THINNING THICKETS

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Abstract

The game of thinning thickets is played on a directed graph and can be regarded as a variant of hackenbush. In this paper, we consider thinning thickets played on cordons. We show that the nim-dimension for green cordons is infinite and characterize those with nim-value 0 and nim-value 1. For red-blue stalks, we show that, up to infinitesimals, these take on only eight values but for blue cordons the values, modulo infinitesimals, are k or switches $\{k+1 \mid k\}$ for all non-negative integers k. Lastly, we show that thinning thickets has arbitrarily high temperature.

1. Introduction

thinning thickets is an offshoot of hackenbush. Both are played on graphs with a set of distinguished vertices (the ground or roots), each arc/edge is colored red, blue or green. In both, there are rules as to what arcs/edges Left and Right can delete, and also, any vertex not connected to the ground is also deleted.

The worth of a tree, and a game, is in the fruit that it bears. hackenbush has many nice features and interesting analyses as do many of its progeny. For example, every red-blue hackenbush position is *cold*, that is, a number ([2], Chapter 7). A long-standing, and difficult, problem is the analysis of sums of flowers (green stalks with a blue or red flower at the top). Other variants include hackendot [16] which has been extended to partially ordered sets in [3] and [5]; timber [11]

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is an (impartial) variant played on directed graphs; TOPPLING DOMINOES [6] every number occurs as exactly one position, proved via ordinal sums; and the nimber $*n$ occurs exactly n times. THINNING THICKETS introduces a parity aspect to the moves.

Definition 1.1. A directed graph G is a *thicket* if there is a subset of vertices $x_1, \ldots, x_k$ called *roots* and every arc is on a directed path to some root.

We will always write an arc as $\overrightarrow{ab}$, where a and b are the initial and terminal vertices respectively. The *in-degree* of an arc $\overrightarrow{ab}$ is the number of arcs with terminal vertex a.

Rules for THINNING THICKETS.

- Board: A finite thicket in which each arc is colored blue, red or green.
- Players: Left and Right, who move alternately.
- Moves: On a move, each player deletes an arc. Left may delete a blue arc with an even in-degree (including 0) or a red arc with an odd in-degree. Similarly, Right may delete a red arc with an even in-degree (including 0) or a blue arc with an odd in-degree. Both may delete a green arc with even in-degree. After the arc is deleted, any arc and vertex not on a directed path to a root is also deleted.
- Winning Condition: If, on their turn to move, a player cannot delete an arc then they lose the game.

In play, when the arc $\overrightarrow{ab}$ is deleted then the in-degree of b changes parity and so the player that can delete the arcs directed out of b also changes. This dynamism exists in only a few analyzed combinatorial games [8], [9], and [13] and all of these are impartial.

We present results for THINNING THICKETS in the cases of cordons, which are tall, thin, singly rooted graphs.

Definition 1.2. A *cordon*[3] consists of i) two sets of vertices $V_1 = \{v_0, v_1, \ldots, v_n\}$, where v_0 is the *root*, v_n is the *top vertex* and the others are called *interior* vertices, and $V_2 = \{l_1, l_2, \ldots, l_k\}$, ii) an increasing sequence $\{a(1), a(2), \ldots, a(k)\}$, where $0 < a(1), a(k) \leq n-1$, and iii) the arcs are $\overrightarrow{v_i v_{i-1}}$, $i = 1, 2, \ldots, n$ and $\overrightarrow{l_j v_{a(j)}}$, $j = 1, \ldots, k$. The vertex $v_{a(i)}$ is called an *attachment vertex.* If V_2 is empty then we call the cordon a *stalk*, hence $\overrightarrow{v_n v_{n-1}}$ is not considered to be a leaf arc.

The *height* of a cordon C, denoted $h(C)$, is the number of arcs in the stalk.

[3] Horticultural definition: A cordon is a tree or shrub, especially a fruit tree, repeatedly pruned and trained to grow on a support as a single rope-like stem.

