A GENERALIZED BINET FORMULA THAT COUNTS THE TILINGS OF A $(2 \times N)$-BOARD

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Abstract

In this paper, we answer an open problem raised by Katz and Stenson. They considered the tilings of a $(2 \times n)$-board using a colors of squares and b colors of dominoes. The number of such tilings, denoted by $K_n^{a,b}$, is a generalization of the Fibonacci numbers. Obtaining a Binet-style formula for these numbers is such problem. We obtain a generalized Binet formula for $K_n^{a,b}$.

1. Introduction

The sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots,$$

denoted by $(F_n)_{n\geq 0}$, of Fibonacci numbers (sequence number A000045 in [19]) is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ and initial values $F_0 = 0$ and $F_1 = 1$ [6]. By substituting $F_n = r^n$, we get the characteristic equation $r^2 - r - 1 = 0$. Its roots are $(1 \pm \sqrt{5})/2$ and so,

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

Now, the assignments $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ give us the representation

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

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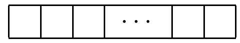

Figure 1: A $(1 \times n)$-board with n cells.

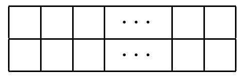

Figure 2: A $(2 \times n)$-board with n columns.

where $\alpha \approx 1.618$ is known as the golden ratio. This is referred to as the Binet formula, although it was known earlier to Euler and de Moivre [14].

Leonardo of Pisa, known as Fibonacci, introduced this sequence to European mathematics by his book Liber Abaci [17]. As he showed in his book, if we assume a newly born pair of rabbits, one male and one female, then the growth of an idealized rabbit population follows the Fibonacci sequence. It is noticeable that this sequence appears in connection with the basic units in Sanskrit prosody and had been presented earlier by Indian mathematician Virahanka [18]. However, Fibonacci numbers appear in many situations and count several combinational problems. For example, F_n is the number of sequences of 1's and 2's that sum to $n-1$ and the sequence $(F_n)_{n\geq 0}$ are used to measure the running time of some algorithms such as the Euclid's algorithm, the Fibonacci search technique and the Fibonacci heap data structure [11, 12].

It is possible to generalize Fibonacci numbers in many ways. One of the most famous generalizations is defined by adding $k \geq 2$ numbers to generate the next number [5] and called the k-generalized Fibonacci numbers, generalized Fibonacci numbers of order k or k-bonacci numbers. In other words, if $(F_n^{(k)})_{n\geq 0}$ be such sequence then the k initial values are $0, 0, \ldots, 0, 1$ and $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$ for any $n \geq k$. The characteristic polynomial of $F_n^{(k)}$ is $\psi_k(r) = r^k - r^{k-1} - \cdots - 1$. It can be shown that $\psi_k(r) \in \mathbb{Q}[r]$ is irreducible with only one zero outside the unit ball [4]. According to [5, Thm 1], we have

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}, \tag{1}$$

where $\alpha_1, \ldots, \alpha_k$ are the roots of the characteristic equation. It is possible to interpret Equation (1) as a generalization of the Binet formula. Notice that $F_n^{(2)} = F_n$ and $F_n^{(3)}$'s are also called the Tribonacci numbers. Also, we can generalize the Fibonacci numbers by preserving the recurrence relation $H_n = H_{n-1} + H_{n-2}$ $(n \geq 2)$ but changing the initial values, i.e., $H_0 = p$ and $H_1 = q$, where p and q are arbitary integers [8]. Now, the sequence

$$p, q, p+q, p+2q, 2p+3q, 3p+5q, 5p+8q, 8p+13q, \ldots$$

is deduced and the Binet's type formula

$$H_n = \frac{(q-p\beta)\alpha^n - (q-p\alpha)\beta^n}{\alpha-\beta}$$

is obtained by the same classical method. Moreover, for any integer $k \geq 1$, the k-Fibonacci sequence $(F_{k,n})_{n\in\mathbb{N}}$ is defined by the recurrence relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ $(n \geq 2)$ and initial values $F_{k,0} = 0$ and $F_{k,1} = 1$ [3]. Hence,

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = (k + \sqrt{k^2+4})/2$ and $\beta = (k - \sqrt{k^2+4})/2$ are the roots of the characteristic equation $r^2 - kr - 1 = 0$. For further generalization of the Fibonacci numbers and their generalized Binet formulas, see [1, 13, 16, 20, 21, 22]. As it is seen, there exists a generalized Binet formula for any of the above generalized Fibonacci numbers.

