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ON OZANAM'S RULE

Adam Krause
Department of Mathematics, University of Rochester, Rochester, New York
akrause5@ur.rochester.edu

Howard Skogman
Department of Mathematics, SUNY Brockport, Brockport, New York
hskogman@brockport.edu

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Abstract

In his book *Récréations Mathématiques et Physiques* (1694), Jacques Ozanam gave a test to determine whether a given integer is triangular or pentagonal. Jean-Ètienne Montucla revised and appended this work, and included a generalization of the test proposed by Ozanam to all polygonal numbers. We show both statements are false in general. In fact, we show this test fails for all polygons with at least five sides at a specific rate depending on the number of sides. We also give a corrected version of the test that accurately detects polygonal numbers.

1. Introduction

The study of polygonal numbers is one of the oldest in number theory and all of mathematics. Given an integer $m > 2$, the m-gonal numbers are the sequence of positive integers $p^m(k)$ for $k \geq 1$, giving the number of points in an m-sided polygon with k points per side. In particular, the sequence $p^m(k)$ may be described as $p^m(1) = 1$, $p^m(2) = m$, and all of the second differences of the sequence are equal to $m - 2$. The method of finite differences gives

$$p^m(k) = 1 * \binom{k-1}{0} + (m-1) * \binom{k-1}{1} + (m-2) * \binom{k-1}{2}.$$

Jacques Ozanam (1640 - 1718) was a french mathematician who is most remembered for a number of books on mathematics including *Cours de Mathématiques* (1693), and *Dictionaire Mathématiques* (1691) (for more information on Ozanam see [1]). Perhaps his most influential work was *Récréations Mathématiques et Physiques* (1694) [4]. *Récréations* was a general mathematics and science textbook covering very specific material followed by applicable problems. Some of the topics covered include arithmetic, geometry, physics, optics, pyrotechnics, construction of sundials and other simple clocks, and mechanics (including a design for a precursor to the bicycle). The book was very popular, and was updated numerous times over the next 150 years. The first English translation appeared in 1708 [5]. In 1778 ([6]) *Récréations* was thoroughly revised and appended by Jean-Ètienne Montucla (1725 - 1799) (author of *Histoire de Mathématiques*). Montucla's version was revised and translated into English by Charles Hutton in 1803 [8]. Hutton's version was again thoroughly revised and appended (in English) by Edward Riddle in 1844 [7].

In the first chapter of the 1694 edition of the text (*Problèmes d'Arithmétique*), Ozanam discusses polygonal numbers and gives the following two rules for determining whether a given integer is triangular or pentagonal.

> *To determine if a given number is triangular, multiply it by* 8 *and add* 1*, if the sum is a square, the given number is triangular.*[1] [4, pp. 16-17]
>
> *. . . pentagonal numbers have the property that if they are multiplied by* 24 *and* 1 *is added to the sum, the sum is a square, this may be used to determine if a given number is pentagonal.*[2] [4, p. 20]

In the 1778 edition of *Récréations*, Montucla generalizes Ozanam's rules for how to determine whether a given number is a polygonal number. In this version the rule has become:

> *Multiply the given number, the number of angles minus two, and* 8*, then add the square of the number of angles minus* 4 *to the product, if the result is a square, the number is a polygonal number of the given type.*[3][6](p. 40)

This statement may be equivalently stated as:

Ozanam's (or **Montucla's**) **Rule**: *If n and m are positive integers and $8n(m-2)+(m-4)^2$ is the square of an integer, then n is an m-gonal number.*

Note that the result is repeated and appears as problem 1 in the later editions by Hutton ([8] p. 38) and Riddle ([7] p. 22).

