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THE PRACTICALITY OF SHIFTED PRIMES

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Abstract

A natural number n is called *practical* if every integer between 1 and n can be expressed as a sum of distinct positive divisors of n. In this paper, given an integer a, we explore the practicality of shifted primes $p-a$. Let

$$P_a(x) := |\{p \leqslant x : p \text{ prime}, \ p-a \text{ practical}\}|.$$

We establish upper and lower bounds for $P_a(x)$. In particular, for any given odd integer a, there are infinitely many primes p such that $p-a$ is practical.

1. Introduction

A natural number n is called *practical* if every integer between 1 and n can be expressed as a sum of distinct positive divisors of n. These numbers were first introduced by Srinivasan [13] and have since been studied by several authors [2, 3, 5, 6, 7, 10, 12, 14, 16, 18]. There are many similarities between the distribution of practical numbers and that of prime numbers. The analogues of Legendre's conjecture [5], Goldbach's conjecture [7], the twin prime conjecture [7], and the prime number theorem [18, 20], are all theorems in the context of practical numbers. More conjectures involving practical numbers can be found in [15].

The sequence of practical numbers, which begins with

$$1, \ 2, \ 4, \ 6, \ 8, \ 12, \ 16, \ 18, \ 20, \ 24, \ 28, \ 30, ...,$$

seems to have many terms in common with the shifted primes $p-1$:

$$1, \ 2, \ 4, \ 6, \ 10, \ 12, \ 16, \ 18, \ 22, \ 28, \ 30, ...$$

This raises the question as to whether there are infinitely many primes p for which $p-1$ is practical. The number of primes below x is $\sim x/\log x$, while the number of practical numbers below x is $\sim cx/\log x$ for some constant $1.311 < c < 1.693$ (see [18, 20]). Thus, it seems reasonable to expect that the number of primes $p \leqslant x$, for which $p-1$ is practical, is asymptotic to $Cx/(\log x)^2$ for some positive constant C. Theorem 1 gives lower and upper bounds for the count of such primes. More generally, given an integer a, we explore the practicality of shifted primes $p-a$. Let

$$P_a(x) := |\{p \leqslant x : p \text{ prime},\ p-a \text{ practical}\}|.$$

Since all practical numbers greater than one are even, only the case of odd a is of interest.

Theorem 1. *Let a be an odd integer. Let $\beta > 2 + 2e\log 2 = 5.7683...$ and $\delta = 1 - (1+\log\log 2)/\log 2 = 0.086071...$ be the Erdős-Tenenbaum-Ford constant. We have*

$$\frac{x}{(\log x)^{\beta}} \ll P_a(x) \ll \frac{x}{(\log x)^{1+\delta}}.$$

We will prove a more general version of Theorem 1, which applies to other sequences besides the practical numbers. Given any arithmetic function θ, define the set $\mathcal{B}_\theta \subseteq \mathbb{N}$ as follows. Let $1 \in \mathcal{B}_\theta$ and let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \in \mathcal{B}_\theta$, with primes $p_1 < \cdots < p_k$, if and only if

$$p_i \leqslant \theta(p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}}), \quad (1 \leqslant i \leqslant k). \tag{1}$$

Sierpinski [12] and Stewart [14] found that the set of practical numbers is precisely the set $\mathcal{B}_\theta$ with $\theta(n) = \sigma(n)+1$, where $\sigma(n)$ denotes the sum of the positive divisors of n.

We shall write $\mathcal{B}_0$ to denote the set $\mathcal{B}_\theta$ with $\theta(n) = \max(2,n)$. Since $\max(2,n) \leqslant \sigma(n)+1$, the set $\mathcal{B}_0$ is a subset of the set of practical numbers.