For a given game, apart from ‘Who wins?’ and ‘How?’, it is interesting to know which values can occur and which cannot. In general, this is a very hard question to answer. This can be broken down into sub-questions:

1. What is the greatest number that can occur?
2. What temperatures can occur?
3. For what positive integers n can $*n$ occur? (Nim-dimension.)
4. Are threats, that is, *tinies*, $\{0 \mid \{0 \mid -n\}\}$, and *minies*, $\{\{n \mid 0\} \mid 0\}$, present?

In the analysis, we find the value $tiny(1) = \{0 \mid \{0 \mid -1\}\}$ occurs but were unable to provide any more insight.

In addressing these subsidiary questions, it became natural to consider restricted versions of THINNING THICKETS.

To answer (3), we consider GREEN CORDONS (all the arcs are green), which is the impartial version of the game. In Section 3, Theorem 3.1, we show that the nim-dimension is infinite (i.e., every $*n$ occurs). In Theorems 3.2 and 3.3 we characterize the GREEN CORDONS positions with values 0 and $*$, and also show a Fibonacci recurrence (Theorem 3.4). The Sprague-Grundy Theory is required and we give an overview of this theory in Section 2.1.

Returning to questions (1) and (2), we consider RED-BLUE CORDONS. Unlike HACKENBUSH, the values are not just numbers, indeed some are hot, and the canonical forms can be quite complicated. However, if the infinitesimal values are ignored then Theorem 4.2 shows that RED-BLUE STALKS positions take on only eight values, specifically 0, 1, -1, $\{1|0\}$, $\{0|-1\}$, $\{1|-1\}$, $\{\{1|0\}|-1\}$, $\{1|\{0|-1\}\}$. We give the necessary background for the reduced canonical form (how to ignore infinitesimals) in Section 2.2. Adding leaves gives a richer set of values. The set of values of BLUE CORDONS positions is infinite but describable if, again, infinitesimal values are ignored. In Theorem 4.1 we show that the value is either k or the switch $\{k+1 \mid k\}$ plus an infinitesimal for any non-negative integer k. Their negatives will be found by the corresponding RED CORDONS.

In Theorem 4.4, we consider a family of cordons where all the arcs but one are blue and the other is red and show that, for any positive integer n, there is a member of this family with temperature greater than n thereby answering question (2).

We conclude the paper with open questions in Section 5.

2. Game Theory Background

A game G is defined in terms of its *Left options*, which are positions Left can reach in one move, and its *Right options*, which are positions Right can reach in one move.

These sets of options are denoted by $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ respectively and G is written as $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$. The *disjunctive sum* of positions G and H, written $G+H$, is the game in which a player may move in either component but not both. In terms of options,

$$G + H = \{G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \mid G^{\mathcal{R}} + H, G + H^{\mathcal{R}}\}.$$

Note, if A is a set of games then $A + H = \{G' + H \ : \ G' \in A\}$.

The *negative* of a game is obtained by interchanging the players. More formally, $-G = \{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\}$. Descriptive, often intuitively obvious, names are given to games. The basic games are *numbers*, which correspond to the number of moves advantage, positive if Left has the advantage, negative if Right. Most games are not numbers. The game with no options is defined as 0. It clearly is of no advantage to either player in any situation. Let A and B be sets of dyadic rationals where $a < b$ for all $a \in A$ and $b \in B$ then either $\{A \mid B\}$ is the integer, n, closest to 0 which satisfies $a < n < b$ or, if no such integer exists, then $\{A \mid B\} = \frac{2p+1}{2^q}$ where q is the smallest positive integer such that there is a unique p with $a < \frac{2p+1}{2^q} < b$.

There is a partial order on games: $G \geqslant H$ if Left can force a win in $G - H$ going second; and $G = H$ if $G - H$ is a second player win, i.e. $G - H = 0$. This equality is an equivalence relation on the set of games. In each equivalence class there is a unique representative that can be obtained by continually applying two operations.