The authors became aware of this statement from Tattersall's text [3] where it appears in the problems for Section 1.1 (problems 27 and 28), and it is called

[1] *Pour connoître si un nombre proposé est Triangulaire, il le faut multiplier par* 8*, & ajoûter* 1 *au produit, car si la somme a sa Racine quarrée, le nombre proposé sera triangulaire.*

[2] *...sont appellez Pentagones, dont la proprieté est telle que chacun étant multiplié par* 24*, & le produit étant augmenté de l'unité, la somme est un nombre quarré, ce qui sert pour connoître quand un nombre proposé est Pentagone.*

[3] *Multipliez par* 8 *le nombre des angles du polygone diminué de* 2*, & par ce premier produit multipliez le nombre proposé, & enfin, à ce nouveau produit ajoutez le quarré du nombre égal à celui des angles du polygone diminué de* 4*; si la somme est un quarré parfait, le nombre proposé est un polygone de l'espece déterminée.*

Ozanam's Rule. The second author assigned these problems as homework in an introductory number theory course in which the first author was a student. The second author (and apparently Tattersall) misread Ozanam's Rule as its converse, however the first author and others tried to prove it as stated. These attempted proofs hit a gap when they required a certain quantity to be an integer only for polygonal numbers. The first author wrote computer programs to check the statement, and found that the test regularly detected numbers which were not polygonal for polygons with more than four sides. In addition, the rate at which the test succeeded in detecting polygonal numbers seemed to be a power of $1/2$ depending on the number of sides in the polygon. The authors were then able to confirm these calculations resulting in the following revision to this rule.

Theorem 1. *The converse of Ozanam's Rule is true, however Ozanam's Rule is only true for triangular and square numbers. Ozanam's Rule is false for all polygons with $m \geq 5$ sides, and fails with a specified frequency. Let c_m be the number of square roots of m^2 modulo $8(m-2)$ that are distinct modulo $2(m-2)$. Let $\mathcal{O}_m$ be the set of positive integers n such that $8n(m-2)+(m-4)^2$ is a square, written as an increasing list. Given any c_m consecutive values in $\mathcal{O}_m$, only one is an m-gonal number. In particular for all $m > 4$, $c_m = 2^t$ for some $t \geq 1$.*

By the proof of Theorem 1, we have the following corollary.

Corollary 1. *(Corrected version of Ozanam's Rule). Let n and m be positive integers. We have $8n(m-2)+(m-4)^2 = c^2$ for $c \in \mathbb{Z}^+$ with $c \equiv m \bmod 2(m-2)$, if and only if n is an m-gonal number.*

Before giving the proofs, we give a couple of examples to illustrate the result.

Example 1. *($m = 5$, Pentagonal Numbers).* The equation $24n+1 = c^2$ has integer solutions n for each $c \equiv 1$ or $5 \bmod 6$. If $c \equiv 5 \bmod 6$, i.e., $c = 5+6k$ for some $k \in \mathbb{Z}^+$, then plugging in for c, and solving for n we get

$$n_{k-1} = \frac{3k^2-k}{2},$$

which is exactly the formula for the k-th pentagonal number.

If $c \equiv 1 \bmod 6$, i.e., $c = 1+6k$ for some $k \in \mathbb{Z}^+$, then plugging in and solving for n we get

$$n_k = \frac{3k^2+k}{2}$$

which gives the sequence $\{2, 7, 15, 26, \ldots\}$, no element of which is pentagonal. For example, $24 * 2 + 1 = 49 = 7^2$, but 2 is certainly not pentagonal. In fact, these numbers are called *generalized pentagonal numbers* (see OEIS A001318 [2]). This sequence arises by evaluating the formula for the k-th pentagonal number at all integers, instead of only positive integers. So for $m = 5$, Ozanam's Rule detects the generalized pentagonal numbers instead of the standard pentagonal numbers.

Example 2. *($m = 14$, Tetradecahedron/Decatetrahedron-al numbers).* The equation $96n + 10^2 = c^2$ has integer solutions n whenever $c \equiv 2, 10, 14$, or $22 \bmod 24$. If $c \equiv 14 \bmod 24$ we get the 14-gonal numbers. Otherwise we get different sequences, all of which have second differences equal to $m - 2 = 12$. However the first two values in these sequence are not 1 and m.

We include the sequence one gets for each possible root and note the first and second differences for each sequence.