Theorem 2. *Let β and δ be as in Theorem 1. Assume θ satisfies $\max(2,n) \leqslant \theta(n) \leqslant n\exp\big(A(\log\log 3n)^{3/(2\delta)}\big)$ for some constant A. Let $\mathcal{S}$ be any set with $\mathcal{B}_0 \subseteq \mathcal{S} \subseteq \mathcal{B}_\theta$ and write $S_a(x) := |\{p \leqslant x : p \text{ prime},\ p-a \in \mathcal{S}\}|$. For any fixed odd integer a, we have*

$$\frac{x}{(\log x)^{\beta}} \ll S_a(x) \ll \frac{x}{(\log x)^{1+\delta}}.$$

These bounds remain valid under the weaker condition

$$\max(2,n) \leqslant \theta(n) \ll n\exp\big((\log n)^{o(1)}\big),$$

provided the exponent $1+\delta$ is replaced by $1+\delta-\varepsilon$ with any $\varepsilon > 0$.

Note that $3/(2\delta) = 17.427....$ Since $\sigma(n)+1 \ll n\log\log 2n$, Theorem 1 follows from Theorem 2.

Besides the practical numbers, we mention two more examples of sets $\mathcal{S}$ to which Theorem 2 applies. Given a fixed paramter $t \geqslant 2$, consider the set of positive integers whose ratios of consecutive divisors do not exceed t (see [10, 16, 18, 20]). These integers are called *t-dense* and are precisely the integers in the set $\mathcal{B}_\theta$ with $\theta(n) = tn$ (see [16, Lemma 2.2]).

Second, the *φ-practical* numbers, i.e., integers n such that the polynomial $X^n - 1$ has a divisor in $\mathbb{Z}[X]$ of every degree below n (see [9, 17]). Although this set is not equal to $\mathcal{B}_\theta$ for any θ, it satisfies $\mathcal{B}_0 \subset \mathcal{B}_{\theta_1} \subset \mathcal{S} \subset \mathcal{B}_{\theta_2}$, where $\theta_1(n) = n + 1$ and $\theta_2(n) = n + 2$.

The proof of the lower bound of Theorem 2 (see Proposition 1) rests on a Bombieri-Vinogradov type theorem (Lemma 1), due to Bombieri, Friedlander and Iwaniec, for primes in arithmetic progressions to large moduli [1]. To control for repeated counting of the same integer, we use a result by Norton [8] for the frequency of large values of the divisor function $\tau(n)$ (Lemma 2). The upper bound in Theorem 2 (see Proposition 2) follows from an upper bound by Ford [4] for the number of shifted primes with a divisor from an interval.

2. The Lower Bound

Proposition 1. *Let a be an odd integer and $\beta > 2 + 2e \log 2 = 5.768...$ The number of primes $p \leqslant x$, for which $p - a \in \mathcal{B}_0$, is*

$$\gg \frac{x}{(\log x)^\beta}.$$

We write $\log_2 x = \log\log x$ and $\log_3 x = \log\log\log x$.

Lemma 1 (Bombieri, Friedlander, Iwaniec [1]). *Let $a \neq 0$ be an integer and $A > 0$, $2 \leqslant Q \leqslant x^{3/4}$ be reals. Let $\mathcal{Q}$ be the set of all integers q, relatively prime to a, from an interval $Q' < q \leqslant Q$. Then*

$$\sum_{q \in \mathcal{Q}} \left| \pi(x; q, a) - \frac{\pi(x)}{\varphi(q)} \right| \leqslant \left\{ K \left(\theta - \frac{1}{2} \right)^2 x L^{-1} + O\left((x L^{-3} (\log_2 x)^2 \right) \right\} \sum_{q \in \mathcal{Q}} \frac{1}{\varphi(q)} + O(x L^{-A}),$$

where $\theta = \log Q / \log x$, $L = \log x$, K is absolute, the first implied constant in O depends only on A, and the second one on a and A.

Lemma 2 (Norton [8]). *Let $\tau(n)$ denote the number of positive divisors of the integer n. For $x \geqslant x_0$, $y \geqslant \log_2 x$,*

$$\#\{n \leqslant x : \tau(n) \geqslant 2^y\} \leqslant \frac{x}{\log x} \exp\left\{ -y \log y + y(\log_3 x + 1) + O\left(\frac{y}{\log_2 x} \right) \right\}.$$

The following observation follows immediately from the definition of the set $\mathcal{B}_0$.