- Suppose $G > H$ then, for Left, G *dominates* H and, for Right, H *dominates* G. Dominated options may be eliminated since they will never be played.
- Given G and a Left option G^L, if there is a Right option X of G^L with $X \leqslant G$ then let H be the game with the same options as G except that G^L is replaced by the Left options of X. Then $G = H$. We say that G^L is *reversible*.

A game with no dominated options and no reversible options is in *canonical form*.

For more details about the analysis of combinatorial games, readers should consult any of [1], [2], [14].

2.1. Sprague-Grundy Theory for Impartial Games

The *minimum excluded value*, *mex*, of a set S is the least non-negative integer not included in S. A game is broken down recursively in terms of its options. Recursively, the *Sprague-Grundy value*, or *nim-value*, of an impartial game H is given by $\mathcal{G}(H) = \text{mex}\{\mathcal{G}(H') \mid H' \text{ is an option of } H\}$. Thus, if a game has no options, it has nim-value 0. Let H be a game, the next player to move can win if and only if $\mathcal{G}(H) > 0$. Let p and q be non-negative integers then $p \oplus q$ signifies the *nim-sum* or *exclusive or* of p and q. It is obtained by writing p and q in binary and adding in binary without carrying. The value of the disjunctive sum of $F + H$ is given by

$\mathcal{G}(F+H) = \mathcal{G}(F) \oplus \mathcal{G}(H)$.

Impartial games are a subset of partizan games and, as such, have values. If $\mathcal{G}(G) = n$ then the value of G, as a partizan game, is denoted $*n$. If $\mathcal{G}(G) = n$ then we say that the nim-value of G is n.

Definition 2.1. The *nim-dimension* [12] of a game H is

$$\max\{k : *2^{k-1} \text{ occurs as a sub-position of } H\}.$$

Note that $0 = *0$ and $* = *1$. A ruleset has dimension 0 if it never has $*$ as a position.

2.2. Reduced Canonical Form

We now discuss the background for *partizan games*, games in which players' options may differ.

Suppose two players play a game and stop as soon as the game is a number. Left attempts to have this stopping position be as large as possible, and Right wants it to be as small as possible. The number arrived at when Left (respectively, Right) moves first is called the Left (Right) stop of the position. More formally,

Definition 2.2. Denote the *Left stop* and *Right stop* of a game G by $\mathrm{LS}(G)$ and $\mathrm{RS}(G)$, respectively. They are defined in a mutually recursive fashion:

$$\mathrm{LS}(G) = \begin{cases} G & \text{if } G \text{ is a number,} \\ \max(\mathrm{RS}(G^L)) & \text{if } G \text{ is not a number;} \end{cases}$$

$$\mathrm{RS}(G) = \begin{cases} G & \text{if } G \text{ is a number,} \\ \min(\mathrm{LS}(G^R)) & \text{if } G \text{ is not a number.} \end{cases}$$

The rest of this section is an amalgam of the comments found around Theorem 8.5 [1], and Proposition 3.17 and Problem 3.11 in [14].

Definition 2.3. Let G and H be short games. Then G and H are *confused* if $G \not\geq H$ and $H \not\geq G$; that is, $G - H$ is a first player win.

Definition 2.4. The *confusion interval* of a game G is defined by

$$\mathrm{C}(G) = \{x : x \text{ is a number that is confused with } G\}.$$

The endpoints of $\mathrm{C}(G)$ are $\mathrm{RS}(G)$ and $\mathrm{LS}(G)$.

It is always the case that $\mathrm{RS}(G) \leqslant \mathrm{LS}(G)$. Now, $\mathrm{LS}(G) \in \mathrm{C}(G)$ if and only if the move that results in reducing the game to the number is a Left move. Left always prefers moving to $\mathrm{LS}(G)$ over letting Right move there. The same is true for the Right stop. The reduced canonical form, however, allows us to ignore the status of the endpoints of the confusion interval.

