# POSITIVE-DEFINITE TERNARY QUADRATIC FORMS WHICH ARE (4,1)-UNIVERSAL AND (4,3)-UNIVERSAL

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**Abstract**
We discuss the problem of determining all positive-definite ternary quadratic forms with integral coefficients which represent all positive integers $n \equiv 1 \pmod 4$ and similarly for all $n \equiv 3 \pmod 4$.

## 1. Introduction

Throughout this paper $F$ denotes a positive-definite ternary quadratic form with integral coefficients. We let

$$F = F(x, y, z) := ax^2 + by^2 + cz^2 + dxy + exz + fyz. \tag{1.1}$$

If $d$, $e$ and $f$ are all even the form $F$ is said to be even. If at least one of $d$, $e$ and $f$ is odd then $F$ is said to be odd. The matrix $A(F)$ of the form $F$ is the $3 \times 3$ symmetric matrix with integer entries given by

$$A(F) := \begin{cases} \begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix} & \text{if } F \text{ is even,} \\ \\ \begin{bmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{bmatrix} & \text{if } F \text{ is odd.} \end{cases} \tag{1.2}$$

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As $F$ is a positive-definite quadratic form, the matrix $A(F)$ is a positive-definite matrix. Thus the determinant of $A(F)$ is a positive integer and we define the discriminant $d(F)$ of the form $F$ to be the positive integer given by

$$d(F) := \det A(F) = \begin{cases} abc + \frac{1}{4}(def - af^2 - be^2 - cd^2) & \text{if } F \text{ is even,} \\ 8abc + 2(def - af^2 - be^2 - cd^2) & \text{if } F \text{ is odd.} \end{cases} \tag{1.3}$$

A positive integer $n$ is said to be represented by the form $F$ if there exist integers $x, y$ and $z$ such that $F(x, y, z) = n$. Let $k$ and $l$ be positive integers with $l \leq k$. If $F$ represents all positive integers $km + l$ $(m = 0, 1, 2, \ldots)$ the form $F$ is said to be $(k, l)$-universal. It is a classical result that a positive-definite ternary quadratic form with integral coefficients cannot represent all positive integers, see for example [1], [3] and [6]. Thus $F$ cannot be $(1, 1)$-universal. In 1995 Kaplansky [13] tackled the problem of determining the positive-definite ternary quadratic forms with integral coefficients which are $(2, 1)$-universal, that is represent all odd positive integers. He proved that there are at most 23 such forms, and proved that 19 of the 23 forms represent all odd positive integers. The remaining four forms he called "plausible candidates" and noted that they represent every odd positive integer less than $2^{14}$. In 1996 Jagy [8] proved that one of these four forms, namely $x^2+3y^2+11z^2+xy+7yz$, does in fact represent all odd positive integers. The remaining three have not yet been shown to represent all odd positive integers and Rouse [16, p. 1695] formally conjectures that they do.

In this paper we begin the problem of determining all the positive-definite ternary quadratic forms with integral coefficients which are $(4, 1)$-universal and those which are $(4, 3)$-universal. This problem cannot be completely solved as the corresponding problem for $(2, 1)$-universal positive-definite integral ternary quadratic forms is not yet solved.

## 2. The Equation $Ar^2 + Ers + Bs^2 = Ct^2$

In Section 3 we need to know that certain equations of the form $Ar^2 + Ers + Bs^2 = Ct^2$ have no solutions in integers $r$, $s$ and $t$ except $(r, s, t) = (0, 0, 0)$. To this end we prove the following result.

**Theorem 1.** *Let $A$, $B$ and $C$ be positive integers. Let $E$ be an integer such that*

$$|E| < 2\sqrt{AB}, \tag{2.1}$$

*so that $4AB - E^2 > 0$. Let $G^2$ $(G > 0)$ be the largest square dividing $4AB - E^2$ and $J^2$ $(J > 0)$ the largest square dividing $4AC$. Let*

$$L := \left( \frac{4AB - E^2}{G^2}, \frac{4AC}{J^2} \right). \tag{2.2}$$

*Define*

$$H := \frac{4AB - E^2}{G^2 L}, \quad K := \frac{4AC}{J^2 L}. \tag{2.3}$$

*Then $G, H, J, K$ and $L$ are positive integers such that*

$$4AB - E^2 = G^2 LH, \quad 4AC = J^2 LK, \tag{2.4}$$

$$L, H, K \text{ squarefree}, \tag{2.5}$$

$$(L, H) = (L, K) = (H, K) = 1. \tag{2.6}$$

*If*

$$\text{- } LH \text{ is a quadratic nonresidue } \pmod{K} \tag{2.7}$$

*or*

$$LK \text{ is a quadratic nonresidue } \pmod{H} \tag{2.8}$$

*or*

$$HK \text{ is a quadratic nonresidue } \pmod{L} \tag{2.9}$$

*then the only solution in integers $r, s$ and $t$ to the equation*

$$Ar^2 + Ers + Bs^2 = Ct^2 \tag{2.10}$$

*is $(r, s, t) = (0, 0, 0)$.*

*Proof.* Suppose that $r, s$ and $t$ are integers such that (2.10) holds. Then, by (2.4) and (2.10), we have

$$\begin{aligned}(2Ar + Es)^2 &= 4A(Ar^2 + Ers + Bs^2 - Ct^2) - (4AB - E^2)s^2 + 4ACt^2 \\ &= -G^2 LHs^2 + J^2 LKt^2 \\ &\equiv 0 \pmod{L}.\end{aligned}$$

As $L$ is squarefree we deduce that

$$2Ar + Es \equiv 0 \pmod{L}.$$

Hence we can define integers $x, y$ and $z$ by

$$x = \frac{2Ar + Es}{L}, \quad y = Gs, \quad z = Jt. \tag{2.11}$$

Then, by (2.4), (2.10) and (2.11), we deduce

$$L(Lx^2 + Hy^2 - Kz^2) = (2Ar + Es)^2 + (4AB - E^2)s^2 - 4ACt^2$$

$$= 4A(Ar^2 + Ers + Bs^2 - Ct^2) = 0$$

so that

$$Lx^2 + Hy^2 - Kz^2 = 0. \tag{2.12}$$

Equation (2.12) is Legendre's equation. As $L$, $H$ and $-K$ are nonzero integers not all of the same sign, $L, H$ and $K$ are all squarefree and coprime in pairs, we know from Legendre's theorem [5, Theorem 91, p. 117] under the assumption of (2.7), (2.8) or (2.9) that (2.12) has only the solution $(x, y, z) = (0, 0, 0)$ in integers $x, y$ and $z$. Hence, by (2.11), $(r, s, t) = (0, 0, 0)$ is the only solution in integers $r, s$ and $t$ to the equation (2.10). □

The following result is a simple application of Theorem 1.

**Corollary 1.** *The only solution in integers $r$, $s$ and $t$ to each of the following equations is $(r, s, t) = (0, 0, 0)$ :*

$$r^2 + wrs + 5s^2 = 13t^2 \ \ (w = 0, \pm 1, \pm 3),$$
$$r^2 + wrs + 5s^2 = 21t^2 \ \ (w = \pm 2, \pm 4),$$
$$3r^2 + wrs + 7s^2 = 11t^2 \ \ (w = 0, \pm 2, \pm 3, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9),$$
$$3r^2 + wrs + 7s^2 = 15t^2 \ \ (w = \pm 1, \pm 4).$$

*Proof.* We just give the details for the equation $3r^2 + 9rs + 7s^2 = 11t^2$. Here $A = 3, B = 7, C = 11, E = 9$, $4AB - E^2 = 3$, $4AC = 132$, $G = 1$, $J = 2$, $L = 3$, $H = 1$ and $K = 11$. Thus, $\left(\frac{-LH}{K}\right) = \left(\frac{-3}{11}\right) = -1$. Hence, by (2.7), $3r^2 + 9rs + 7s^2 = 11t^2$ has only the solution $(r, s, t) = (0, 0, 0)$. □

We deduce the following result from Theorem 1.

**Theorem 2.** *Let $A$ and $B$ be positive integers. Let $E$ be an integer such that (2.1) holds. Then there exists a positive integer $C$ such that the only solution to (2.10) is $(r, s, t) = (0, 0, 0)$.*

*Proof.* As $4AB - E^2 > 0$, $E^2 - 4AB$ is not a perfect square so there exists a prime $p$ such that [7, p. 57]

$$\left(\frac{E^2 - 4AB}{p}\right) = -1.$$

Clearly $p \neq 2$, $p \nmid A$, $p \nmid B$ and $p \nmid 4AB - E^2$. Choose $C = p$. Then $p \nmid G$, $p || 4AC$, $p \nmid J$, $p \nmid L$, $p \nmid H$ and $p || K$. Moreover

$$\left(\frac{-LH}{p}\right) = \left(\frac{-(4AB - E^2)/G^2}{p}\right) = \left(\frac{E^2 - 4AB}{p}\right) = -1.$$

Hence $-LH$ is a quadratic nonresidue (mod $p$) and so, as $p | K$, $-LH$ is quadratic nonresidue (mod $K$). Thus, by Theorem 1, the only solution in integers $r, s$ and $t$ to (2.10) is $(r, s, t) = (0, 0, 0)$. □

## 3. Bound for $d(F)$

Given that the positive-definite integral ternary quadratic form $F$ represents certain positive integers we give an upper bound for $d(F)$. As usual $\mathbb{Z}$ and $\mathbb{Q}$ denote the sets of integers and rational numbers respectively.