Root	Sequence	First Differences	Second Differences
2	$6, 25, 56, 99, 154, ...$	$19, 32, 43, 55, ...$	$12, 12, 12, ...$
10	$11, 34, 69, 116, 175, ...$	$23, 35, 47, 59, ...$	$12, 12, 12, ...$
14	$1, 14, 39, 76, 125, ...$	$13, 25, 37, 49, ...$	$12, 12, 12, ...$
22	$4, 21, 50, 91, 144, ...$	$17, 29, 41, 53, ...$	$12, 12, 12, ...$

Note that the sequence corresponding to the root 14 is the 14-gonal numbers, and the sequence for the root 10 (or $-14 \bmod 24$) gives the rest of the *generalized* 14-*gonal numbers* [2](A195818). The other two sequences do not seem to have a natural interpretation in this context.

2. Proofs

For simplicity, we assume congruence class representatives modulo n are taken in $\{0, 1, 2 \ldots n-1\}$ throughout the following argument.

Proof. To verify the converse, we note that we may rewrite the above formula for an m-gonal number as $p^m(k) = \frac{k}{2}(2 + (m-2)(k-1))$. Hence

$$8p^m(k)(m-2) + (m-4)^2 = 8\left(\frac{k}{2}(2 + (m-2)(k-1))\right)(m-2) + (m-4)^2 =$$

$$4k^2(m-2)^2 - 4k(m-2)(m-4) + (m-4)^2 = (2k(m-2) - (m-4))^2.$$

For the original implication in the case $m = 3$, the rule reduces down to $8n+1 = c^2$. This implies only that c is odd, say $c = 2k+1$. Thus

$$n = \frac{c^2 - 1}{8} = \frac{4k^2 + 4k + 1 - 1}{8} = \frac{k(k+1)}{2},$$

in other words n is the k-th triangular number.

The case when $m = 4$ is even more trivial (and probably why it is not even stated in the original version). In particular, the rule in this case states that if $16n$ is a square integer, then n is a square integer.

For $m > 4$, we find all values of n such that $8n(m-2)+(m-4)^2$ is a square, and determine exactly which such n are m-gonal. First note that

$$8n(m-2)+(m-4)^2 = c^2 \text{ for } c \in \mathbb{Z} \text{ if and only if } (m-4)^2 \equiv c^2 \bmod 8(m-2).$$

That is, $8n(m-2)+(m-4)^2$ is a square, say c^2 with $c \in \mathbb{Z}$, if and only if $(m-4)^2 \equiv c^2 \bmod 8(m-2)$. Also note that $(m-4)^2 \equiv m^2 \bmod 8(m-2)$.

Now assume c is a positive integer such that $c^2 \equiv (m-4)^2 \bmod 8(m-2)$. Further assume $c \equiv r \bmod 2(m-2)$ with $r^2 \equiv (m-4)^2 \bmod 8(m-2)$. Thus $c = r+2(m-2)k$ for some integer k. Consider the resulting sequence of corresponding values of n, for a fixed such congruence class r modulo $2(m-2)$.

$$n_k^{(r)} = \frac{c^2-(m-4)^2}{8(m-2)} = \frac{(r+2(m-2)k)^2-(m-4)^2}{8(m-2)}$$

and we have

$$n_k^{(r)} = \frac{r^2-(m-4)^2}{8(m-2)} + \frac{rk+(m-2)k^2}{2}. \tag{1}$$

Note that to insure $n_k^{(r)} > 0$, if $r \le m-4$ then $k > 0$, and if $r > (m-4)$ then $k \ge 0$.

We now argue that this sequence gives m-gonal numbers if and only if $r \equiv m \bmod 2(m-2)$. First note that when $r \equiv m \bmod 2(m-2)$,

$$n_{k-1}^{(m)} = \frac{m^2-(m-4)^2}{8(m-2)} + \frac{m(k-1)+(m-2)(k-1)^2}{2} = \frac{k}{2}\left(mk-m-2k+4\right),$$

which is the k-th m-gonal number $p^m(k)$.