Definition 2.5. A game G is an *infinitesimal* if, for every positive number x, we have $-x < G < x$. Let I denote the set of infinitesimals. When $G - H$ is infinitesimal, we say that G and H are *infinitesimally close*, and write $G \equiv_{\mathrm{I}} H$. We will sometimes say that H is G-ish (G *I*nfinitesimally *SH*ifted).

Definition 2.6. $G \geq_I H$ if $G \geq H + \epsilon$ for some infinitesimal ϵ; $G \leq_I H$ is defined similarly.

An alternate definition of an infinitesimal is: *G is an infinitesimal if $LS(G) = RS(G) = 0$.* This, together with the Number Translation Principle, gives the next result, which will be used often in this paper.

Lemma 2.7. *Let G be a game. If $LS(G) = RS(G) = x$, for some number x, then $G \equiv_I x$.*

The *Number Translation Principle* [1] states that if x is a number and G is not, then $G + x = \left\{G^L + x \mid G^R + x\right\}$. Closely related is the *Number Avoidance Theorem* [1] which states: Suppose that x is a number in canonical form with a Left option and that G is not a number. Then, there exists a G^L such that $G^L + x > G + x^L$. That is, in the disjunctive sum of a number and a non-number, the best move is always in the non-number.

The next result is new but follows easily from existing results.

Theorem 2.8. *Let G and H be games. If* $\mathrm{RS}(G) \geqslant \mathrm{LS}(H)$ *then $G - H \geqslant \epsilon$ for some infinitesimal ϵ.*

Proof. If $\mathrm{RS}(G) > x > y > \mathrm{LS}(H)$, for some numbers x, y, then $G - H > x - y$ and $x - y$ is bigger than any infinitesimal. Thus we may assume that $\mathrm{RS}(G) = \mathrm{LS}(H) = x$. By the Number Avoidance Theorem, we have $\mathrm{RS}(G - x) = 0$ and $LS(x - H) = 0$ and therefore, by Lemma 6.4 [14], $G - x \geqslant \epsilon$ and $\delta \geqslant H - x$ for some infinitesimals ϵ, δ. Together they yield $G - H \geqslant \epsilon - \delta$. □

As a consequence, we obtain a result first found in [10].

Corollary 2.9. *Let a and b be numbers with $a \geqslant b$ then $a \geqslant_I \{a \mid b\} \geqslant_I b$.*

The results in the remainder of this section all originate in [7] and can also be found in [14].

Definition 2.10. Let G be any game.

- A Left option G^L is *Inf-dominated* if $G^L \leq_I G^{L'}$ for some other Left option $G^{L'}$.
- A Left option G^L is *Inf-reversible* if $G^{LR} \leq_I G$ for some G^{LR}.

The definitions for Right options are similar.

For example, let $G = \{1, \{1 \mid 0\} \mid 0\}$. Then $\{1 \mid 0\}$ is an Inf-dominated Left option of G, since $\{1 \mid 0\} \leq 1+\uparrow$. In the reduced canonical form, the Inf-dominated options are removed (originally defined in [4], notation has been reformulated as used in [7]).

Definition 2.11. A game G is said to be in *reduced canonical form* provided that, for every sub-position H of G, either:

- H is a number in simplest form; or
- H is not a number or a number plus an infinitesimal, and contains no Inf-dominated or Inf-reversible options.

Theorem 2.12. *For any game G, there is a game G' in reduced canonical form with $G \equiv_{\mathrm{I}} G'$.*

Theorem 2.13. *Suppose that G and H are in reduced canonical form. If $G \equiv_{\mathrm{I}} H$, then $G = H$.*

This then shows that the reduced canonical form of a game G is well-defined and unique. In our paper, we will use the $\equiv_{\mathrm{I}}$ notation instead of defining a function $Rcf(G)$.