**Theorem 3.** *Let $A$ and $B$ be fixed positive integers such that*

$$A < B, \quad AB \neq M^2 \text{ for any integer } M.$$

*For each integer $E$ with*

$$\begin{cases} |E| < 2\sqrt{AB}, \ \ E \equiv 0 \pmod 2 & \text{if } F \text{ is even}, \\ |E| < 2\sqrt{AB} & \text{if } F \text{ is odd}, \end{cases}$$

*there exists by Theorem 2 a positive integer $C(E)$ such that the only solution in integers $r, s$ and $t$ to*

$$Ar^2 + Ers + Bs^2 = C(E)t^2$$

*is $(r, s, t) = (0, 0, 0)$. As $Ar^2 + Ers + Bs^2$ and $Ar^2 - Ers + Bs^2$ represent exactly the same integers we may choose $C(E) = C(-E)$. Suppose $F$ represents all of the integers*

$$\begin{cases} A, B, C(E) \ \ (0 \leq E < 2\sqrt{AB}, \ E \equiv 0 \pmod 2) & \text{if } F \text{ is even}, \\ A, B, C(E) \ \ (0 \leq E < 2\sqrt{AB}) & \text{if } F \text{ is odd}. \end{cases}$$

*Then*

$$d(F) \leq \begin{cases} AB \displaystyle\max_{\substack{0 \leq E < 2\sqrt{AB} \\ E \equiv 0 \pmod 2}} C(E) & \text{if } F \text{ is even}, \\ 8AB \displaystyle\max_{0 \leq E < 2\sqrt{AB}} C(E) & \text{if } F \text{ is odd}. \end{cases}$$

*Proof.* As $F(x, y, z)$ represents both the integers $A$ and $B$ there exist $(x_1, y_1, z_1) \in \mathbb{Z}^3$ and $(x_2, y_2, z_2) \in \mathbb{Z}^3$ such that

$$F(x_1, y_1, z_1) = A, \ \ F(x_2, y_2, z_2) = B. \tag{3.1}$$

Since $F(0, 0, 0) = 0$ and $A, B > 0$ we have

$$(x_1, y_1, z_1) \neq (0, 0, 0), \ \ (x_2, y_2, z_2) \neq (0, 0, 0). \tag{3.2}$$

We now show that $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are linearly independent over $\mathbb{Q}$. Suppose not. Then there exist $u, v \in \mathbb{Q}$ with $(u, v) \neq (0, 0)$ such that

$$u(x_1, y_1, z_1) + v(x_2, y_2, z_2) = (0, 0, 0).$$

If $v = 0$ then $u \neq 0$ and we have $(x_1, y_1, z_1) = (0, 0, 0)$, contradicting (3.2). Hence $v \neq 0$. Thus

$$(x_2, y_2, z_2) = -\frac{u}{v}(x_1, y_1, z_1).$$

Appealing to (1.1) and (3.1), we have

$$B = F(x_2, y_2, z_2) = \left(\frac{-u}{v}\right)^2 F(x_1, y_1, z_1) = \frac{u^2}{v^2} A$$

so that $AB = \left(\frac{uA}{v}\right)^2$. As $AB$ is a positive integer, the rational number $\frac{uA}{v}$ must be an integer $M$, so $AB = M^2$, a contradiction. This proves the linear independence of $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ over $\mathbb{Q}$.

For integers $r$ and $s$ we have

$$F(r(x_1, y_1, z_1) + s(x_2, y_2, z_2)) = Ar^2 + Ers + Bs^2, \tag{3.3}$$

where the integer $E$ is given by

$$E = 2ax_1x_2 + 2by_1y_2 + 2cz_1z_2 + d(x_1y_2 + x_2y_1) + e(x_1z_2 + x_2z_1) + f(y_1z_2 + y_2z_1). \tag{3.4}$$

We note that

$$E \equiv 0 \pmod 2 \text{ if } F \text{ is even.} \tag{3.5}$$

As $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are linearly independent over $\mathbb{Q}$ (and thus over $\mathbb{Z}$), we have for $r, s \in \mathbb{Z}$

$$r(x_1, y_1, z_1) + s(x_2, y_2, z_2) = (0, 0, 0) \iff (r, s) = (0, 0). \tag{3.6}$$

Thus, as $F(x, y, z)$ is positive-definite, we see from (3.3) and (3.6) that the integral binary quadratic form $Ar^2 + Ers + Bs^2$ is also positive-definite. Hence the discriminant $E^2 - 4AB$ of $Ar^2 + Ers + Bs^2$ is negative and so $|E| < 2\sqrt{AB}$.

For integers $E$ satisfying

$$\begin{cases} |E| < 2\sqrt{AB},\ E \equiv 0 \pmod 2 & \text{if } F \text{ is even,} \\ |E| < 2\sqrt{AB} & \text{if } F \text{ is odd,} \end{cases} \tag{3.7}$$

$F$ represents $C(E) = C(-E)$, and so there exists $(x(E), y(E), z(E)) \in \mathbb{Z}^3$ with $(x(E), y(E), z(E)) = (x(-E), y(-E), z(-E))$ such that

$$F(x(E), y(E), z(E)) = C(E). \tag{3.8}$$

Thus for all $t \in \mathbb{Z}$ we have

$$F(tx(E), ty(E), tz(E)) = C(E)t^2. \tag{3.9}$$

Now by the definition of $C(E)$ we have

$$Ar^2 + Ers + Bs^2 = C(E)t^2 \implies (r, s, t) = (0, 0, 0) \tag{3.10}$$

so appealing to (3.3), (3.9) and (3.10) we deduce that

$$(tx(E), ty(E), tz(E)) \neq r(x_1, y_1, z_1) + s(x_2, y_2, z_2) \tag{3.11}$$

for all $r, s, t \in \mathbb{Z}$ with $(r, s, t) \neq (0, 0, 0)$. Thus the system of three linear equations in $r$, $s$ and $t$

$$\begin{cases} x_1 r + x_2 s - x(E)t = 0, \\ y_1 r + y_2 s - y(E)t = 0, \\ z_1 r + z_2 s - z(E)t = 0, \end{cases} \tag{3.12}$$

has only the trivial solution $(r, s, t) = (0, 0, 0)$. Let

$$X(E) := \begin{bmatrix} x_1 & x_2 & x(E) \\ y_1 & y_2 & y(E) \\ z_1 & z_2 & z(E) \end{bmatrix} \in M_3(\mathbb{Z}). \tag{3.13}$$

Clearly

$$X(E) = X(-E), \quad \det X(E) \neq 0. \tag{3.14}$$

As $\det X(E) \in \mathbb{Z}$ we see from (3.14) that

$$(\det X(E))^2 \geq 1. \tag{3.15}$$

From (1.1) and (1.2) we have

$$F(x, y, z) = \begin{cases} \underline{x}^T A(F)\underline{x} & \text{if } F \text{ is even}, \\ \frac{1}{2}\underline{x}^T A(F)\underline{x} & \text{if } F \text{ is odd}, \end{cases} \tag{3.16}$$

where $\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. A short calculation using (1.2), (3.1), (3.4), (3.8) and (3.13) shows that

$$X(E)^T A(F) X(E) = \begin{cases} \begin{bmatrix} A & E/2 & k \\ E/2 & B & l \\ k & l & C(E) \end{bmatrix} & \text{if } F \text{ is even}, \\ \\ \begin{bmatrix} 2A & E & k \\ E & 2B & l \\ k & l & 2C(E) \end{bmatrix} & \text{if } F \text{ is odd}, \end{cases} \tag{3.17}$$

for some integers $k$ and $l$. If $F$ is even, as

$$A > 0, \quad \begin{vmatrix} A & E/2 \\ E/2 & B \end{vmatrix} = \frac{1}{4}(4AB - E^2) > 0$$

and

$$\begin{vmatrix} A & E/2 & k \\ E/2 & B & l \\ k & l & C(E) \end{vmatrix} = \det(X(E)^T A(F) X(E)) = \det(X(E))^2 \det(A(F)) > 0,$$

the matrix $X(E)^T A(F) X(E)$ is positive-definite. If $F$ is odd, as

$$2A > 0, \quad \begin{vmatrix} 2A & E \\ E & 2B \end{vmatrix} = 4AB - E^2 > 0$$

and

$$\begin{vmatrix} 2A & E & k \\ E & 2B & l \\ k & l & 2C(E) \end{vmatrix} = \det(X(E)^T A(F) X(E)) = \det(X(E))^2 \det(A(F)) > 0,$$

the matrix $X(E)^T A(F) X(E)$ is positive-definite in this case too. Thus in both cases $X(E)^T A(F) X(E)$ is a positive-definite symmetric matrix and we have that $\det(X(E)^T A(F) X(E))$ is less than or equal to the product of the diagonal entries of $X(E)^T A(F) X(E)$ [14, Theorem 13.5.2, p. 417]. Thus if $F$ is even we have by (1.3), (3.15) and (3.17)

$$\begin{aligned} d(F) = \det A(F) &\leq (\det(X(E))^2 \det A(F) = \det(X(E)^T A(F) X(E)) \\ &\leq AB \max_{\substack{0 \leq E < 2\sqrt{AB} \\ E \equiv 0 \pmod 2}} C(E) \end{aligned}$$

and if $F$ is odd we have

$$\begin{aligned} d(F) = \det A(F) &\leq (\det(X(E))^2 \det A(F) = \det(X(E)^T A(F) X(E)) \\ &\leq (2A)(2B)\left(2 \max_{0 \leq E < 2\sqrt{AB}} C(E)\right) \\ &= 8AB \max_{0 \leq E < 2\sqrt{AB}} C(E) \end{aligned}$$

as asserted. □

Our next theorem, which is a simple application of Theorem 3, bounds the discriminant of a $(4, 1)$-universal ternary quadratic form and the discriminant of a $(4, 3)$-universal ternary quadratic form.