Next we show that if $r_1 \not\equiv r_2 \bmod 2(m-2)$ then $n_k^{(r_1)} < n_k^{(r_2)} < n_{k+1}^{(r_1)}$. This implies that if $r \not\equiv m \bmod 2(m-2)$, then $n_k^{(r)}$ cannot be a m-gonal number. Note again that all congruence class representatives are taken to be least non-negative.

Assume $r_2 \not\equiv r_1 \bmod 2(m-2)$ and $r_2 > r_1$. Then we have (recall $k \ge 0$)

$$n_k^{(r_2)} - n_k^{(r_1)} = \frac{r_2^2-r_1^2}{8(m-2)} + \frac{(r_2-r_1)k}{2} > 0. \tag{2}$$

Also note that

$$n_{k+1}^{(r_1)} - n_k^{(r_1)} = \frac{r+(m-2)}{2} + k(m-2). \tag{3}$$

From the conditions on r_1, r_2 we have

$$\frac{r_2^2-r_1^2}{8(m-2)} + \frac{(r_2-r_1)k}{2} \le \frac{(m-1)^2-1}{8(m-2)} + \frac{(m-2)k}{2} = \frac{m}{8} + \frac{(m-2)k}{2}.$$

Also we have

$$\frac{m}{8} \le \frac{m-2}{2} \text{ if and only if } 8 \le 3m.$$

Since $m \geq 5$, we have

$$\frac{m}{8} + \frac{m-2}{2}k \leq \frac{(m-2)(k+1)}{2} \leq \frac{(m-2)(2k+1)}{2} < \frac{r+(m-2)}{2} + k(m-2),$$

so $n_k^{(r_2)} - n_k^{(r_1)} < n_{k+1}^{(r_1)} - n_k^{(r_1)}$, and thus $n_k^{(r_1)} < n_k^{(r_2)} < n_{k+1}^{(r_1)}$.

To show that Ozanam's Rule is false for all $m > 4$, we show that the number of distinct congruence classes r modulo $2(m-2)$ such that $r^2 \equiv m^2 \bmod 8(m-2)$, is always at least 2. Thus there is always a sequence of integers n for which $8n(m-2)+(m-4)^2$ is a square, but n is not m-gonal. For all positive integers m, n let

$$R(m^2, n) = \{r \bmod n \mid r^2 \equiv m^2 \bmod n\},$$

let $S(m) = \{r \bmod 2(m-2) \mid r^2 \equiv m^2 \bmod 8(m-2)\}$, and set $c_m = |S_m|$.

Lemma 1. *With the notation as above,*

$$c_m = \begin{cases} |R(m^2, 2(m-2)| & \text{if } m \equiv 1 \bmod 2 \text{ or } m \equiv 6 \bmod 8 \\ \frac{1}{2}|R(m^2, 2(m-2)| & \text{if } m \equiv 0 \bmod 4 \\ 2|R(m^2, j)| & \text{if } m \equiv 2 \bmod 8, m = 2 + 2^t j, j \equiv 1 \bmod 2. \end{cases} \quad (4)$$

Proof. (Lemma.) We prove each case separately.

`Case 1.` If m is odd, then $m-2$ is also odd. By the Chinese Remainder Theorem the congruence $r^2 \equiv m^2 \bmod 2(m-2)$ is equivalent to the same two congruences modulo 2 and $m-2$. Since $r^2 \equiv m^2 \equiv 1 \bmod 2$, r is also odd. Thus $r^2 \equiv m^2 \equiv 1 \bmod 8$, and combined with the same equivalence modulo $m-2$ we have $r^2 \equiv m^2 \bmod 8(m-2)$. So r^2 is equivalent to m^2 modulo $2(m-2)$ if and only if the same congruence holds modulo $8(m-2)$.