Lemma 2.14. *If G is not a number and G' is obtained from G by eliminating an Inf-dominated option, then $G' \equiv_{\mathrm{I}} G$.*

Theorem 2.15. *If $G = \{G^L|G^R\}$ is not a number and $G' = \{G^{L'}|G^{R'}\}$ is a game with $G^{L'} \equiv_{\mathrm{I}} G^L$ and $G^{R'} \equiv_{\mathrm{I}} G^R$, then $G' \equiv_{\mathrm{I}} G$.*

3. green cordons

Recall that a GREEN CORDONS position is a game on a cordon where all arcs are green and directed towards the ground. On a player's turn, one may remove an arc which has even in-degree. In Figure 1 do you want to play first or second?

Theorem 3.1 shows that a GREEN CORDONS position can have arbitrarily high nim-values. In Section 3.2, we characterize the positions with nim-values 0 and 1 and in Section 3.3 we show there is a Fibonacci recurrence associated with each.

3.1. Nim-dimension

Let $L(n)$ be the GREEN CORDONS position with stalk vertices $\{v_0, v_1, \ldots, v_n\}$ and with a leaf arc at v_i, for $1 \leqslant i \leqslant n-2$. Note that $L(0)$, $L(1)$ and $L(2)$ are the stalks with 0, 1 and 2 arcs, respectively.

Theorem 3.1. *The nim-dimension of* GREEN CORDONS *and* THINNING THICKETS *is infinite.*

Proof. From $L(n)$, cutting $\overrightarrow{v_i v_{i-1}}$, $2 \leqslant i \leqslant n-1$, results in $L(i)$. Hence, all $L(k)$ where $k = 2n$ for $n = 1, 2, \ldots$ are distinct. Therefore, the nim-dimension of GREEN CORDONS, and hence THINNING THICKETS, is infinite. □

3.2. Characterizations for Positions of Value 0 and $*$

Leaves are ordered starting closest to the root. Recall that $a(i)$ is the index of the stalk vertex to which the ith leaf arc is attached; that is, leaf l_i is attached to the stalk vertex $v_{a(i)}$. For example, in Figure 1, components A, B and C have $a(1)$ equal to 2, 1, and 1 respectively.

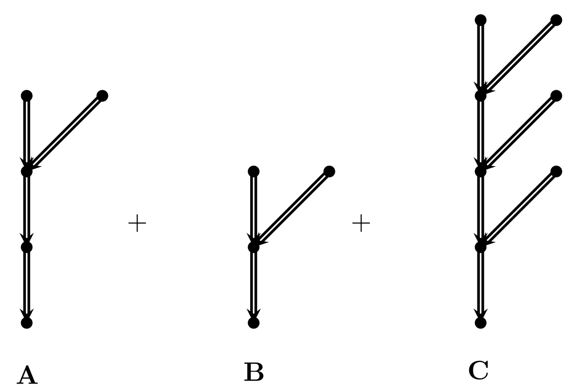

Figure 1: A GREEN CORDONS position, where ⇒ is a green arc.

Theorem 3.2. *Let T be a* GREEN CORDONS *position, of height n. Then $\mathcal{G}(T) = 0$ if and only if either i) T is a stalk and n is even, or, ii) $a(1)$ is even, and all $a(i+1) - a(i)$, $1 \leq i < k$, and $n - a(k)$ are odd, where k is the number of leaf arcs.*

Proof. Let T be a GREEN CORDONS position with stalk vertices $\{v_0, v_1, \ldots, v_n\}$, thus $h(T) = n$. Let A be the set of GREEN CORDONS positions with the following properties: If T is a stalk, then n is even, or, if there are leaves, $a(1)$ is even, and $a(i+1) - a(i)$, $1 \leq i < k$, and $n - a(k)$ are all odd. To prove $\mathcal{G}(T) = 0$ only for $T \in A$, we must show that any option from T is not in A and if $S \notin A$ then S has an option in A (Theorem 2.13[1]).

