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POLYNOMIALS RELATED TO POWERS OF THE DEDEKIND ETA FUNCTION

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Abstract

The vanishing properties of Fourier coefficients of integral powers of the Dedekind eta function correspond to the existence of integral roots of integer-valued polynomials $P_n(x)$ introduced by M. Newman. In this paper we study the derivatives of these polynomials. We obtain non-vanishing results at integral points. As an application we prove that integral roots are simple if the index n of the polynomial is equal to a prime power p^m or to $p^m + 1$. We obtain a formula for the derivative of $P_n(x)$ involving the polynomials of lower degree.

1. Introduction

In 1955, Newman [7] studied a family of polynomials $P_n(x)$ with remarkable properties. These polynomials are recursively defined by $P_0(x) := 1$ and

$$P_n(x) := \frac{x}{n} \sum_{k=1}^{n} \sigma(k)\, P_{n-k}(x). \tag{1}$$

Here $\sigma(m) := \sum_{d|m} d$ is the divisor sum. The polynomials P_n have degree n and are integer-valued functions.

Let $\eta(\tau)$ be the Dedekind eta function ([1], [4]):

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - q^n\right), \tag{2}$$

where $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$. One puts $q := e^{2\pi i\,\tau}$. Then

$$\sum_{n=0}^{\infty} P_n(z)\, q^n = \prod_{n=1}^{\infty} (1-q^n)^{-z}\,. \tag{3}$$

Let r be an integer. Then $P_n(-r)$ is essentially equal to the n-th Fourier coefficient of η^r (see also [12], introduction). The number of partitions [10] of n is given by

$$P_1(1) = 1,\; P_2(1) = 2,\; P_3(1) = 3,\; P_4(1) = 5, \ldots, P_{200}(1) = 3972999029388, \ldots$$

Euler already studied the values of the Fourier coefficients for $r = -1$, which are related to pentagonal numbers [8]. The first few are given by

$$P_1(-1) = -1,\; P_2(-1) = -1,\; P_3(-1) = 0,\; P_4(-1) = 0,\; P_5(-1) = 1.$$

In contrast to partition numbers, where r is negative, the Fourier coefficients for r positive have sign changes and can vanish. The study of these properties, especially the vanishing property is of special interest (see for example [13], [2] (page 94), [3]).

Let $\tau(n)$ be the Ramanujan tau-function [6, 11], the coefficients of the discriminant function Δ (see Section 2 for more details). Then the values of $P_n(x)$ at -24 are given by

$$\begin{aligned} P_0(-24) &= \tau(1) = 1, \\ P_1(-24) &= \tau(2) = -24, \\ P_2(-24) &= \tau(3) = 252. \end{aligned}$$

The roots of $P_n(x)$ dictate the vanishing of the Fourier coefficients of the corresponding powers of the Dedekind eta function. For example

$$P_{19}(x) = \frac{x}{19!}\;(x+1)(x+3)(x+4)(x+6)(x+8)(x+14)\,Q(x), \tag{4}$$

where $Q(x)$ is an irreducible polynomial over $\mathbb{Q}$. Hence only the 19-th Fourier coefficient of the zeroth, first, third, sixth, eighth, and fourteenth integral power of $\eta(\tau)$ vanishes. This implies that $\tau(20) \neq 0$. Actually the Lehmer conjecture [6] predicts that $\tau(n)$ never vanishes. It is known that the sum Σ_n and product Π_n of the roots of $\frac{n!}{x}\;P_n(x)$ are given by

$$\Sigma_n = \frac{3n(n-1)}{2}, \quad \Pi_n = (n-1)!\;\sigma(n). \tag{5}$$

Note that several roots as $-1, -2, -3, -4, -6, -8, -10, -14, -26$ occur frequently (see [3]) and precise criteria for the related Fourier coefficients exist [13]. Nevertheless up to the moment only $-5, -7$ and -15 have been identified as further integral roots. To make use of the explicit formula for Σ_n, Π_n to study the distribution of the roots, especially the integral roots, it is essential to know if all non-trivial roots have negative real part and their multiplicity. Note that the converse is not true. For example

$$(X+3)\,(X-1-3i)\,(X-1+3i) = X^3 + X^2 + 4X + 30.$$

Numerical calculations show that the polynomials $\frac{n!}{x}\, P_n(x)$ for $n \leq 1000$ are Hurwitz polynomials, which supports the assumption that all roots have negative real part.