`Case 2.` If $m \equiv 6 \bmod 8$, then $2^2||(m-2)$, and thus $2^3||2(m-2)$. The congruence $r^2 \equiv m^2 \bmod 2(m-2)$ is equivalent to the two congruences modulo 8 and $\frac{m-2}{4}$. Since $r^2 \equiv m^2 \equiv 4 \bmod 8$, $2||r$ and thus $r \equiv 2$ or $6 \bmod 8$. If we write $m = 6+8x$, and $r = 2+8y$ or $r = 6+8y$ for $x, y \in \mathbb{Z}$, a simple calculation shows that $32|(r^2-m^2)$ is both cases. Hence $r^2 \equiv m^2 \bmod 32$, and combined with the same equivalence modulo $\frac{m-2}{4}$, we have $r^2 \equiv m^2 \bmod 8(m-2)$.

`Case 3.` If 4 divides m, then $2||(m-2)$, and $2^2||2(m-2)$. Hence the congruence $r^2 \equiv m^2 \bmod 2(m-2)$ is equivalent to the same congruence modulo 4 and $\frac{m-2}{2}$. Since $r^2 \equiv m^2 \equiv 0 \bmod 4$, we have $r \equiv 0$ or 2 modulo 4. In order to have $r^2 \equiv m^2 \equiv 0 \bmod 16$, r must be 0 modulo 4. Thus only $\frac{1}{2}$ of the solutions to $r^2 \equiv m^2 \bmod 4$ satisfy the same condition modulo 16. Combined again with the same equivalence modulo $\frac{m-2}{2}$, this shows one half of the elements in $R(m^2, 2(m-2))$ are also in S_m.

`Case 4.` If $m \equiv 2 \bmod 8$, write $m-2 = 2^t j$ where j is odd, and $t \geq 3$. The congruence $r^2 \equiv m^2 \bmod 2(m-2)$ is equivalent to the same congruences modulo 2^{t+1} and j. A quick calculation shows that $m^2 \equiv 4 \bmod 2^{t+1}$, and hence $2||r$.

Also note that $m^2 \equiv 4 + 2^{t+2} \bmod 2^{t+3}$. Now $m^2 \equiv 4 \bmod 2^{t+1}$ if and only if $\left(\frac{m}{2}\right)^2 \equiv 1 \bmod 2^{t-1}$. The congruence $x^2 \equiv 1 \bmod 2^{t-1}$ has two solutions $\pm 1 \bmod 4$ when $t = 3$, and four solutions $\pm 1, \pm 1 + 2^{t-2} \bmod 2^{t-1}$ when $t \geq 4$. When $t = 3$, $x^2 \equiv 36 \bmod 64$ has solutions $x \equiv 6$ or $10 \bmod 16$. When $t \geq 4$, $r^2 \equiv 4 \bmod 2^{t+1}$ has solutions r when $r \equiv \pm 2$, or $\pm 2 + 2^{t-1} \bmod 2^t$. Of these solutions, only $r \equiv \pm 2 + 2^t \equiv \pm m \bmod 2^{t+1}$ satisfy $r^2 \equiv 4 + 2^{t+2} \bmod 2^{t+3}$. Thus there are 2 such values of r modulo 2^{t+1} that satisfy $r^2 \equiv 4 + 2^{t+2} \bmod 2^{t+3}$. Therefore for all $t \geq 3$ we have two solutions r modulo 2^{t+1} that satisfy $r^2 \equiv m^2 \bmod 2^{t+3}$. So we have 2 choices for r modulo 2^{t+1}, and $|R(m^2, j)|$ possibilities modulo j, that satisfy $r^2 \equiv m^2 \bmod 8(m-2)$. $\square$

To complete the proof that Ozanam's Rule is false for $m > 4$, we need to check that $c_m \geq 2$. This is obvious in the case $m \equiv 2 \bmod 8$ from the lemma above. First note that $\gcd(m, m-2) \leq 2$, hence any odd primes dividing $m-2$ do not divide m. Also note that for any odd prime p dividing $m-2$, $x^2 \equiv m^2 \bmod p$ has exactly two solutions. Therefore by Hensel's Lemma, $x^2 \equiv m^2 \bmod p^k$ has exactly two solutions for any odd prime p dividing $m-2$, and any positive integer k.