Lemma 3. *If $n \in \mathcal{B}_0$ and $1 \leqslant k \leqslant n$, then $nk \in \mathcal{B}_0$.*

Lemma 4. *Let a be an odd integer. There is a constant $\kappa = \kappa(a) > 1$, such that the number of integers $n \in \mathcal{B}_0 \cap [x, \kappa x]$, which are relatively prime to a, is*

$$\gg_a \frac{x}{\log x}.$$

Proof. Define

$$\mathcal{D}_{a,h}(x) = \{m = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \leqslant x2^{-h} : P^+(|a|) < p_i \leqslant 2^h p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}},\ 1 \leqslant i \leqslant k\},$$

where $p_1 < p_2 < \cdots < p_k$ are the prime factors of m. Here $P^+(n)$ denotes the largest prime factor of n and $P^+(1) = 1$. For each $m \in \mathcal{D}_{a,h}(x)$, we have $m2^h \in \mathcal{B}_0 \cap [1, x]$ and $\gcd(a, m2^h) = 1$, since a is odd and $P^+(|a|) < p_1$. The result now follows from an estimate due to Saias [11, Theorem 1], which shows that

$$|\mathcal{D}_{a,h}(x)| \asymp_{a,h} \frac{x}{\log x},$$

provided $h > h_0$ and $x > x_0$. □

Proof of Proposition 1. Let $\beta > 2 + 2e \log 2$ and put $c = (\beta - 2)/2 > e \log 2$. Let $\kappa > 1$ be the constant from Lemma 4. Define

$$\mathcal{Q}_a = \{q \in (\sqrt{x}, \kappa\sqrt{x}] : q \in \mathcal{B}_0,\ \gcd(q, a) = 1,\ \tau(q) \leqslant (\log x)^c\}.$$

By Lemma 2, the number of $n \leqslant \kappa\sqrt{x}$ with $\tau(n) > (\log x)^c$ is $o(\sqrt{x}/\log x)$. Together with Lemma 4, this implies

$$|\mathcal{Q}_a| \gg_a \frac{\sqrt{x}}{\log x}.$$

By Lemma 1, we have

$$\sum_{q \in \mathcal{Q}_a} \left| \pi(x; q, a) - \frac{\pi(x)}{\varphi(q)} \right| \ll_a \frac{x(\log_2 x)^3}{(\log x)^3},$$

since $\varphi(q) \gg q/\log_2 q$. It follows that the number of $q \in \mathcal{Q}_a$ for which

$$\left| \pi(x; q, a) - \frac{\pi(x)}{\varphi(q)} \right| > \frac{\pi(x)}{2\varphi(q)} \gg \frac{x}{q \log x} \asymp \frac{\sqrt{x}}{\log x},$$

is

$$\ll_a \frac{\sqrt{x}(\log_2 x)^3}{(\log x)^2}.$$

Consequently,

$$|\mathcal{Q}_a'| \gg_a \frac{\sqrt{x}}{\log x},$$

where

$$\mathcal{Q}_a' = \left\{ q \in \mathcal{Q}_a : \left| \pi(x;q,a) - \frac{\pi(x)}{\varphi(q)} \right| \leqslant \frac{\pi(x)}{2\varphi(q)} \right\}.$$

For each $q \in \mathcal{Q}_a'$, the number of k such that $qk + a \leqslant x$ is prime, is

$$\pi(x;q,a) \geqslant \frac{\pi(x)}{2\varphi(q)} \gg \frac{x}{q \log x} \asymp \frac{\sqrt{x}}{\log x}.$$