Suppose $T \in A$. If T is a stalk, then the only move is to remove $\overrightarrow{v_n v_{n-1}}$, resulting in a stalk T' and $h(T')$ is odd. If T has a leaf arc, let T' be the resulting tree after a move, then the moves are as follows.
i) Remove $\overrightarrow{v_n v_{n-1}}$ (where $\overrightarrow{l_k v_{n-1}} \notin E(T)$) and thus in T' we have $h(T') - a(k) = n - 1 - a(k)$ is even.
ii) Remove $\overrightarrow{v_n v_{n-1}}$ (where $\overrightarrow{l_k v_{n-1}} \in E(T)$). Then $h(T') = h(T)$ however, $h(T') - a(k-1)$ is even.
iii) Remove $\overrightarrow{v_{a(i)} v_{a(i)-1}}$, where $i > 1$. Now $h(T') - a(i-1)$ is even.
iv) Remove $\overrightarrow{v_{a(1)} v_{a(1)-1}}$. Now T' is a stalk and $h(T')$ is odd.
v) Remove $\overrightarrow{l_i v_{a(i)}}$, where $1 < i < k$. Then $a(i+1) - a(i-1)$ is even.
vi) Remove $\overrightarrow{l_1 v_{a(1)}}$. Then $a(2)$ (which is $a(1)$ for T') is odd.
vii) Remove $\overrightarrow{l_k v_{a(k)}}$. Then $h(T') - a(k-1)$ is even.

Now, consider $S \notin A$. Then,
i) If S is a stalk, then $h(S)$ is odd and we remove $\overrightarrow{v_n v_{n-1}}$ leaving $h(S')$ even, i.e., $S' \in A$.
ii) If $a(1)$ is odd then remove $\overrightarrow{v_{a(1)} v_{a(1)-1}}$ to leave S', a stalk of even height.
iii) If $a(1)$ is even and there exists $a(i+1) - a(i)$ even, where j is the least such i which satisfies this property. Remove $\overrightarrow{v_{a(j+1)} v_{a(j+1)-1}}$, Then S' satisfies: $a(1)$ even, $a(i+1) - a(i)$ odd, for $1 \leq i < j$, and $h(S') - a(j) = a(j+1) - 1 - a(j)$ which is odd.
iv) If $a(1)$ is even, $a(i+1) - a(i)$, for all i, is odd but $n - a(k)$ is even, then delete $\overrightarrow{v_n v_{n-1}}$ to result in $S' \in A$. □

The classification for cordons with nim-value 1 is similar to that for the classification of cordons with nim-value 0 with the roles of nim-values 0 and 1 interchanged. Thus, the details of Theorem 3.3 are left to the reader.

Theorem 3.3. *Let T be a* GREEN CORDONS *position, of height n. Then $\mathcal{G}(T) = 1$ if and only if either (i) T is a stalk and n is odd; or (ii) all of $a(1)$, $a(i+1) - a(i)$, $1 \leq i < k$, and $n - a(k)$ are odd.*

3.3. Fibonacci Connection

Let F_n be the set of GREEN CORDONS positions of height n and nim-value 0. Let $f_n = |F_n|$. Note that $f_0 = 1$, $f_1 = 0$ and $f_2 = 1$.

Theorem 3.4. *The value f_n is given by the recurrence relation $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 1$ and $f_1 = 0$.*

Proof. Let $A_n \subset F_n$ be the subset of positions of F_n with no leaf arc at height 2. Let $C_n \subset F_n$ be the subset of positions of F_n with the first leaf arc at height 2 (i.e., $a(1) = 2$). Note that $A_n \cap C_n = \emptyset$ and $F_n = A_n \cup C_n$.

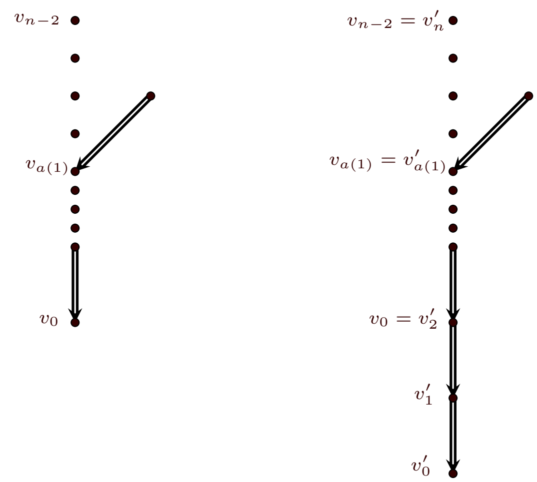

Figure 2: $T \in F_{n-2}$ (left) and $T' \in A_n$ (right).