For any positive integer $n \geq 2$, let $P^O(n)$ denote the number of distinct odd primes that divide n.

`Case 1.` If m (and $m-2$) is odd, then $c_m = |R(m^2, 2(m-2)|$. There is one solution modulo 2 to $r^2 \equiv m^2 \equiv 1 \bmod 2$, and for every odd prime p such that $p^k || (m-2)$ there are exactly 2 solutions to $r^2 \equiv m^2 \bmod p^k$. Thus in this case $c_m = 2^{P^O(m-2)}$ and since $m \geq 5$, $m-2$ has at least one odd prime factor. So if m is odd and $m \geq 5$, then $c_m \geq 2$.

`Case 2.` If $m \equiv 6 \bmod 8$, then as in Case 1, $c_m = |R(m^2, 2(m-2)|$. Recall from above that $2(m-2) = 8\left(\frac{m-2}{4}\right)$ where $\frac{m-2}{4}$ is odd. There are two solutions to $r^2 \equiv 4 \bmod 8$ (namely 2 and 6), and as above there are 2 solutions modulo p^k for every odd prime p dividing $\frac{m-4}{2}$. Hence in this case $c_m = 2(2^{P^O\left(\frac{m-2}{4}\right)}) = 2^{P^O\left(\frac{m-2}{4}\right)+1} \geq 2$.

`Case 3.` If 4 divides m, $c_m = \frac{1}{2}|R(m^2, 2(m-2)|$. Recall from above that $2(m-2) = 4\left(\frac{m-2}{2}\right)$ where $\frac{m-2}{2}$ is odd. Again note there are two solutions to $r^2 \equiv m^2 \equiv 0 \bmod 4$, and 2 solutions for each odd distinct prime dividing $\frac{m-2}{2}$. Hence $c_m = \frac{1}{2}2(2^{P^O\left(\frac{m-2}{2}\right)}) = 2^{P^O\left(\frac{m-2}{2}\right)}$. Since $m \neq 4$, $m \geq 8$, and thus $\frac{m-2}{2}$ has at least one odd factor, so again we have $c_m \geq 2$.

Finally given m, let $\mathcal{O}_m = \{n \in \mathbb{Z}^+ \mid 8n(m-2) + (m-4)^2 \text{ is a square}\}$. Note that by the above arguments there are c_m sequences $n_k^{(r)}$ in $\mathcal{O}_m$, only one of which is the m-gonal numbers. As was shown above if $r_2 > r_1$, then $n_k^{(r_1)} < n_k^{(r_2)} < n_{k+1}^{(r_1)}$, hence only one out of every c_m consecutive elements in $\mathcal{O}_m$ is m-gonal. $\square$

Recall that the sequence of *generalized m-gonal numbers* arises by evaluating the formula for the k-th m-gonal number at all integers. From the proof above we see that the generalized m-gonal numbers will be those with $8n(m-2)+(m-4)^2 = c^2$ for $c \in \mathbb{Z}^+$, with $c \equiv \pm m \bmod 2(m-2)$.

For completeness we include the value of c_m, where $\frac{1}{c_m}$ is the rate at which the original Ozanam's Rule succeeded in detecting m-gonal numbers. From the proof above, c_m is the number of square roots of m^2 modulo $8(m-2)$ that are distinct modulo $2(m-2)$.

Corollary 2. *With all of the notation as above,*

$$c_m = \begin{cases} 2^{P^O(m-2)} & m \equiv 1 \bmod 2 \\ 2^{P^O\left(\frac{m-2}{4}\right)+1} & m \equiv 6 \bmod 8 \\ 2^{P^O\left(\frac{m-2}{2}\right)} & 4|m \\ 2^{P^O(j)+1} & m \equiv 2 \bmod 8, m = 2+2^t j, j \text{ odd} \end{cases}.$$

The calculation for Case 4 follows the same argument as in the first 3 cases given in the above proof.

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