**Theorem 4.** *(i) If $F$ is even and represents*

$$1, 5, 13, 21,$$

*then*

$$d(F) \leq 105.$$

*(ii) If $F$ is odd and represents*

$$1, 5, 13, 21,$$

*then*

$$d(F) \leq 840.$$

*(iii) If $F$ is even and represents*

$$3, 7, 11, 15,$$

*then*

$$d(F) \leq 315.$$

*(iv) If $F$ is odd and represents*

$$3, 7, 11, 15,$$

*then*

$$d(F) \leq 2520.$$

*Proof.* (i) We chose $A = 1$ and $B = 5$. The integers $E$ such that $|E| < 2\sqrt{AB} = 2\sqrt{5} \approx 4.47$ with $E \equiv 0 \pmod 2$ are $E = 0, \pm 2, \pm 4$. By Corollary 1 we may take

$$C(0) = 13,\ C(\pm 2) = C(\pm 4) = 21.$$

Hence if $F$ is even and represents

$$1, 5, 13 \text{ and } 21,$$

then by Theorem 3 we have

$$d(F) \leq 1 \times 5 \times 21 = 105.$$

(ii) We choose $A = 1$ and $B = 5$. The integers $E$ such that $|E| < 2\sqrt{AB} = 2\sqrt{5} \approx 4.47$ are $E = 0, \pm 1, \pm 2, \pm 3, \pm 4$. By Corollary 1 we may take

$$C(0) = C(\pm 1) = C(\pm 3) = 13,\ \ C(\pm 2) = C(\pm 4) = 21.$$

Hence if $F$ is odd and represents

$$1, 5, 13 \text{ and } 21,$$

then by Theorem 3 we have $d(F) \leq 8 \times 1 \times 5 \times 21 = 840$.

(iii) We chose $A = 3$ and $B = 7$. The integers $E$ such that $|E| < 2\sqrt{AB} = 2\sqrt{21} \approx 9.16$ with $E \equiv 0 \pmod 2$ are $E = 0, \pm 2, \pm 4, \pm 6, \pm 8$. By Corollary 1 we may take

$$C(0) = C(\pm 2) = C(\pm 6) = C(\pm 8) = 11, \ \ C(\pm 4) = 15.$$

Hence if $F$ is even and represents $3, 7, 11$ and $15$, then by Theorem 3 we have

$$d(F) \leq 3 \times 7 \times 15 = 315.$$

(iv) We choose $A = 3$ and $B = 7$. The integers in $E$ such that $|E| < 2\sqrt{AB} = 2\sqrt{21} \approx 9.16$ are $E = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9$. By Corollary 1 we may take

$$C(0) = C(\pm 2) = C(\pm 3) = C(\pm 5) = C(\pm 6) = C(\pm 7) = C(\pm 8) = C(\pm 9) = 11$$

and $C(\pm 1) = C(\pm 4) = 15$. Hence if $F$ is odd and represents $3, 7, 11$ and $15$, then by Theorem 3 we have $d(F) \leq 8 \times 3 \times 7 \times 15 = 2520$. □

## 4. Method for Determining All $(4, 1)$-universal Ternaries and All $(4, 3)$-universal Ternaries

It is convenient to let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We recall that a positive-definite integral ternary quadratic form $f(x, y, z)$ is said to be regular if there exists a finite number of progressions $\left\{A^k(Bl + C) \,|\, k, l \in \mathbb{N}_0\right\}$ such that $n \in \mathbb{N}$ is represented by $f(x, y, z)$ if and only if $n$ does not belong to any of these progressions. For example, by Legendre's theorem $x^2 + y^2 + z^2$ is regular as $n$ is represented by $x^2 + y^2 + z^2$ if and only if $n \neq 4^k(8l + 7)$ for any $k, l \in \mathbb{N}_0$.

The progressions corresponding to regular ternaries can be found in [4, pp. 112 - 113], [12] or deduced from Jones' theorem [10, Theorem, p. 99], [11, Theorem 5, p. 123]. The gcd of the coefficients of an integral ternary quadratic form divides any integer represented by the form. Thus, if the form represents 1 or two coprime integers the gcd of the coefficients of the form must be 1, that is, the form is primitive. Hence $(4, 1)$-universal and $(4, 3)$-universal integral ternary quadratic forms must be primitive. Thus to determine all $(4, 1)$-universal ternary quadratic forms we need only consider integral ternary quadratic forms which are positive-definite and primitive. Similarly for $(4, 3)$-universal forms. It is a result of Schiemann [17] that a positive-definite primitive integral ternary quadratic form is equivalent to one and only one reduced form, that is, a form $ax^2 + by^2 + cz^2 + dxy + exz + fyz$ satisfying the following thirteen conditions:

1. $\gcd(a, b, c, d, e, f) = 1$,
2. $0 < a \leq b \leq c$,

3. $-b < f \leq b$,

4. $0 \leq e \leq a$,

5. $0 \leq d \leq a$,

6. $a + b \geq d + e - f$,

7. if $d = 0$ or $e = 0$ then $f \geq 0$,

8. if $a = b$ then $\mid f \mid \leq e$,

9. if $b = c$ then $e \leq d$,

10. if $d = a$ then $e \leq 2f$,

11. if $e = a$ then $d \leq 2f$,

12. if $f = b$ then $d \leq 2e$,

13. if $a + b - d - e + f = 0$ then $2a - 2e - d \leq 0$.

Since equivalent forms represent the same integers in order to determine all $(4,1)$-universal and all $(4,3)$-universal ternaries, it suffices to consider only reduced primitive positive-definite integral ternary quadratic forms. By Theorem 4, we need only consider such forms with discriminant less than or equal to 105 in the case of $(4,1)$-universal even forms, with discriminant less than or equal to 315 in the case of $(4,3)$-universal even forms, with discriminant less than or equal to 840 in the case of $(4,1)$-universal odd forms, and with discriminant less than or equal to 2520 in the case of $(4,3)$-universal odd forms.

Our computer search for even reduced primitive positive-definite integral ternary quadratic forms of discriminant $\leq 105$ representing all the integers $1, 5, 9, \ldots, 1001$ yielded 24 forms. Thus there are at most 24 equivalence classes of positive -definite primitive integral ternary quadratic forms which are $(4,1)$-universal. The table of regular ternary quadratic forms due to Jagy, Kaplansky and Schiemann [9] showed that 19 of these 24 forms are regular. A check of the congruence conditions (given in the comments column) for the representability of integers by these 19 regular forms showed that they all represent every positive integer $\equiv 1 \pmod 4$ and so are $(4,1)$-universal. All $(4,1)$-universal even reduced primitive positive-definite integral ternary quadratic forms $ax^2 + by^2 + cz^2 + dxy + exz + fyz$ are contained in Table 1. "$R$" indicates that the form is regular and that it is $(4,1)$-universal. For example, the form $x^2 + 3y^2 + 3z^2 + 2yz$ is regular (see the table of Jagy, Kaplansky and Schiemann [9]) and the positive integers $n$ represented by this form are precisely those $n$ for which $n \neq 4l + 2$, $4^k(16l + 14)$ for any $k, l \in \mathbb{N}_0$. "$NR$" indicates that the form is not regular. A theorem in the comments column by a non-regular form gives a proof that the form is $(4,1)$-universal. An asterisk by a non-regular form indicates that there is as yet no proof of its $(4,1)$-universality and that the form has been checked to represent all $n \in \mathbb{N}$ satisfying $n \equiv 1 \pmod 4$ and $n \leq 100,000$.

Table 1: (4,1)-universal even ternary quadratic forms

| disc | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | R/NR | comments |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | R | $4^k(8l+7)$ |
| 2 | 1 | 1 | 2 | 0 | 0 | 0 | R | $4^k(16l+14)$ |
| 4 | 1 | 1 | 4 | 0 | 0 | 0 | R | $8l+3, 4^k(8l+7)$ |
| 4 | 1 | 2 | 2 | 0 | 0 | 0 | R | $4^k(8l+7)$ |
| 5 | 1 | 1 | 5 | 0 | 0 | 0 | R | $4^k(8l+3)$ |
| 6 | 1 | 2 | 3 | 0 | 0 | 0 | R | $4^k(16l+10)$ |
| 8 | 1 | 1 | 8 | 0 | 0 | 0 | R | $4l+3, 16l+6, 4^k(16l+14)$ |
| 8 | 1 | 2 | 4 | 0 | 0 | 0 | R | $4^k(16l+14)$ |
| 8 | 1 | 3 | 3 | 0 | 0 | 2 | R | $4l+2, 4^k(16l+14)$ |
| 9 | 1 | 2 | 5 | 0 | 0 | 2 | R | $4^k(8l+7)$ |
| 14 | 1 | 3 | 5 | 0 | 0 | 2 | R | $4^k(16l+2)$ |
| 16 | 1 | 1 | 16 | 0 | 0 | 0 | R | $4l+3, 8l+6, 32l+12, 4^k(8l+7)$ |
| 16 | 1 | 4 | 4 | 0 | 0 | 0 | R | $4l+2, 4l+3, 4^k(8l+7)$ |
| 16 | 1 | 4 | 5 | 0 | 0 | 4 | R | $8l+2, 8l+3, 32l+12, 4^k(8l+7)$ |
| 20 | 1 | 4 | 5 | 0 | 0 | 0 | NR | Theorem 5 |
| 24 | 1 | 5 | 5 | 0 | 0 | 2 | R | $4l+3, 16l+2, 4^k(16l+10)$ |
| 29 | 1 | 5 | 6 | 0 | 0 | 2 | NR | * |
| 32 | 1 | 4 | 8 | 0 | 0 | 0 | R | $4l+2, 4l+3, 4^k(16l+14)$ |
| 32 | 1 | 4 | 9 | 0 | 0 | 4 | NR | Theorem 6 |
| 36 | 1 | 5 | 8 | 0 | 0 | 4 | NR | Theorem 7 |
| 56 | 1 | 5 | 12 | 0 | 0 | 4 | R | $4l+3, 8l+2, 4^k(16l+2)$ |
| 64 | 1 | 4 | 16 | 0 | 0 | 0 | R | $4l+2, 4l+3, 16l+2, 4^k(8l+7)$ |
| 64 | 1 | 4 | 17 | 0 | 0 | 4 | NR | Theorem 8 |
| 64 | 1 | 5 | 13 | 0 | 0 | 2 | R | $8l+2, 8l+3, 32l+8, 32l+12,$ $128l+48, 4^k(8l+7)$ |