In this paper we study the multiplicities of possible integral roots of $P_n(x)$. Our main ingredient is a new formula for the derivatives of $P_n(x)$. Here we obtain results for specific indices n and general x. Additionally we study the cases

$$x = -1, -2, \ldots, -10 \text{ and } x = -24.$$

For example we have $P_n'(-1) \notin \mathbb{Z}$ for $2 \leq n \leq 10^7$.

2. Results

The Dedekind eta function $\eta(\tau)$ is a modular form of weight $1/2$ for $SL_2(\mathbb{Z})$. Due to Euler, the Fourier coefficients are easy to calculate. The same is true for $\eta(\tau)^3$ due to Jacobi (see also [9], [5]) and

$$\prod_{n=1}^{\infty} (1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n \, q^{\frac{3n^2+n}{2}}, \tag{6}$$

$$\prod_{n=1}^{\infty} (1-q^n)^3 = \sum_{n=0}^{\infty} (-1)^n \, (2n+1) \, q^{\frac{n^2+n}{2}}. \tag{7}$$

Here $(3n^2 - n)/2$ are the pentagonal numbers. Note that $\eta(\tau)$ and $\eta(\tau)^3$ are the only superlacunary odd integral η powers.

For general powers the situation is much more complicated. Let $\Delta(\tau)$ be the discriminant function with Fourier coefficients $\tau(n)$:

$$\Delta(\tau) := \eta(\tau)^{24} = \sum_{n=1}^{\infty} \tau(n)\, q^n. \tag{8}$$

This is the unique newform of weight 12 for $SL_2(\mathbb{Z})$. The $\tau(n)$ have many remarkable properties. They are multiplicative with an additional law for $\tau(p^m)$ (p prime and $m \in \mathbb{N}$) since Δ is a Hecke eigenform. Further Δ satisfies the Ramanujan conjecture proven by Deligne, which implies a precise estimation on the growth of the numbers $\tau(n)$. It is also known that the Fourier coefficients of Δ have infinitely many sign changes. But it is still an open problem (Lehmer conjecture) if any coefficient vanishes. Lehmer proved that the smallest n, for which $\tau(n)$ is zero, has to be a prime [6].

Let $n \in \mathbb{N}$. We note that

$$\frac{n!}{x} P_n(x) = \sum_{k=0}^{n-1} a_{n,k} \, x^k \in \mathbb{Z}[x]$$

is a normalized polynomial of degree $n-1$ and that all its coefficients $a_{n,k}$ are positive. It is easy to show that $a_{n,0} = (n-1)! \, \sigma(n)$ and $a_{n,n-2} = 3\frac{n(n-1)}{2}$. See also [7]. These simple formulas are misleading, since there is no explicit closed formula known for $a_{n,k}$ in general.

Our first result on the derivatives of $P_n(x)$ is the following one.

Theorem 1. *Let $n = p^m$, where p is a prime and $m \in \mathbb{N}$. Then*

$$P_n'(x_0) \in \mathbb{Q} \setminus \mathbb{Z} \tag{9}$$

for any integer x_0.