A combinatorial interpretation for the Fibonacci numbers is the tiling of a $(1\times n)$-board (see Figure 1) by squares and dominoes. If f_n denotes the number of such tiling then $f_n = F_{n+1}$ [2]. By this interpretation, Benjamin and Quinn [2] proved many identities for Fibonancci numbers. Moreover, McQuistan and Lichtman [15] used such tilings for placing dumbbells on a lattice. They found a recurrence relation and Grimson [7] studied the generating function for dumbbells on a $(2 \times n)$-board (see Figure 2). Katz and Stenson [10] also expanded the board and studied the tiling of a $(2 \times n)$-board with colored squares and dominoes. They showed that if $K_n^{a,b}$ counts the tilings of a $(2 \times n)$-board using a colors of squares and b colors of dominoes, where a and b are nonnegative integers, then

$$K_n^{a,b} = (a^2+2b)K_{n-1}^{a,b} + a^2bK_{n-2}^{a,b} - b^3K_{n-3}^{a,b} \quad (n \geq 3) \tag{2}$$

with the initial values $K_0^{a,b} = 1$, $K_1^{a,b} = a^2 + b$ and $K_2^{a,b} = a^4 + 4a^2 + 2b^2$. Since $K_n^{0,1} = f_n$, we can say that $K_n^{a,b}$ is a generalization of the Fibonancci numbers and it is possible to generalize some Fibonancci identities for the numbers $K_n^{a,b}$ [10]. Recently, the first author has obtained some identities for the sequence $(K_n^{a,b})_{(n\geq 0)}$ [9]. Katz and Stenson then proposed an open problem: Is it possible to obtain an expression for $K_n^{a,b}$ which is analogous to the Binet formula for the Fibonacci numbers? In this paper, we answer this open problem and find a generalized Binet formula for $K_n^{a,b}$.

2. A Generalized Binet Formula

Consider the recurrence relation (2) and let $K_n^{a,b} = r^n$. Upon substitution, we get the characteristic equation

$$r^3 - (a^2+2b)r^2 - a^2br + b^3 = 0. \tag{3}$$

Set $y(r) = r^3 - (a^2+2b)r^2 - a^2br + b^3$. Since $y(r)$ is a cubic, it has at most three real roots. If $b = 0$ then it is easy to see that $K_n^{a,b} = K_n^{a,0} = a^{2n}$. Now, suppose that $b > 0$. Then, we have $y \to -\infty$ as $r \to -\infty$, $y(0) = b^3 > 0$,

$$y\left(2(a^2+2b)/3\right) = -(4a^6 + 42a^4b + 84a^2b^2 + 5b^3)/27 < 0$$

and $y \to \infty$ as $r \to \infty$. This shows that y has at least three real roots, and hence it has exactly three distinct real roots $\alpha < 0 < \beta < \gamma$, and so the general solution is

$$K_n^{a,b} = c_1\alpha^n + c_2\beta^n + c_3\gamma^n. \tag{4}$$

To solve for c_1, c_2 and c_3, we use the given initial values. Hence,

$$\begin{cases} c_1 + c_2 + c_3 = 1, \\ c_1\alpha + c_2\beta + c_3\gamma = a^2 + b, \\ c_1\alpha^2 + c_2\beta^2 + c_3\gamma^2 = a^4 + 4a^2b + 2b^2, \end{cases}$$

and we have

$$\begin{aligned} c_1 &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\left(\beta\gamma - (a^2+b)(\beta+\gamma) + a^2 + 4a^2b + 2b^2\right), \\ c_2 &= \frac{1}{(\beta-\alpha)(\beta-\gamma)}\left(\alpha\gamma - (a^2+b)(\alpha+\gamma) + a^2 + 4a^2b + 2b^2\right), \\ c_3 &= \frac{1}{(\gamma-\alpha)(\gamma-\beta)}\left(\alpha\beta - (a^2+b)(\alpha+\beta) + a^2 + 4a^2b + 2b^2\right). \end{aligned}$$