From Table 1 we see that the only diagonal ternaries which are $(4,1)$-universal are the 13 forms

$$\begin{gathered}x^2+y^2+z^2,\ x^2+y^2+2z^2,\ x^2+y^2+4z^2,\ x^2+2y^2+2z^2,\\ x^2+y^2+5z^2,\ x^2+2y^2+3z^2,\ x^2+y^2+8z^2,\ x^2+2y^2+4z^2,\\ x^2+y^2+16z^2,\ x^2+4y^2+4z^2,\ x^2+4y^2+5z^2,\ x^2+4y^2+8z^2,\\ x^2+4y^2+16z^2.\end{gathered}$$

This result was proved by the authors in [15].

A computer search for the even reduced primitive positive-definite integral ternary quadratic forms of discriminant $\leq 315$ representing all the integers $3, 7, 11, 15, \ldots, 1003$ yielded 57 forms. Thus there are at most 57 equivalence classes of primitive, positive-definite, integral ternary quadratic forms which are $(4,3)$-universal. From the table of Jagy, Kaplansky and Schiemann [9], we see that 28 of these forms are regular. A check of the congruence conditions for the representability of positive integers by these 28 forms showed that they all represent every positive integer $n \equiv 3 \pmod 4$ and so are $(4,3)$-universal. For example, $2x^2+3y^2+7z^2+2xy+2xz+2yz$ of discriminant 32 is regular and represents every positive integer not of the types $4l+1, 8l+6, 16l+4, 32l+24, 4^k(256l+224)$ for any $k, l \in \mathbb{N}_0$. Since $4l+3$ is not of any of these types, the form is $(4,3)$-universal. All $(4,3)$-universal even reduced primitive positive-definite integral ternary quadratic forms $ax^2+by^2+cz^2+dxy+exz+fyz$ are contained in Table 2. "R" indicates that the form is regular and that it is $(4,3)$-universal. "NR" indicates that the form is not regular. A theorem in the comments column by a non-regular form gives a proof that the form is $(4,3)$-universal. If the letter (c) follows the theorem number the proof is conditional upon another form of smaller discriminant being $(4,3)$-universal. An asterisk by a non-regular form indicates that there is as yet no proof of the $(4,3)$-universality of the form and that the form has been checked to represent all $n \in \mathbb{N}$ satisfying $n \equiv 3 \pmod 4$ and $n \leq 100{,}000$.

Table 2: (4,3)-universal even ternary quadratic forms

| disc | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | R/NR | comments |
|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 2 | 0 | 0 | 0 | R | $4^k(16l+14)$ |
| 3 | 1 | 2 | 2 | 0 | 0 | 2 | R | $4^k(8l+5)$ |
| 6 | 1 | 2 | 3 | 0 | 0 | 0 | R | $4^k(16l+10)$ |
| 7 | 2 | 2 | 3 | 2 | 2 | 2 | R | $4^k(8l+1)$ |
| 8 | 1 | 2 | 4 | 0 | 0 | 0 | R | $4^k(16l+14)$ |
| 8 | 1 | 3 | 3 | 0 | 0 | 2 | R | $4l+2, 4^k(16l+14)$ |
| 8 | 2 | 2 | 3 | 0 | 2 | 2 | R | $4l+1, 16l+6, 4^k(16l+14)$ |
| 11 | 1 | 2 | 6 | 0 | 0 | 2 | R | $4^k(8l+5)$ |
| 12 | 1 | 2 | 6 | 0 | 0 | 0 | R | $4^k(8l+5)$ |
| 12 | 2 | 3 | 3 | 2 | 2 | 2 | R | $8l+1, 4^k(8l+5)$ |
| 14 | 1 | 3 | 5 | 0 | 0 | 2 | R | $4^k(16l+2)$ |
| 15 | 2 | 3 | 3 | 2 | 0 | 0 | R | $4^k(8l+1)$ |
| 23 | 2 | 3 | 5 | 2 | 0 | 2 | R | $4^k(8l+1)$ |
| 24 | 3 | 3 | 3 | 2 | 0 | 0 | R | $4l+1, 16l+2, 4^k(16l+10)$ |
| 26 | 1 | 3 | 9 | 0 | 0 | 2 | NR | * |
| 28 | 2 | 3 | 5 | 0 | 0 | 2 | R | $4^k(8l+1)$ |
| 28 | 2 | 3 | 6 | 2 | 0 | 2 | R | $8l+5, 4^k(8l+1)$ |
| 31 | 2 | 3 | 6 | 0 | 2 | 2 | NR | * |
| 32 | 1 | 3 | 11 | 0 | 0 | 2 | NR | Theorem 9 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 32 | 2 | 3 | 7 | 2 | 2 | 2 | R | $4l+1, 8l+6, 16l+4, 32l+24,$ $4^k(256l+224)$ |
| 32 | 3 | 3 | 4 | 2 | 0 | 0 | R | $4l+1, 4l+2, 4^k(16l+14)$ |
| 32 | 3 | 4 | 4 | 2 | 2 | 4 | NR | Theorem 10 |
| 39 | 3 | 3 | 5 | 0 | 2 | 2 | NR | * |
| 44 | 3 | 3 | 6 | 2 | 2 | 2 | NR | * |
| 46 | 3 | 4 | 5 | 2 | 2 | -2 | NR | * |
| 47 | 2 | 3 | 10 | 2 | 2 | 2 | NR | * |
| 48 | 2 | 3 | 10 | 2 | 0 | 2 | R | $8l+1, 8l+6, 16l+4, 4^k(8l+5)$ |
| 48 | 3 | 3 | 6 | 0 | 2 | 2 | R | $8l+1, 8l+2, 32l+4, 4^k(8l+5)$ |
| 48 | 3 | 3 | 7 | 2 | 2 | -2 | R | $4l+1, 4l+2, 4^k(8l+5)$ |
| 50 | 3 | 4 | 5 | 2 | 2 | 2 | NR | * |
| 56 | 3 | 4 | 6 | 0 | 2 | 4 | R | $4l+1, 16l+10, 4^k(16l+2)$ |
| 60 | 3 | 3 | 7 | 0 | 2 | 0 | NR | Theorem 12 (c) |
| 63 | 3 | 5 | 6 | 2 | 2 | -4 | NR | * |
| 71 | 3 | 5 | 6 | 2 | 2 | 4 | NR | * |
| 79 | 3 | 5 | 6 | 2 | 2 | 0 | NR | * |
| 92 | 3 | 6 | 7 | 2 | 2 | 6 | NR | * |
| 95 | 3 | 5 | 7 | 2 | 0 | 2 | NR | * |
| 104 | 3 | 4 | 10 | 0 | 2 | 4 | NR | Theorem 13 (c) |
| 112 | 3 | 6 | 7 | 2 | 2 | 2 | R | $8l+2, 8l+5, 16l+4, 4^k(8l+1)$ |
| 112 | 3 | 7 | 7 | 2 | 2 | 6 | R | $4l+2, 8l+5, 4^k(8l+1)$ |
| 124 | 3 | 6 | 8 | 2 | 0 | 4 | NR | Theorem 14 (c) |
| 128 | 3 | 4 | 11 | 0 | 2 | 0 | R | $4l+1, 8l+2, 16l+6, 16l+8,$ $4^k(16l+14)$ |
| 128 | 3 | 7 | 7 | 2 | 0 | 4 | NR | Theorem 11 |
| 128 | 3 | 7 | 7 | 2 | 2 | -2 | R | $4l+1, 16l+2, 16l+4, 16l+6,$ $16l+10, 32l+24, 4^k(16l+14)$ |
| 131 | 3 | 6 | 9 | 2 | 2 | -4 | NR | * |
| 147 | 3 | 6 | 9 | 2 | 2 | 0 | NR | * |
| 156 | 3 | 6 | 10 | 2 | 2 | 4 | NR | Theorem 15 (c) |
| 176 | 3 | 6 | 11 | 2 | 2 | -2 | NR | * |
| 176 | 3 | 7 | 10 | 0 | 2 | 6 | NR | Theorem 16 (c) |
| 184 | 3 | 6 | 11 | 2 | 0 | 2 | NR | Theorem 17 (c) |
| 188 | 3 | 7 | 10 | 2 | 2 | -2 | NR | * |
| 192 | 3 | 6 | 12 | 2 | 0 | 4 | NR | * |
| 192 | 3 | 7 | 10 | 2 | 2 | 2 | NR | * |
| 192 | 3 | 7 | 11 | 2 | 2 | 6 | R | $4l+2, 8l+1, 32l+4$ $4^k(8l+5)$ |
| 200 | 3 | 7 | 10 | 0 | 2 | 2 | NR | Theorem 18 (c) |
| 224 | 3 | 6 | 14 | 2 | 2 | 4 | R | $4l+1, 16l+4, 16l+10$ $64l+40, 4^k(16l+2)$ |
| 252 | 3 | 6 | 15 | 2 | 0 | 2 | NR | * |

From Table 2 we see that the only diagonal ternaries which are $(4,3)$-universal are the four forms

$$x^2+y^2+2z^2,\ x^2+2y^2+3z^2,\ x^2+2y^2+4z^2,\ x^2+2y^2+6z^2.$$

This result was proved by the authors in [15]. The only diagonal ternaries which are both $(4,1)$-universal and $(4,3)$-universal, that is are $(2,1)$-universal, are $x^2+y^2+2z^2$, $x^2+2y^2+3z^2$ and $x^2+2y^2+4z^2$. This was proved in [18]. There are five ternaries common to both Tables 1 and 2. These are the $(2,1)$-universal even ternaries given by Kaplansky [13].