In the proof we use the property that for these specific n we have that $\sigma(n)$ and n are coprime. We hope to extend our theorem to all n with this property. On the other hand, our theorem is true for all integers x_0, which has not been expected. Let n be a prime power that is an element of the set

$$\{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, \ldots\}, \tag{10}$$

then $P_n'(x_0)$ is not an integer. This implies:

Corollary 1. *Suppose $n = p^m$, where p is a prime and $m \in \mathbb{N}$. Then all integral roots of $P_n(x)$ are simple.*

In the course of the proof we obtain the following useful formula, expressing the derivatives of $P_n(x)$ in terms of linear combinations of $P_m(x)$ for $m \leq n-1$:

$$P_n'(x) = \sum_{k=1}^{n} \frac{\sigma(k)}{k} \, P_{n-k}(x) \qquad (n \in \mathbb{N}). \tag{11}$$

Inverting this formula leads to:

Corollary 2. *There is a sequence of b_k, $k \geq -1$, so that we can express $P_n(x)$ as a linear combination of derivatives $P_m'(x)$, where*

$$P_n(x) = \sum_{k=-1}^{n-1} b_k P_{n-k}'(x).$$

The coefficients are obtained recursively from $b_{-1} = 1$. For $k \geq 0$:

$$b_k = -\sum_{m=-1}^{k-1} b_m \, \frac{\sigma(k+1-m)}{k+1-m}. \tag{12}$$

If we further let $n = p^m + 1$, then we do not get the full result of Theorem 1, but still an integral result for the derivative of the related polynomial

$$F_n(x) := \frac{n}{x} P_n(x)$$

at integral points x_0. This leads to the simplicity of integral roots at $n = p^m + 1$ independent of x_0.

Theorem 2. *Let $n = p^m + 1$, where p is a prime and $m \in \mathbb{N}$. Then*

$$F_n'(x_0) \in \mathbb{Q} \setminus \mathbb{Z} \tag{13}$$

for any integer x_0.

This shows immediately:

Corollary 3. *Suppose $n = p^m + 1$, where p is a prime and $m \in \mathbb{N}$. Then all integral roots of $P_n(x)$ are simple.*

This adds the following n to the above list (10):

$$6, 10, 12, 14, 18, 20, 24, 26, 28, 30, 33, \ldots, \tag{14}$$

Hence the first index not covered by these two corollaries is $n = 15$.

3. Proofs of Theorem 1 and Theorem 2

Before we prove the two theorems, we list the derivatives of $P_n(x)$ for $n \leq 10$. For a list of the polynomials for $n \leq 20$ we refer the reader to [3].

n	$P_n'(x)$
0	0
1	1
2	$(2\,x+3)\,/2$
3	$\left(3\,x^2+18\,x+8\right)/3!$
4	$\left(4\,x^3+54\,x^2+118\,x+42\right)/4!$
5	$\left(5\,x^4+120\,x^3+645\,x^2+900\,x+144\right)/5!$
6	$\left(6\,x^5+225\,x^4+2260\,x^3+7425\,x^2+6788\,x+1440\right)/6!$
7	$\left(7\,x^6+378\,x^5+6125\,x^4+37380\,x^3+84882\,x^2+61824\,x+5760\right)/7!$
8	$(8\,x^7+588\,x^6+14028\,x^5+138600\,x^4+591556\,x^3+1020348\,x^2+586584\,x$ $+\,75600)/8!$
9	$(9\,x^8+864\,x^7+28518\,x^6+417312\,x^5+2896845\,x^4+9365328\,x^3$ $+\,13006788\,x^2+6064416\,x+524160)/9!$
10	$(10\,x^9+1215\,x^8+53040\,x^7+1080450\,x^6+11145078\,x^5+58723875\,x^4$ $+\,152199680\,x^3+173321100\,x^2+72581472\,x+6531840)/10!$

It is well-known and not difficult to show that the derivative of a Hurwitz polynomial is again a Hurwitz polynomial. Hence we expect that all roots of $P_n'(x)$ have negative real part. Similar to the $P_n(x)$, the $P_n'(x)$ can also have non-real roots. For example for $n = 21$.