Now, the relations between coefficients and roots of the characteristic equation gives

$$\begin{cases} \alpha + \beta + \gamma = a^2 + 2b, \\ \alpha\beta + \beta\gamma + \gamma\alpha = -a^2b, \\ \alpha\beta\gamma = -b^3. \end{cases}$$

This implies that $a^2 + b = \alpha + \beta + \gamma + \sqrt[3]{\alpha\beta\gamma}$ and $a^4 + 4a^2b + 2b^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta\gamma)^{2/3}$. Then,

$$\begin{aligned} \left(\beta\gamma - (a^2+b)(\beta+\gamma) + a^2 + 4a^2b + 2b^2\right) &= \alpha(\alpha + \sqrt[3]{\alpha\beta\gamma}), \\ \left(\alpha\gamma - (a^2+b)(\alpha+\gamma) + a^2 + 4a^2b + 2b^2\right) &= \beta(\beta + \sqrt[3]{\alpha\beta\gamma}), \\ \left(\alpha\beta - (a^2+b)(\alpha+\beta) + a^2 + 4a^2b + 2b^2\right) &= \gamma(\gamma + \sqrt[3]{\alpha\beta\gamma}) \end{aligned}$$

and so, our main result is deduced:

Theorem 1. *Let $K_n^{a,b}$ be the number of the tilings of a $(2\times n)$-board using a colors of squares and b colors of dominoes, where $a \geq 0$ and $b > 0$. Let $\alpha < \beta < \gamma$ be the roots of characteristic Equation (3). Then,*

$$K_n^{a,b} = \frac{\alpha^{n+1}(\alpha + \sqrt[3]{\alpha\beta\gamma})}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}(\beta + \sqrt[3]{\alpha\beta\gamma})}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}(\gamma + \sqrt[3]{\alpha\beta\gamma})}{(\gamma-\alpha)(\gamma-\beta)}. \tag{5}$$

In the special case $a = 0$ and $b = 1$, our formula still holds true; the characteristic equation is $r^3 - 2r^2 + 1 = 0$ and hence, its roots are $\alpha = (1 - \sqrt{5})/2$, $\beta = 1$ and $\gamma = (1 + \sqrt{5})/2$. Since $\alpha + \gamma = 1$ and $\alpha\gamma = -1$, by replacing them in Equation (5), we get the Binet formula $F_{n+1} = K_n^{0,1} = \frac{\gamma^{n+1} - \alpha^{n+1}}{\gamma - \alpha}$. Thus, we can say that Equation (5) is a generalized Binet formula.

Example. In [10, Table 1], Katz and Stenson obtained some values of $K_n^{2,3}$ by the recurrence relation (2). In this example, we obtain these same values by the formula (5). Characteristic Equation (3) becomes $r^3 - 10r^2 - 12r + 27 = 0$ and the roots are easily calculated to be $\alpha \approx -2.0729$, $\beta \approx 1.1977$, and $\gamma \approx 10.8751$. Formula (5) thus becomes

$$\begin{aligned} K_n^{a,b} \approx & \frac{(-2.0729)^{n+1}(-2.0729-3)}{42.3477} + \frac{(1.1977)^{n+1}(1.1977-3)}{-31.6509} \\ & + \frac{(10.8751)^{n+1}(10.8751-3)}{125.3030}, \end{aligned}$$

where we have used that $\alpha\beta\gamma = -b^3$ and so in this specific example $\sqrt[3]{\alpha\beta\gamma} = -3$. This gives us $K_1^{2,3} \approx 7$, $K_2^{2,3} \approx 82$, $K_3^{2,3} \approx 877$, $K_4^{2,3} \approx 9565$, and if done with enough precision we could also obtain the subsequent numbers 103960, 1130701, and so on. See the sequence number A253265 in [19].