A computer search for the odd reduced primitive positive-definite integral ternary quadratic forms of discriminant $\leq 840$ representing all the integers $1,5,9,13,\ldots,1001$ yielded 31 forms. Thus there are at most 31 equivalence classes of primitive, positive-definite, integral ternary quadratic forms which are $(4,1)$-universal. From the table of Jagy, Kaplansky and Schiemann [9], we see that 14 of these forms are regular. Checking the congruence conditions for the representability of positive integers by these 14 forms showed that they all represent every positive integer $\equiv 1 \pmod 4$ and so are $(4,1)$-universal. For example, $x^2+3y^2+7z^2+xy+xz+2yz$ of discriminant 144 represents every positive integer not of the types $4l+2$, $4^k(64l+56)$ for any $k,l\in\mathbb{N}_0$. Since $4l+1$ is not of any of these types, the form is $(4,1)$-universal. All $(4,1)$-universal odd reduced primitive positive-definite integral ternary quadratic forms $ax^2+by^2+cz^2+dxy+exz+fyz$ are contained in Table 3. "R" indicates that the form is regular and that it is $(4,1)$-universal. "NR" indicates that the form is not regular. Just one of the non-regular forms is known to be $(4,1)$-universal (in fact $(2,1)$-universal) from the work of Kaplansky [13]. An asterisk by a non-regular form indicates that there is as yet no proof of the $(4,1)$-universality of the form and that the form has been checked to represent all $n\in\mathbb{N}$ satisfying $n\equiv 1 \pmod 4$ and $n\leq 100,000$.

Table 3: (4,1)-universal odd ternary quadratic forms

| disc | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | R/NR | comments |
|---|---|---|---|---|---|---|---|---|
| 4 | 1 | 1 | 1 | 1 | 1 | 1 | R | $4^k(16l+14)$ |
| 12 | 1 | 1 | 2 | 1 | 0 | 0 | R | $4^k(16l+10)$ |
| 16 | 1 | 1 | 3 | 1 | 1 | 1 | R | $4l+2, 4^k(64l+56)$ |
| 20 | 1 | 1 | 3 | 0 | 1 | 1 | R | $4^k(16l+6)$ |
| 28 | 1 | 1 | 5 | 1 | 1 | 1 | R | $4^k(16l+2)$ |
| 36 | 1 | 2 | 3 | 0 | 1 | 2 | R | $4^k(16l+14)$ |
| 44 | 1 | 2 | 3 | 0 | 1 | 0 | R | $4^k(16l+10)$ |
| 48 | 1 | 3 | 3 | 1 | 1 | 3 | R | $4l+2, 4^k(64l+40)$ |
| 52 | 1 | 1 | 7 | 0 | 1 | 1 | NR | * |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 60 | 1 | 3 | 3 | 1 | 1 | 1 | R | $4^k(16l+2)$ |
| 64 | 1 | 3 | 3 | 1 | 0 | 1 | R | $4l+2, 16l+8, 4^k(256l+224)$ |
| 68 | 1 | 1 | 9 | 0 | 1 | 1 | NR | * |
| 76 | 1 | 2 | 5 | 0 | 1 | 0 | NR | * |
| 80 | 1 | 3 | 4 | 1 | 0 | 2 | R | $4l+2, 4^k(64l+24)$ |
| 92 | 1 | 3 | 5 | 1 | 1 | 3 | R | $4^k(16l+2)$ |
| 100 | 1 | 3 | 5 | 1 | 1 | 2 | NR | $(2,1)$-universal [13] |
| 112 | 1 | 3 | 5 | 0 | 1 | 1 | R | $4l+2, 4^k(64l+8)$ |
| 124 | 1 | 3 | 6 | 1 | 0 | 2 | NR | * |
| 144 | 1 | 3 | 7 | 1 | 1 | 2 | R | $4l+2, 4^k(64l+56)$ |
| 148 | 1 | 3 | 7 | 1 | 1 | 1 | NR | * |
| 172 | 1 | 5 | 5 | 1 | 0 | 3 | NR | * |
| 180 | 1 | 3 | 9 | 1 | 1 | 3 | NR | * |
| 188 | 1 | 3 | 9 | 1 | 1 | 2 | NR | * |
| 208 | 1 | 4 | 7 | 0 | 1 | 2 | NR | * |
| 212 | 1 | 3 | 10 | 1 | 0 | 2 | NR | * |
| 236 | 1 | 3 | 11 | 1 | 1 | 1 | NR | * |
| 240 | 1 | 3 | 11 | 1 | 0 | 1 | NR | * |
| 252 | 1 | 5 | 7 | 0 | 1 | 3 | NR | * |
| 272 | 1 | 4 | 9 | 0 | 1 | 2 | NR | * |
| 284 | 1 | 3 | 13 | 1 | 0 | 1 | NR | * |
| 348 | 1 | 5 | 9 | 0 | 1 | 1 | NR | * |

A computer search for the odd reduced primitive positive-definite integral ternary quadratic forms of discriminant $\leq 2520$ representing all the integers $3, 7, 11, 15, \ldots, 1003$ yielded 80 forms. The two forms $3x^2+5y^2+7z^2+2xz+yz$ and $3x^2+5y^2+8z^2+xy+3xz+3yz$ fail to represent 1191 and 1227 respectively and were eliminated. The remaining 78 forms are listed in Table 4. A check of the table of Jagy, Kaplansky and Schiemann [9] showed that just 14 of these forms are regular and a check of the congruence conditions for the representability of positive integers by these 14 forms showed that they all represent every positive integer $\equiv 3 \pmod 4$ and so are $(4,3)$-universal. The remaining 64 forms are non-regular and only one of them is known to be $(4,3)$-universal (in fact $(2,1)$-universal) from the work of Kaplansky [13]. However, it has been verified that they represent all $n \in \mathbb{N}$ satisfying $n \equiv 3 \pmod 4$ and $n \leq 100{,}000$.

