Remainder Theorem, there exists a positive integer s satisfying the following system of linear congruences:

(*i*) $s \equiv 11 \pmod{12}$.

(*ii*) $s \equiv 3 \pmod 8$.

(*iii*) For each p_i, $s \equiv 1 \pmod{p_i}$.

(*iv*) For each q_j, $s \equiv -1 \pmod{q_j}$.

Note that $\gcd(r, s) = 1$. By Theorem 9, the arithmetic progression $s, s+r, s+2r, \ldots$ contains infinitely many primes. Let t be a prime in this sequence. Corollary 1 guarantees that the Diophantine equation $12n^2 - tk^2 = 1$ is solvable. So in Equation (9), we may replace the $(12n^2 - 1)$ coefficient with (tk^2), then "absorb" the k^2 to arrive at the equation

$$rx^2 + y^2 - t(a^2 + b^2)z^2 = 0, \tag{10}$$

whose solvability implies the solvability of Equation (9).

Regarding Theorem 4, to establish the solvability of Equation 10, we need to show that $t(a^2 + b^2)$ is a residue of r and that $-r$ is a residue of both t and $(a^2 + b^2)$. However, in Section 2 we already observed that $\pm r$ is a residue of $(a^2 + b^2)$ and $-(a^2+b^2)$ is a residue of r. So really, we only need that $-t$ is a residue of r and that $-r$ is a residue of t. For any p_i, we have $\left(\frac{-t}{p_i}\right) = \left(\frac{-1}{p_i}\right)\left(\frac{1}{p_i}\right) = 1$. For any q_j, we have $\left(\frac{-t}{q_j}\right) = \left(\frac{-1}{q_j}\right)\left(\frac{-1}{q_j}\right) = 1$. Thus $-t$ is a residue of r. We also have that $\left(\frac{-r}{t}\right) = \left(\frac{-1}{t}\right)\left(\frac{2}{t}\right)\left(\frac{p_1}{t}\right)\cdots\left(\frac{p_\alpha}{t}\right)\left(\frac{q_1}{t}\right)\cdots\left(\frac{q_\beta}{t}\right) = (-1)(-1)(1)\cdots(1)(1)\cdots(1) = 1$. This shows that $-r$ is a residue of t, guaranteeing that Equation 10 is solvable and completing the proof of the theorem. $\square$

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