The number of $k \leqslant \sqrt{x}$ for which $\tau(k) > (\log x)^c$ is $o(\sqrt{x}/\log x)$, by Lemma 2. Hence there are $\gg \sqrt{x}/\log x$ values of k for which $qk + a \leqslant x$ is prime and $\tau(k) \leqslant (\log x)^c$. Since $k \leqslant q$ and $q \in \mathcal{B}_0$, Lemma 3 shows that $qk \in \mathcal{B}_0$. Thus the number of pairs (q,k) such that $q \in \mathcal{Q}_a'$, $\tau(k) \leqslant (\log x)^c$ and $qk + a \leqslant x$ is prime, is

$$\gg \left(\frac{\sqrt{x}}{\log x} \right) \left(\frac{\sqrt{x}}{\log x} \right) = \frac{x}{(\log x)^2}.$$

For each of these pairs (q,k), we have

$$\tau(qk) \leqslant \tau(q)\tau(k) \leqslant (\log x)^{2c},$$

which implies that there are at most $(\log x)^{2c}$ pairs $(\tilde{q}, \tilde{k})$, such that $qk = \tilde{q}\tilde{k}$. It follows that the number of distinct elements of $\mathcal{B}_0$ of the form qk, for which $qk + a$ is prime, is

$$\gg \frac{x}{(\log x)^{2+2c}} = \frac{x}{(\log x)^{\beta}}.$$

$\square$

3. The Upper Bound

Proposition 2. *Let $a \neq 0$ be fixed and $\delta = 1 - (1 + \log\log 2)/\log 2 = 0.086071\ldots$. Let θ be an arithmetic function and $\theta(n)/n \leqslant h(n)$ for some non-decreasing h with $2 \leqslant h(x) \ll \sqrt{x}$. The number of primes $p \leqslant x$ such that $p - a \in \mathcal{B}_\theta$ is*

$$\ll \frac{x}{\log x} \left(\frac{2 \log h(x)}{\log x} \right)^{\delta} \left(\log \left(\frac{\log x}{\log h(x)} \right) \right)^{-3/2}.$$

The upper bounds in Theorem 2 follow from taking $h(n) = A \exp\big((\log n)^{o(1)}\big)$ and $h(n) = \exp\big(A(\log\log 3n)^{3/(2\delta)}\big)$.

Proof. Let $\tilde{\theta}(n) = nh(n)$ such that $\theta(n) \leqslant \tilde{\theta}(n)$ for all $n \geqslant 1$. Write $\mathcal{B}_1$ to denote $\mathcal{B}_{\tilde{\theta}}$. Since $h(n)$ is non-decreasing, Theorem 4 of [19] shows that $\mathcal{B}_1$ is precisely the set of integers n whose divisors $1 = d_1 < d_2 < \ldots < d_{\tau(n)} = n$ satisfy

$$d_{j+1} \leqslant \tilde{\theta}(d_j) \qquad (1 \leqslant j < \tau(n)).$$

We have $d_{j+1} \leqslant \tilde{\theta}(d_j) = d_j h(d_j) \leqslant d_j h(x)$ for every $n \in [1, x] \cap \mathcal{B}_1$, $1 \leqslant j < \tau(n)$. It follows that every $n \in (\sqrt{x}, x] \cap \mathcal{B}_1$ has a divisor in the interval $(\sqrt{x}, \sqrt{x}h(x)]$. We get

$$\begin{aligned}
&|\{p \leqslant x : p \text{ prime},\ p - a \in \mathcal{B}_\theta\}| \\
&\leqslant |\{p \leqslant x : p \text{ prime},\ p - a \in \mathcal{B}_1\}| \\
&= |\{n \leqslant x - a : n + a \text{ prime},\ n \in \mathcal{B}_1\}| \\
&\leqslant \sqrt{x} + |\{n \leqslant x + |a| : n + a \text{ prime},\ n \text{ has a divisor in } (\sqrt{x}, \sqrt{x}h(x)]\}| \\
&\ll \frac{x}{\log x} \left(\frac{2 \log h(x)}{\log x} \right)^{\delta} \left(\log \left(\frac{\log x}{\log h(x)} \right) \right)^{-3/2},
\end{aligned}$$

where the last estimate follows from combining Theorem 1(v) and Theorem 6 of Ford [4]. □

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