Table 4: (4,3)-universal odd ternary quadratic forms

| disc | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | R/NR | comments |
|---|---|---|---|---|---|---|---|---|
| 4 | 1 | 1 | 1 | 1 | 1 | 1 | R | $4^k(16l+14)$ |
| 12 | 1 | 1 | 2 | 1 | 0 | 0 | R | $4^k(16l+10)$ |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 16 | 1 | 1 | 3 | 1 | 1 | 1 | R | $4l+2, 4^k(64l+56)$ |
| 20 | 1 | 1 | 3 | 0 | 1 | 1 | R | $4^k(16l+6)$ |
| 28 | 1 | 1 | 5 | 1 | 1 | 1 | R | $4^k(16l+2)$ |
| 36 | 1 | 2 | 3 | 0 | 1 | 2 | R | $4^k(16l+14)$ |
| 44 | 1 | 2 | 3 | 0 | 1 | 0 | R | $4^k(16l+10)$ |
| 48 | 1 | 3 | 3 | 1 | 1 | 3 | R | $4^k(4l+2)$ |
| 52 | 1 | 1 | 9 | 1 | 1 | 1 | NR | * |
| 60 | 1 | 3 | 3 | 1 | 1 | 1 | R | $4^k(16l+2)$ |
| 64 | 1 | 1 | 11 | 1 | 1 | 1 | NR | * |
| 64 | 1 | 3 | 3 | 1 | 0 | 1 | R | $4^k(4l+2)$ |
| 68 | 1 | 2 | 5 | 0 | 1 | 2 | NR | * |
| 76 | 1 | 2 | 5 | 0 | 1 | 0 | NR | * |
| 80 | 1 | 3 | 4 | 1 | 0 | 2 | R | $4l+2, 4^k(64l+24)$ |
| 84 | 2 | 3 | 3 | 2 | 2 | 3 | NR | * |
| 92 | 1 | 3 | 5 | 1 | 1 | 3 | R | $4^k(16l+2)$ |
| 100 | 1 | 3 | 5 | 1 | 1 | 2 | NR | $(2,1)$-universal)[13] |
| 108 | 1 | 3 | 5 | 1 | 0 | 1 | NR | * |
| 112 | 1 | 3 | 5 | 0 | 1 | 1 | R | $4l+2, 4^k(64l+8)$ |
| 116 | 2 | 3 | 3 | 2 | 0 | 1 | NR | * |
| 124 | 1 | 3 | 6 | 1 | 0 | 2 | NR | * |
| 144 | 1 | 3 | 7 | 1 | 1 | 2 | R | $4l+2, 4^k(64l+56)$ |
| 144 | 3 | 3 | 3 | 3 | 2 | 1 | NR | * |
| 148 | 1 | 3 | 7 | 1 | 1 | 1 | NR | * |
| 156 | 3 | 3 | 3 | 3 | 1 | 1 | NR | * |
| 164 | 2 | 3 | 5 | 2 | 2 | 3 | NR | * |
| 176 | 3 | 3 | 3 | 2 | 1 | -1 | NR | * |
| 204 | 3 | 3 | 3 | 1 | 1 | 0 | NR | * |
| 208 | 1 | 3 | 9 | 0 | 1 | 1 | NR | * |
| 212 | 2 | 3 | 5 | 0 | 2 | 1 | NR | * |
| 236 | 2 | 3 | 5 | 0 | 0 | 1 | NR | * |
| 256 | 1 | 3 | 11 | 0 | 1 | 1 | NR | * |
| 256 | 3 | 3 | 4 | 1 | 2 | 0 | NR | * |
| 260 | 3 | 3 | 5 | 1 | 3 | 3 | NR | * |
| 284 | 3 | 3 | 5 | 1 | 3 | 2 | NR | * |
| 316 | 3 | 3 | 5 | 1 | 2 | -1 | NR | * |
| 320 | 3 | 4 | 5 | 2 | 3 | 4 | NR | * |
| 324 | 3 | 3 | 5 | 1 | 2 | 1 | NR | * |
| 332 | 2 | 3 | 7 | 0 | 0 | 1 | NR | * |
| 368 | 3 | 4 | 5 | 2 | 3 | 2 | NR | * |
| 380 | 3 | 5 | 5 | 3 | 2 | 5 | NR | * |
| 396 | 3 | 3 | 6 | 1 | 2 | 0 | NR | * |
| 400 | 3 | 4 | 5 | 2 | 1 | -2 | NR | * |
| 428 | 3 | 5 | 5 | 3 | 1 | 4 | NR | * |
| 432 | 3 | 4 | 5 | 2 | 1 | 0 | NR | * |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 436 | 3 | 3 | 7 | 1 | 3 | 1 | NR | * |
| 452 | 3 | 5 | 5 | 3 | 2 | 3 | NR | * |
| 464 | 3 | 3 | 7 | 1 | 2 | 1 | NR | * |
| 464 | 3 | 5 | 5 | 3 | 1 | 3 | NR | * |
| 484 | 3 | 3 | 7 | 1 | 1 | 0 | NR | * |
| 500 | 3 | 5 | 5 | 3 | 1 | 1 | NR | * |
| 556 | 3 | 5 | 6 | 1 | 2 | -4 | NR | * |
| 564 | 3 | 3 | 9 | 1 | 3 | 2 | NR | * |
| 572 | 3 | 5 | 6 | 3 | 2 | 2 | NR | * |
| 576 | 3 | 3 | 9 | 1 | 3 | 1 | NR | * |
| 576 | 3 | 4 | 7 | 2 | 1 | -2 | NR | * |
| 588 | 3 | 5 | 6 | 1 | 2 | 4 | NR | * |
| 592 | 3 | 4 | 7 | 2 | 1 | 2 | NR | * |
| 596 | 3 | 3 | 9 | 1 | 2 | -1 | NR | * |
| 604 | 3 | 3 | 9 | 1 | 2 | 1 | NR | * |
| 604 | 3 | 5 | 7 | 2 | 3 | 5 | NR | * |
| 624 | 3 | 3 | 9 | 1 | 1 | 0 | NR | * |
| 636 | 3 | 5 | 6 | 1 | 2 | -2 | NR | * |
| 652 | 3 | 5 | 6 | 1 | 2 | 2 | NR | * |
| 656 | 3 | 5 | 7 | 1 | 2 | 5 | NR | * |
| 656 | 3 | 5 | 7 | 3 | 2 | 3 | NR | * |
| 668 | 3 | 5 | 7 | 3 | 1 | 3 | NR | * |
| 676 | 3 | 3 | 10 | 1 | 2 | 0 | NR | * |
| 708 | 3 | 5 | 7 | 2 | 1 | -3 | NR | * |
| 724 | 3 | 5 | 7 | 1 | 3 | 2 | NR | * |
| 788 | 3 | 5 | 7 | 1 | 1 | -2 | NR | * |
| 796 | 3 | 6 | 7 | 2 | 3 | 4 | NR | * |
| 1028 | 3 | 5 | 10 | 1 | 2 | -4 | NR | * |
| 1044 | 3 | 5 | 9 | 1 | 1 | -1 | NR | * |
| 1156 | 3 | 7 | 7 | 1 | 0 | 1 | NR | * |
| 1332 | 3 | 7 | 9 | 1 | 3 | 3 | NR | * |
| 1472 | 3 | 7 | 9 | 1 | 1 | -1 | NR | * |

There are eighteen ternaries common to both Tables 3 and 4. These are the $(2,1)$-universal odd ternaries given by Kaplansky [13]. The form $x^2+3y^2+5z^2+xy+xz+2yz$ of discriminant 100 is equivalent to Kaplansky's form $x^2+3y^2+5z^2+xy+xz-yz=(x+z)^2+3y^2+5(-z)^2+(x+z)y+(x+z)(-z)+2y(-z)$ and so is $(2,1)$-universal, and thus both $(4,1)$-universal and $(4,3)$-universal.

## 5. Proofs of the $(4,1)$-universality or $(4,3)$-universality of Certain Non-regular Ternary Quadratic Forms

In this section we prove the $(4,1)$-universality of the four non-regular forms $x^2+4y^2+5z^2$ (Theorem 5), $x^2+4y^2+9z^2+4yz$ (Theorem 6), $x^2+5y^2+8z^2+4yz$ (Theorem 7), $x^2+4y^2+17z^2+4yz$ (Theorem 8) and the $(4,3)$-universality of the three non-regular forms $x^2+3y^2+11z^2+2yz$ (Theorem 9), $3x^2+4y^2+4z^2+2xy+2xz+4yz$ (Theorem 10), $3x^2+7y^2+7z^2+2xy+4yz$ (Theorem 11).

**Theorem 5.** *The form $x^2+4y^2+5z^2$ is $(4,1)$-universal.*

*Proof.* The form $x^2+y^2+5z^2$ is regular and it represents all positive integers $n \neq 4^k(8l+3)$ for any $k,l \in \mathbb{N}_0$ [4, Table 5, p. 112]. A positive integer $n \equiv 1 \pmod 4$ is not of this form so there exist integers $a,b$ and $c$ such that $a^2+b^2+5c^2=n$. If $a \equiv b \equiv 1 \pmod 2$ then $c^2 \equiv 5c^2 \equiv n-a^2-b^2 \equiv 1-1-1 \equiv 3 \pmod 4$, which is impossible. Hence $a$ or $b \equiv 0 \pmod 2$. Interchanging $a$ and $b$ if necessary we may suppose that $b \equiv 0 \pmod 2$. We then define integers $x,y$ and $z$ by $x=a$, $y=b/2$, $z=c$ so that

$$n=a^2+b^2+5c^2=x^2+4y^2+5z^2,$$

showing that the form $x^2+4y^2+5z^2$ is $(4,1)$-universal. □

**Theorem 6.** *The form $x^2+4y^2+9z^2+4yz$ is $(4,1)$-universal.*

*Proof.* The form $x^2+y^2+8z^2$ is regular and the positive integers it represents are precisely those which are not of the forms $4l+3$, $16l+6$, $4^k(16l+14)$ $(k,l \in \mathbb{N}_0)$ [4, Table 5, p. 112]. A positive integer $n \equiv 1 \pmod 4$ is not of these forms so there exist integers $a,b$ and $c$ such that

$$a^2+b^2+8c^2=n.$$

Clearly $a$ and $b$ are of opposite parity and so we can interchange $a$ and $b$ if necessary so that

$$b \equiv c \pmod 2.$$

We then define integers $x,y$ and $z$ by

$$x=a,\ y=\frac{b-c}{2},\ z=c$$

so that

$$n=a^2+b^2+8c^2=x^2+(2y+z)^2+8z^2=x^2+4y^2+9z^2+4yz.$$

Thus $x^2+4y^2+9z^2+4yz$ is $(4,1)$-universal. □

**Theorem 7.** *The form $x^2+5y^2+8z^2+4yz$ is $(4,1)$-universal.*

*Proof.* The form $x^2 + 2y^2 + 5z^2 + 2yz$ is $(4,1)$-universal (see Table 1). Let $n \in \mathbb{N}$ be such that $n \equiv 1 \pmod 4$. Then there exist integers $a, b$ and $c$ such that

$$n = a^2 + 2b^2 + 5c^2 + 2bc.$$

If $b \equiv 0 \pmod 2$ we can define integers $x, y$ and $z$ by

$$x = a,\ y = c,\ z = \frac{b}{2}.$$

Then

$$n = x^2 + 5y^2 + 8z^2 + 4yz.$$

If $b \equiv 1 \pmod 2$ then

$$1 \equiv a^2 + 2 + c^2 + 2c \pmod 4$$

so that

$$a^2 + (c+1)^2 \equiv 0 \pmod 4$$

and thus

$$a \equiv 0 \pmod 2,\ c \equiv 1 \pmod 2,$$

so

$$b \equiv c \pmod 2.$$

We define integers $x, y$ and $z$ by

$$x = -a,\ y = c,\ z = \frac{-b-c}{2},$$

so

$$a = -x,\ b = -y - 2z,\ c = y.$$

Hence

$$\begin{aligned} n &= a^2 + 2b^2 + 5c^2 + 2bc = (-x)^2 + 2(-y-2z)^2 + 5y^2 + 2(-y-2z)y \\ &= x^2 + 5y^2 + 8z^2 + 4yz, \end{aligned}$$