Proof of Theorem 1. We deduce from [7] that

$$\sum_{n=0}^{\infty} P_n(z)\, q^n = \prod_{n=1}^{\infty} (1-q^n)^{-z} \qquad (z \in \mathbb{C}). \tag{15}$$

See also Section 2, where some further remarks are given. The proof is based on examining the logarithmic derivative of (15), certain grouping of terms and using the uniqueness of the coefficients of the involved power series in q. We have analytic functions on the upper half space, since $|q| < 1$. It is well known that

$$\prod_{n=1}^{\infty} (1-q^n) \quad = \quad \exp\left(- \sum_{n=1}^{\infty} \sigma(n)\, \frac{q^n}{n} \right).$$

Note the logarithmic derivative of the Dedekind eta function is essentially the holomorphic Eisenstein series of weight 2. Putting both sides of our equation to the power of $-z$ leads to

$$\prod_{n=1}^{\infty} (1-q^n)^{-z} = \exp\left(z \sum_{n=1}^{\infty} \sigma(n)\, \frac{q^n}{n} \right).$$

Hence we obtain the useful identity

$$\sum_{n=0}^{\infty} P_n(z)\, q^n = \exp\left(z \sum_{n=1}^{\infty} \sigma(n)\, \frac{q^n}{n} \right). \tag{16}$$

Now we differentiate both sides of (16) with respect to z. The right side of the formula reproduces the power series multiplied with

$$\sum_{n=1}^{\infty} \sigma(n)\, \frac{q^n}{n}.$$

After comparing the Fourier coefficients with respect q^n of both sides we obtain the following identity:

$$P_n'(x) = \sum_{k=1}^{n} \frac{\sigma(k)}{k}\; P_{n-k}(x) \qquad (n \in \mathbb{N}). \tag{17}$$

It expresses the derivative of the polynomial $P_n(x)$ in terms of $P_{n-k}(x)$ and $\sigma(k)/k$ instead of $\sigma(k)$ in the recursion formula of $P_n(x)$. To examine the possible denominator of the right side, we write down the sum explicitly:

$$P_n'(x) = P_{n-1}(x) + \frac{\sigma(2)}{2} P_{n-2}(x) + \ldots + \frac{\sigma(n-1)}{n-1}\, x + \frac{\sigma(n)}{n}. \tag{18}$$

Recall that the polynomial $P_n(x) \in \mathbb{Q}[x]$ is an integer-valued function, hence possible denominators can only be obtained by the quotients $\sigma(k)/k$. Let $n = p^m$. Then the greatest common divisor of $\sigma(n)$ and n is equal to 1 ($\gcd(\sigma(n), n) = 1$). Further we note that $n = p^m$ is exactly the highest power of p appearing in the denominator of $P_n'(x_0)$, where $x_0 \in \mathbb{Z}$. Hence $P_n'(x_0)$ is a rational number, but not an integer. $\square$

The same argument is expected to work if we assume that $\sigma(n)$ and n are coprime. But if n splits into different coprime numbers, the denominator could be eliminated by other terms in the sum (18). We put this as an open problem.

Proof of Theorem 2. For all natural numbers n we have:

$$F_n(x) = \sum_{k=1}^{n} \sigma(k)\; P_{n-k}(x). \tag{19}$$

Taking the derivative with respect to x leads to

$$\begin{aligned} F_n'(x) &= \sum_{k=1}^{n-1} \sigma(k)\; P_{n-k}'(x) \\ &= P_{n-1}'(x) + \sigma(2) P_{n-2}'(x) + \ldots + \sigma(n-1)\, P_1'(x). \end{aligned}$$

Note that $P_0'(x) = 0$ and $P_1'(x) = 1$. Using the formula (17) leads to

$$F_n'(x) = \sum_{k=1}^{n-1} \sum_{l=1}^{n-k} \sigma(k)\; \frac{\sigma(l)}{l}\, P_{n-k-l}(x). \tag{20}$$