We will show that there is a bijection between 1) A_n and F_{n-2}, and 2) C_n and F_{n-1}.

1) Consider $T \in F_{n-2}$ of height $n-2$ with $\mathcal{G}(T) = 0$. Add two arcs below the root vertex of T, to get T'. Since $a(1)$ of T was even, adding these two arcs below the root will result in the height of T' being n and $a(1)$ for T' is even at an attachment vertex value greater than 2. Hence $T' \in A_n$. See Figure 2. Conversely, if $T \in A_n$ then the induced subgraph starting at the stalk vertex of height 2 gives $T' \in F_{n-2}$. Hence there is a bijection between A_n and F_{n-2}.

2) Consider $T \in F_{n-1}$ of height $n-1$ with $\mathcal{G}(T) = 0$, $a(2) - a(1)$ is odd, and $a(1)$ is even. Consider T' which we define to be T with an additional arc emanating from the root vertex of T and an arc at the second attachment vertex of T', $\overrightarrow{bv'_2}$. This gives $a(1) = 2$ for T' and $a(2) - a(1)$ is odd. Hence $T' \in C_n$. See Figure 3. Conversely, consider $T \in C_n$ with stalk vertices $\{v_0, v_1, \ldots, v_n\}$ and $a(1) = 2$ and let $\overrightarrow{bv_2}$ define the first leaf arc. Then taking the induced subgraph on stalk vertices $\{v_1, \ldots, v_n\}$ (so v_1 is the new root) and all leaves without b we obtain a GREEN CORDONS position T' of height $n-1$ with $a(1)$ even. Hence $T' \in F_{n-1}$.

Thus $f_n = |F_n| = |A_n| + |C_n| = |F_{n-2}| + |F_{n-1}| = f_{n-2} + f_{n-1}$, where $|F_0| = 1$ and $|F_1| = 0$. □

As the classification for nim-value 0 and nim-value 1 are symmetric, the following theorem is immediate.

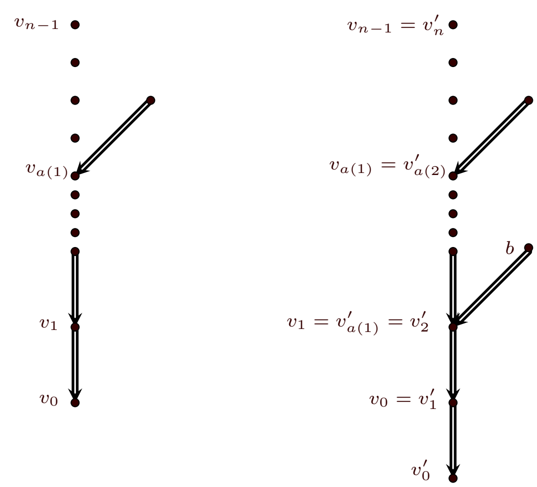

Figure 3: $T \in F_{n-1}$ (left) and $T' \in C_n$ (right).

Theorem 3.5. *Let H_n be the set of* GREEN CORDONS *positions of height n with nim-value 1. Then $|H_n| = |H_{n-1}| + |H_{n-2}|$, where $|H_0| = 0$, and $|H_1| = 1$.*

Consider Figure 1. Using Theorems 3.2, 3.3, we see that the nim-values for A, B and C are 0, 1 and 1 respectively. So $\mathcal{G}(A + B + C) = \mathcal{G}(A) \oplus \mathcal{G}(B) \oplus \mathcal{G}(C) = 0 \oplus 1 \oplus 1 = 0$. Hence this is a second player win.