proving that $x^2 + 5y^2 + 8z^2 + 4yz$ is $(4,1)$-universal. $\square$

**Theorem 8.** *The form $x^2 + 4y^2 + 17z^2 + 4yz$ is $(4,1)$-universal.*

*Proof.* The form $x^2 + y^2 + 16z^2$ is regular [4, Table 5, p. 112] and the positive integers it represents are precisely those not of any of the forms

$$4l + 3,\ 8l + 6,\ 32l + 12,\ 4^k(8l + 7)\ (k, l \in \mathbb{N}_0).$$

Thus $x^2+y^2+16z^2$ is $(4,1)$-universal. Hence, for any positive integer $n \equiv 1 \pmod 4$ there exist integers $a$, $b$ and $c$ such that

$$a^2+b^2+16c^2=n.$$

Clearly $a$ and $b$ are of opposite parity so we can interchange $a$ and $b$ if necessary so that

$$b \equiv c \pmod 2.$$

We then define integers $x, y$ and $z$ by

$$x=a,\ y=\frac{b-c}{2},\ z=c$$

so that

$$n=a^2+b^2+16c^2=x^2+(2y+z)^2+16z^2=x^2+4y^2+17z^2+4yz,$$

and thus $x^2+4y^2+17z^2+4yz$ is $(4,1)$-universal. □

**Theorem 9.** *The form $x^2+3y^2+11z^2+2yz$ is $(4,3)$-universal.*

*Proof.* The genus of discriminant 32 containing the class of the form $x^2+3y^2+11z^2+2yz$ contains exactly one other class, namely the class of the form $3x^2+4y^2+4z^2+2xy+2xz+4yz$ [2]. By Jones' theorem [12, Theorem 5, p. 123], the set of positive integers represented by either $x^2+3y^2+11z^2+2yz$ or $3x^2+4y^2+4z^2+2xy+2xz+4yz$ or both is precisely the set of positive integers not of the forms

$$4l+2,\ 16l+8,\ 4^k(256l+224)\,(k,l \in \mathbb{N}_0). \tag{5.1}$$

Let $n \in \mathbb{N}$ be such that $n \equiv 3 \pmod 4$. As $n$ is not of any of the forms in (5.1) it is represented by either $x^2+3y^2+11z^2+2yz$ or $3x^2+4y^2+4z^2+2xy+2xz+4yz$. We show that $n$ is always represented by $x^2+3y^2+11z^2+2yz$. Suppose $n$ is represented by $3x^2+4y^2+4z^2+2xy+2xz+4yz$. Then there are integers $u, v$ and $w$ such that

$$n=3u^2+4v^2+4w^2+2uv+2uw+4vw.$$

Taking this equation modulo 2 we see that $u \equiv 1 \pmod 2$. Next taking it modulo 4 we deduce that $v \equiv w \pmod 2$. Thus we can define integers $g, h$ and $k$ with $g$ even by

$$g=v-w,\ h=\frac{2u+v+w}{2},\ k=\frac{-v-w}{2},$$

so

$$u=h+k,\ v=\frac{g}{2}-k,\ w=\frac{-g}{2}-k.$$

Hence $n=g^2+3h^2+11k^2+2hk$ so that $n$ is represented by $x^2+3y^2+11z^2+2yz$. Hence $x^2+3y^2+11z^2+2yz$ is $(4,3)$-universal. □

**Theorem 10.** *The form* $3x^2 + 4y^2 + 4z^2 + 2xy + 2xz + 4yz$ *is* $(4, 3)$*-universal.*

*Proof.* The form $x^2 + 3y^2 + 3z^2 + 2yz$ is $(2, 1)$-universal (see Table 1). Hence for any positive integer $n \equiv 3 \pmod 4$ there exist integers $a, b$ and $c$ such that

$$n = a^2 + 3b^2 + 3c^2 + 2bc.$$

An examination of this equation modulo 4 yields

$$(a, b, c) \equiv (0, 0, 1) \text{ or } (0, 1, 0) \pmod 2.$$

Permuting $b$ and $c$ if necessary we may suppose that $a \equiv b \pmod 2$. We define integers $x, y$ and $z$ by

$$x = c,\ y = \frac{-a+b}{2},\ z = \frac{a+b}{2},$$

so that

$$\begin{aligned} n &= a^2 + 3b^2 + 3c^2 + 2bc = (z-y)^2 + 3(y+z)^2 + 3x^2 + 2(y+z)x \\ &= 3x^2 + 4y^2 + 4z^2 + 2xy + 2xz + 4yz, \end{aligned}$$

and thus the form $3x^2 + 4y^2 + 4z^2 + 2xy + 2xz + 4yz$ is $(4, 3)$-universal. □

**Theorem 11.** *The form* $3x^2 + 7y^2 + 7z^2 + 2xy + 4yz$ *is* $(4, 3)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 4$. As $2x^2 + 3y^2 + 7z^2 + 2xy + 2xz + 2yz$ (discriminant $= 32$) is $(4, 3)$-universal (see Table 2), there exist integers $a, b$ and $c$ such that

$$n = 2a^2 + 3b^2 + 7c^2 + 2ab + 2ac + 2bc.$$

Reducing this equation modulo 2 we see that

$$1 \equiv b + c \pmod 2$$

so that

$$b \not\equiv c \pmod 2.$$

Hence we have either

$$a \equiv b \pmod 2 \text{ or } a \equiv c \pmod 2.$$

We define integers $x, y$ and $z$ by

$$\begin{cases} x = \frac{-a+b}{2},\ y = \frac{a+b}{2},\ z = c & \text{if } a \equiv b \pmod 2, \\ x = \frac{-a-2b-c}{2},\ y = \frac{a+c}{2},\ z = -c & \text{if } a \equiv c \pmod 2, \end{cases}$$

so that

$$\begin{cases} a = y - x,\ b = x + y,\ c = z & \text{if } a \equiv b \pmod 2, \\ a = 2y + z,\ b = -x - y,\ c = -z & \text{if } a \equiv c \pmod 2, \end{cases}$$

and thus in both cases we have

$$n = 3x^2 + 7y^2 + 7z^2 + 2xy + 4yz,$$

proving that $3x^2 + 7y^2 + 7z^2 + 2xy + 4yz$ is $(4, 3)$-universal. □

## 6. Conditional Results

In this section we establish the $(4, 1)$-universality of seven forms from the conjectured such universality of seven other forms.

**Theorem 12.** *Assume that*

$$2x^2 + 3y^2 + 3z^2 + 2xy \quad (\textit{discriminant} = 15)$$

*is* $(4, 3)$*-universal. Then*

$$3x^2 + 3y^2 + 7z^2 + 2xz \quad (\textit{discriminant} = 60)$$

*is* $(4, 3)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 4$. By assumption there exist integers $a, b$ and $c$ such that

$$n = 2a^2 + 3b^2 + 3c^2 + 2ab.$$

Suppose first that $a$ is even. We define integers $x, y$ and $z$ by

$$x = \frac{-a - 2b}{2},\ y = -c,\ z = \frac{a}{2},$$

so that

$$a = 2z,\ b = -x - z,\ c = -y,$$

and thus

$$\begin{aligned} n &= 2(2z)^2 + 3(-x - z)^2 + 3(-y)^2 + 2(2z)(-x - z) \\ &= 3x^2 + 3y^2 + 7z^2 + 2xz. \end{aligned}$$

Now suppose that $a$ is odd. Then

$$3 \equiv 2 + 3b^2 + 3c^2 + 2b \pmod 4$$

so that
$$3((b+1)^2+c^2) \equiv 3b^2+3c^2+2b+3 \equiv 0 \pmod 4,$$
and thus
$$(b+1)^2+c^2 \equiv 0 \pmod 4$$
so
$$b+1 \equiv c \equiv 0 \pmod 2.$$
Hence
$$b \equiv 1 \pmod 2$$
and thus
$$a \equiv b \pmod 2.$$
We define integers $x, y$ and $z$ by
$$x = \frac{a-b}{2},\ y = c,\ z = \frac{-a-b}{2}$$
so that
$$a = x - z,\ b = -x - z,\ c = y,$$
and thus
$$\begin{aligned} n &= 2(x-z)^2+3(-x-z)^2+3y^2+2(x-z)(-x-z) \\ &= 3x^2+3y^2+7z^2+2xz. \end{aligned}$$
This completes the proof that
$$3x^2+3y^2+7z^2+2xz$$
is $(4,3)$-universal under the stated assumption. □

**Theorem 13.** *Assume that*
$$x^2+3y^2+9z^2+2yz \quad (\textit{discriminant} = 26)$$
*is* $(4,3)$*-universal. Then*
$$3x^2+4y^2+10z^2+2xz+4yz \quad (\textit{discriminant} = 104)$$
*is* $(4,3)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 4$. Then, by assumption, there exist integers $a, b$ and $c$ such that
$$n = a^2+3b^2+9c^2+2bc.$$
Hence
$$a^2+3b^2+c^2+2bc \equiv 3 \pmod 4.$$

If $b \equiv 0 \pmod 2$ then
$$a^2 + c^2 \equiv 3 \pmod 4,$$
which is impossible. Hence $b \equiv 1 \pmod 2$. Thus $a^2 + c^2 + 2c \equiv 0 \pmod 4$ so that
$$a \equiv c \pmod 2.$$

Define integers $x, y$ and $z$ by
$$x = -b,\ y = \frac{c-a}{2},\ z = -c.$$

Then $a = -2y - z$, $b = -x$, $c = -z$ and
$$\begin{aligned} n &= a^2 + 3b^2 + 9c^2 + 2bc = (-2y - z)^2 + 3(-x)^2 + 9(-z)^2 + 2(-x)(-z) \\ &= 3x^2 + 4y^2 + 10z^2 + 2xz + 4yz \end{aligned}$$

so $3x^2 + 4y^2 + 10z^2 + 2xz + 4yz$ is $(4, 3)$-universal under the stated assumption. □