Then the claim follows from taking special values for k and l. We observe that for $k = 1$ and $l = n-1$, the summand $\sigma(n-1)/\,(n-1)$ has the denominator p^m, which dominates the denominator of $F_n'(x_0)$ for $x_0 \in \mathbb{Z}$. Here it is essential that $P_n(x)$ is an integer-valued function. $\square$

4. On the Derivatives of $P_n(x)$ at $x = -1$

We have proven that $P_n(x)$ does not have multiple zeros for n or $n-1$ equal to prime powers at integral points. It would definitely be interesting to understand what happens for general n. Fixing the argument $x \in \mathbb{Z}$ and varying the n gives some new insight.

In this section we study $P_n(x_0)$ at $x_0 = -1$. Our Theorems already show that $P_n'(-1) \neq 0$ for $n \leq 14$, if n is a prime power. Hence -1 is a single root for $n \leq 14$. A calculation shows that

$$P_{15}'(-1) = \frac{12563}{10920} = \frac{17 \cdot 739}{2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13}.$$

Our interest is the multiplicity of (-1) as a root of $P_n(x)$. The n's label in (6) the Fourier coefficients of

$$\begin{aligned}\prod_{m=1}^{\infty}(1-q^m) &= 1+\sum_{m=1}^{\infty}(-1)^m q^{(3m+1)m/2}=\sum_{n=0}^{\infty}P_n(-1)\ q^n \\ &= 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+\dots\ .\end{aligned}$$

Let $P_n(-1)=1$ then $P_n(x)$ is forced to have a root in $(-1,0)$. Further investigations in the values of the derivatives of $P_n(x)$ at -1 lead to:

Theorem 3. *If $n \leq N = 10^7$, then $P_n'(-1) \neq 0$. Refined calculations show that $P_n'(-1) \notin \mathbb{Z}$ for $2 \leq n \leq N$.*

If $x_0 \in \{-2,-3,\dots,-10\}$, then $P_n'(x_0) \notin \mathbb{Z}$ for $2 \leq n \leq 10^4$. We also checked the special case $x_0=-24$. We have that $P_n'(-24)$ is not an integer for $2 \leq n \leq 10^5$.

For further studies and for the convenience of the reader we record the explicit factorization of the values $P_n'(-1)$ for $n \leq 20$. They reveal sign changes. Further note that 2 and 3 do not always appear in the denominator.

n	1	2	3	4	5
$P_n'(-1)$	1	$\frac{1}{2}$	$-\frac{7}{2\cdot3}$	$-\frac{13}{2^2\cdot3}$	$-\frac{113}{2^2\cdot3\cdot5}$
$P_{n+5}'(-1)$	$\frac{1}{2^2\cdot5}$	$-\frac{3\cdot13}{2\cdot5\cdot7}$	$\frac{179}{2^3\cdot3\cdot7}$	$\frac{5\cdot13^2}{2^3\cdot3^2\cdot7}$	$\frac{73}{2^3\cdot3^2}$
$P_{n+10}'(-1)$	$\frac{29\cdot109}{2^2\cdot3^2\cdot5\cdot11}$	$\frac{2\cdot1031}{3\cdot5\cdot7\cdot11}$	$\frac{1811}{2^3\cdot3\cdot11\cdot13}$	$-\frac{997}{2\cdot3^2\cdot7\cdot13}$	$\frac{17\cdot739}{2^3\cdot3\cdot5\cdot7\cdot13}$
$P_{n+15}'(-1)$	$\frac{-83\cdot1063}{2^4\cdot3^2\cdot5\cdot7\cdot11}$	$\frac{-853}{2^4\cdot3\cdot17}$	$\frac{-232811}{2^4\cdot3\cdot11\cdot13\cdot17}$	$\frac{-23\cdot1201}{2^2\cdot3\cdot7\cdot17\cdot19}$	$\frac{-5\cdot7\cdot257}{2^3\cdot3\cdot13\cdot19}$

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