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References

[1] M. Akbulak and D. Bozkurt, On the order-m generalized Fibonacci k-numbers, *Chaos Solitons Fractals* **42** (2009), 1347-1355.

[2] A. T. Benjamin and J. J. Quinn, *Proofs That Really Count: The Art of Combinatorial Proof*, The Mathematical Association of America, Washington, D.C., 2003.

[3] P. Catarino, A note on $h(x)$-Fibonacci quaternion polynomials, *Chaos Solitons Fractals* **77** (2015), 1-15.

[4] A. P. Chaves and D. Marques, A Diophantine equation related to the sum of powers of two consecutive generalized Fibonacci numbers, *J. Number Theory* **156** (2015), 1-14.

[5] G. P. B. Dresden and Z. Du, A simplified Binet formula for k-generalized Fibonacci numbers, *J. Integer Seq.* **17** (2014), Article 14.4.7.

[6] R. P. Grimaldi, *Discrete and Combinatorial Mathematics*, 5th Edition, Pearson Addison Wesley, Boston, 2004.

[7] R. C. Grimson, Exact formulas for $2 \times n$ arrays of dumbbells, *J. Math. Phys.* **15** (1974), 214-216.

[8] A. F. Horadam, A generalized Fibonacci sequences, *Amer. Math. Monthly* **68** (1961), no. 5, 455-459.

[9] R. Kahkeshani, The tilings of a $(2\times n)$-board and some new combinatorial identities, *J. Integer Seq.* **20** (2017), Article 17.5.4.

[10] M. Katz and C. Stenson, Tiling a $(2 \times n)$-board with squares and dominoes, *J. Integer Seq.* **12** (2009), Article 9.2.2.

[11] D. E. Knuth, *The Art of Computer Programming*, Vol. 1, 3rd Edition, Addison-Wesley, Amsterdam, 1997.

[12] D. E. Knuth, *The Art of Computer Programming*, Vol. 3, 2nd Edition, Addison-Wesley, Amsterdam, 1998.

[13] G. Y. Lee, S. G. Lee, J. S. Kim, and H. K. Shin, The Binet formula and representations of k-generalized Fibonacci numbers, *Fibonacci Quart.* **39** (2001), 158-164.

[14] M. Livio, *The Golden Ratio: The Story of Phi, the World's Most Astonishing Number*, Broadway Books, New York, 2002.

[15] R. B. McQuistan and S. J. Lichtman, Exact recursion relation for $2 \times N$ arrays of dumbbells, *J. Math. Phys.* **11** (1970), 3095-3099.

[16] J. Petronilho, Generalized Fibonacci sequences via orthogonal polynomials, *Appl. Math. Comput.* **218** (2012), 9819-9824.

[17] L. E. Sigler, *Fibonacci's Liber Abaci*, Springer-Verlag, New York, 2002.

[18] P. Singh, The so-called Fibonacci numbers in ancient and medieval India, *Historia Math.* **12** (1985), 229-244.

[19] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences* (2008), available at `https://oeis.org/`.

[20] W. R. Spickerman and R. N. Joyner, Binet's formula for the recursive sequence of order k, *Fibonacci Quart.* **22** (1984), 327-331.

[21] A. Stakhov and B. Rozin, Theory of Binet formulas for Fibonacci and Lucas p-numbers, *Chaos Solitons Fractals* **27** (2006), 1162-1177.

[22] K. Uslu, N. Taskara, and H. Kose, The generalized k-Fibonacci and k-Lucas numbers, *Ars Combin.* **99** (2011), 25-32.