**Theorem 14.** *Assume that*
$$2x^2 + 3y^2 + 6z^2 + 2xz + 2yz \quad (\textit{discriminant} = 31)$$
*is* $(4, 3)$*-universal. Then*
$$3x^2 + 6y^2 + 8z^2 + 2xy + 4yz \quad (\textit{discriminant} = 124)$$
*is* $(4, 3)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 4$. By assumption, there exist integers $a, b$ and $c$ such that
$$n = 2a^2 + 3b^2 + 6c^2 + 2ac + 2bc.$$

Clearly $b \equiv 1 \pmod 2$. Then
$$3 \equiv 2a^2 + 3 + 2c^2 + 2ac + 2c \pmod 4$$
so $0 \equiv a(a + c) \pmod 2$. Hence
$$a \equiv 0 \pmod 2 \ \text{ or } a \equiv c \pmod 2.$$

We define integers $x, y$ and $z$ by
$$\begin{cases} x = b, \quad y = c, \quad z = a/2 & \text{if } a \equiv 0 \pmod 2, \\ x = -b,\ y = -c,\ z = \frac{a+c}{2} & \text{if } a \equiv c \equiv 1 \pmod 2, \end{cases}$$

so that
$$\begin{cases} a = 2z, \qquad b = x, \quad c = y & \text{if } a \equiv 0 \pmod 2, \\ a = y + 2z,\ b = -x,\ c = -y & \text{if } a \equiv c \equiv 1 \pmod 2, \end{cases}$$

and thus in both cases we have

$$n = 3x^2 + 6y^2 + 8z^2 + 2xy + 4yz,$$

proving that $3x^2 + 6y^2 + 8z^2 + 2xy + 4yz$ is $(4, 3)$-universal under the stated assumption. □

**Theorem 15.** *Assume that*

$$3x^2 + 3y^2 + 5z^2 + 2xz + 2yz \quad (\textit{discriminant} = 39)$$

*is* $(4, 3)$*-universal. Then*

$$3x^2 + 6y^2 + 10z^2 + 2xy + 2xz + 4yz \quad (\textit{discriminant} = 156)$$

*is* $(4, 3)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 4$. By assumption there exist integers $a, b$ and $c$ such that

$$n = 3a^2 + 3b^2 + 5c^2 + 2ac + 2bc.$$

If $c \equiv 0 \pmod 2$ then $3 \equiv 3a^2 + 3b^2 \pmod 4$ so that $a^2 + b^2 \equiv 1 \pmod 4$ and thus $(a, b) \equiv (0, 1)$ or $(1, 0) \pmod 2$. We may permute $a$ and $b$ so that $b \equiv 0 \pmod 2$. Then $b \equiv c \equiv 0 \pmod 2$.

If $c \equiv 1 \pmod 2$ then

$$3 \equiv 3a^2 + 3b^2 + 1 + 2a + 2b \equiv 3a^2 + 3b^2 + 1 + 2a^2 + 2b^2 \pmod 4,$$

so $a^2 + b^2 \equiv 2 \pmod 4$ and thus $a \equiv b \equiv 1 \pmod 2$. Hence $b \equiv c \equiv 1 \pmod 2$.

In both cases we have $b \equiv c \pmod 2$ and we can define integers $x, y$ and $z$ by

$$x = -a,\ y = \frac{b-c}{2},\ z = \frac{-b-c}{2},$$

so that $a = -x$, $b = y - z$, $c = -y - z$, and hence

$$\begin{aligned} n &= 3a^2 + 3b^2 + 5c^2 + 2ac + 2bc \\ &= 3x^2 + 3(y-z)^2 + 5(y+z)^2 + 2x(y+z) - 2(y-z)(y+z) \\ &= 3x^2 + 6y^2 + 10z^2 + 2xy + 2xz + 4yz \end{aligned}$$

proving that $3x^2 + 6y^2 + 10z^2 + 2xy + 2xz + 4yz$ is $(4, 3)$-universal under the stated assumption. □

**Theorem 16.** *Assume that*

$$3x^2 + 3y^2 + 6z^2 + 2xy + 2xz + 2yz \quad (\textit{discriminant} = 44)$$

*is* $(4, 3)$*-universal. Then the form*

$$3x^2 + 7y^2 + 10z^2 + 2xz + 6yz \quad (\textit{discriminant} = 176)$$

*is* $(4, 3)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 4$. As $3x^2+3y^2+6z^2+2xy+2xz+2yz$ is assumed to be $(4,3)$-universal there are integers $a, b$ and $c$ such that

$$n = 3a^2 + 3b^2 + 6c^2 + 2ab + 2ac + 2bc.$$

Taking this equation modulo 2, we see that $a$ and $b$ are of opposite parity. We can interchange $a$ and $b$ if necessary so that $b \equiv c \pmod 2$. Then we can define integers $u, v$ and $w$ by

$$u = \frac{-2a-b-c}{2},\ v = \frac{-b+c}{2},\ w = \frac{b+c}{2}.$$

Hence

$$a = -u-w,\ b = -v+w,\ c = v+w,$$

and so

$$\begin{aligned} n =& 3(-u-w)^2 + 3(-v+w)^2 + 6(v+w)^2 + 2(-u-w)(-v+w) \\ &+ 2(-u-w)(v+w) + 2(-v+w)(v+w) \\ =& 3u^2 + 7v^2 + 10w^2 + 2uw + 6vw. \end{aligned}$$

Hence $3x^2+7y^2+10z^2+2xz+6yz$ is $(4,3)$-universal under the stated assumption. □

**Theorem 17.** *Assume that*

$$3x^2 + 4y^2 + 5z^2 + 2xy + 2xz - 2yz \quad (\textit{discriminant} = 46)$$

*is* $(4,3)$*-universal. Then*

$$3x^2 + 6y^2 + 11z^2 + 2xy + 2yz \quad (\textit{discriminant} = 184)$$

*is* $(4,3)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 4$. By assumption there exist integers $a, b$ and $c$ such that

$$n = 3a^2 + 4b^2 + 5c^2 + 2ab + 2ac - 2bc.$$

Clearly $1 \equiv a + c \pmod 2$. If $(a,c) \equiv (0,1) \pmod 2$ then $3 \equiv 1 + 2b \pmod 4$ so $b \equiv 1 \pmod 2$. If $(a,c) \equiv (1,0) \pmod 2$ then $3 \equiv 3+2b \pmod 4$ so $b \equiv 0 \pmod 2$. Thus $b \equiv c \pmod 2$. We define integers $x, y$ and $z$ by

$$x = \frac{-2a-b-c}{2},\ y = \frac{b+c}{2},\ z = \frac{-b+c}{2},$$

so that

$$a = -x-y,\ b = y-z,\ c = y+z,$$

and thus

$$n = 3(-x-y)^2 + 4(y-z)^2 + 5(y+z)^2 + 2(-x-y)(y-z)$$

$$
\begin{aligned}
&+ 2(-x-y)(y+z) - 2(y-z)(y+z) \\
=&3x^2 + 6y^2 + 11z^2 + 2xy + 2yz
\end{aligned}
$$

proving that $3x^2 + 6y^2 + 11z^2 + 2xy + 2yz$ is $(4,3)$-universal under the stated assumption. ☐

**Theorem 18.** *Assume that*

$$3x^2 + 4y^2 + 5z^2 + 2xy + 2xz + 2yz \quad (\textit{discriminant} = 50)$$

*is* $(4,3)$*-universal. Then*

$$3x^2 + 7y^2 + 10z^2 + 2xz + 2yz \quad (\textit{discriminant} = 200)$$

*is* $(4,3)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 4$. Then by assumption there exist integers $a, b$ and $c$ such that

$$n = 3a^2 + 4b^2 + 5c^2 + 2ab + 2ac + 2bc.$$

Taking this equation modulo 2, we obtain $1 \equiv a + c \pmod 2$ so that

$$(a, c) \equiv (0, 1) \text{ or } (1, 0) \pmod 2.$$

If $(a, c) \equiv (0, 1) \pmod 2$ then $3 \equiv 1+2b \pmod 4$ so $b \equiv 1 \pmod 2$. If $(a, c) \equiv (1, 0) \pmod 2$ then $3 \equiv 3 + 2b \pmod 4$ so $b \equiv 0 \pmod 2$. Hence in both cases we have $b \equiv c \pmod 2$. Define integers $x, y$ and $z$ by

$$x = \frac{-2a-b-c}{2},\ y = \frac{-b+c}{2},\ z = \frac{b+c}{2}$$

so that $a = -x - z$, $b = -y + z$, $c = y + z$, and thus

$$
\begin{aligned}
n =&3(-x-z)^2 + 4(-y+z)^2 + 5(y+z)^2 + 2(-x-z)(-y+z) \\
&+ 2(-x-z)(y+z) + 2(-y+z)(y+z) \\
=&3x^2 + 7y^2 + 10z^2 + 2xz + 2yz,
\end{aligned}
$$

proving that $3x^2 + 7y^2 + 10z^2 + 2xz + 2yz$ is $(4,3)$-universal under the stated assumption. ☐

## 7. Concluding Remarks

In Tables 1 - 4 we list all possible $(4,1)$- and $(4,3)$-universal reduced primitive positive-definite integral ternary quadratic forms. As there is no known algorithm for determining the integers represented by an arbitrary non-regular ternary quadratic form, it remains a difficult problem to prove the $(4, 1)$-universality or $(4, 3)$-universality of the forms marked with an asterisk in these tables.

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