A NOTE ON THE CONVERSE OF WOLSTENHOLME'S THEOREM

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Abstract

Given a prime p and a positive integer m satisfying a certain inequality, the converse of Wolstenholme's Theorem is shown to hold for the product mp^k where k is any positive integer, generalizing a result by Helou and Terjanian.

1. Jones' Conjecture

For $n \in \mathbb{N}$, let $w_n = \binom{2n-1}{n-1} = \frac{1}{2}\binom{2n}{n}$. In 1862, Wolstenholme [6] proved the following:

Theorem 1 (Wolstenholme's Theorem). *If $p \geq 5$ is prime, then*

$$w_p \equiv 1 \pmod{p^3}.$$

James P. Jones conjectures that no other solutions exist.

Conjecture 1 (Jones' conjecture). *The integer $n \geq 5$ is prime if and only if*

$$w_n \equiv 1 \pmod{n^3}.$$

Jones' conjecture is true for even integers and for powers of primes up to 10^9 ([2], [5]). Computations also show Jones' conjecture holds for all integers not exceeding 10^9. See the expository paper [3] for further background.

Recall that the p-adic valuation of an integer is the exponent of the highest power of the prime p that divides the integer.

Theorem 2 (Kummer's Theorem for binomial coefficients [4]). *Given integers $n \geq m \geq 0$ and a prime p, the p-adic valuation of $\binom{n}{m}$ is equal to the number of carries when m is added to $n-m$ in base p.*

Notice the p-adic valuation of w_n and $2w_n$ are equal when $p \neq 2$. The integers $2w_n = \binom{2n}{n}$ are known as the *central binomial coefficients*.

Proposition 1. *For any odd prime p and $m \in \mathbb{N}$ such that*

$$p^a < m < p^{a+1} < 2m$$

for some integer $a \geq 0$, the product $n = mp^b$ satisfies $w_n \not\equiv 1 \pmod{n}$ for any $b \in \mathbb{N}$ and therefore Jones' conjecture holds for n.

Proof. Let $m = n_0 + n_1 p + n_2 p^2 + \cdots + n_k p^k$ be the p-adic expansion of m. This means for each coefficient n_i, $0 \leq n_i \leq p-1$ with $n_k \neq 0$. Assume there exists an integer $a \geq 0$ such that

$$p^a < m < p^{a+1} < 2m. \tag{1}$$

Then

$$p^a < n_0 + n_1 p + n_2 p^2 + \cdots + n_k p^k < p^{a+1},$$

so $k < a+1$. Also, by the definition of a p-adic expansion, $k \geq a$. To see this explicitly, the formula for the sum of a geometric series gives us

$$\begin{aligned} n_0 + n_1 p + n_2 p^2 + \cdots + n_k p^k &\leq (p-1)(1 + p + p^2 + \cdots + p^k) \\ &= (p-1)\left(\frac{p^{k+1}-1}{p-1}\right) \\ &= p^{k+1} - 1. \end{aligned}$$

Therefore $p^a < p^{k+1} - 1$ and so $k \geq a$. Since $k < a+1$, $k = a$.

Now let

$$m_0 + m_1 p + m_2 p^2 + \cdots + m_l p^l$$

be the p-adic expansion of $2m$. By the same argument as before,

$$m_0 + m_1 p + m_2 p^2 + \cdots + m_l p^l \leq p^{l+1} - 1,$$

so by (1), $p^{a+1} < p^{l+1} - 1$ and so $l \geq a+1$. Therefore $l > k$, meaning there is at least one carry when adding m and $2m - m = m$ in base p. Since $w_m = \frac{1}{2}\binom{2m}{m}$ and $p \neq 2$, theorem 2 implies $p \mid w_m$. Finally, the coefficients in the p-adic expansion of m and $n = mp^b$, $b \in \mathbb{N}$, are the same, so $p \mid w_n$ and so $w_n \not\equiv 1 \pmod{p}$. Therefore $w_n \not\equiv 1 \pmod{n}$ and the proof is complete. □

Corollary 1 (Proposition 5, part 4 [2]). *For any odd prime p and $m \in \mathbb{N}$ such that*

$$m < p < 2m,$$

the product $n = mp^k$ satisfies $w_n \not\equiv 1 \pmod{n}$ for any $k \in \mathbb{N}$ and therefore Jones' conjecture holds for n.

Remark 1. If p and q are consecutive primes with $2 \neq q < p$, then Chebyshev's Theorem [1] (also known as Bertrand's postulate) implies

$$q < p < 2q.$$

Therefore by corollary 1, Jones' conjecture holds for the product of two consecutive primes.

Remark 2. For powers of a prime, proposition 1 clearly does not apply. In this case since $n = p^k$, adding n and n in base p gives us $2p^k$ and so there are no carries when $p \neq 2$. Therefore by theorem 2, $p \nmid w_n$.

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