4. Multicolored Cordons

A BLUE (RED) CORDONS position is a cordon where all the arcs are blue (red). We are only required to analyze BLUE CORDONS positions because RED CORDONS positions are their negatives.

The stalks seem recursively simple. Let $B(n)$ be a stalk with n blue arcs. Recall, Left can only remove $\overrightarrow{v_n v_{n-1}}$ and Right can remove any of the others: for $n > 1$,

$B(n) = \{B(n-1) \mid B(0), B(1), \ldots, B(n-2)\}$. The canonical forms are

$$\begin{aligned}
B(0) &= 0\\
B(1) &= 1\\
B(2) &= \{B(1) \mid B(0)\} = \{1 \mid 0\}\\
B(3) &= \{B(2) \mid B(0), B(1)\} = \{\{1 \mid 0\} \mid 0, 1\} = \{\{1 \mid 0\} \mid 0\}\\
B(4) &= \{B(3) \mid B(0), B(1), B(2)\} = \{\{\{1 \mid 0\} \mid 0\} \mid 0\}.
\end{aligned}$$

It is not too difficult to show that $B(n) = \{B(n-1) \mid 0\}$[4]. The canonical forms will get longer as n increases. However, starting at $n = 3$, the games are infinitesimal since $LS(B(n)) = RS(B(n)) = 0$. The reduced canonical forms are much simpler: $Rcf(B(n)) = B(n)$ for $n = 0, 1, 2$ and $Rcf(B(n)) = 0$ for $n \geq 3$.

The analysis of BLUE CORDONS, in general, is made simpler by using the reduced canonical forms since there are only a few cases to consider.

First, we mention our notation. A position is denoted by a tuple, each entry of which represents the presence (1), or absence (0), of a leaf arc on v_i, $i > 0$, where left to right in the tuple is top to bottom on the cordon. A position will always start with a 0, since there is no leaf arc at the top vertex (it would be part of the stalk instead) but it is useful to indicate the moves. We do not include the ground vertex, so the empty game is $G = [\,]$ and the stalk with one arc is $G = [0]$.

Suppose there is a leaf arc at v_i for some $i > 0$. Our notation would have a 1 in the ith place, as in $[0 \ldots 1 \ldots]$ or, better, $[\alpha 1 \beta]$ for some arbitrary strings α and β. Around this ith vertex, Left has several cases to consider. Left can move to $[\alpha 0 \beta]$ (note that α starts with 0) by removing the leaf arc at v_i. If the position is $[\alpha 11 \beta]$ Left can remove the stalk arc $v_i v_{i-1}$ to leave $[00\beta]$. Otherwise the position is $[\alpha 10 \beta]$ and Left can remove the stalk arc $v_i v_{i-1}$ to leave $[0\beta]$. For $n > i > 0$, if there is no leaf arc at v_i, the position can be expressed as one of $[\alpha 00 \beta]$, $[\alpha 01 \beta]$ or $[\alpha 0]$, where α has a leading 0. Right has the move to delete the arc below v_i giving the positions, respectively, $[0\beta]$, $[00\beta]$ or $[\,]$. For a simple example, in Figure 4, reading from left to right, we have $[010] + [001] - [0101]$.

To emphasize, α indicates a section of the cordon which is arbitrary, and starts with 0 (i.e., non-empty).

Theorem 4.1. *The value of a* BLUE CORDONS *position is:*

1. $[0] = 1$, $[00] = \{1 \mid 0\}$, $[\alpha 10] \equiv_I \{1 \mid 0\}$, $[\alpha 00] \equiv_I 0$
2. $[\alpha 01^{2k+1}] \equiv_I k+1, k \geq 0$
3. $[\alpha 001^{2k}] \equiv_I k, k \geq 1$
4. $[\alpha 101^{2k}] \equiv_I \{k+1 \mid k\}, k \geq 1$

[4]For those conversant with CGSuite [15] notation, $B(n) = \{-_1 \mid 0^{n-2}\}$.

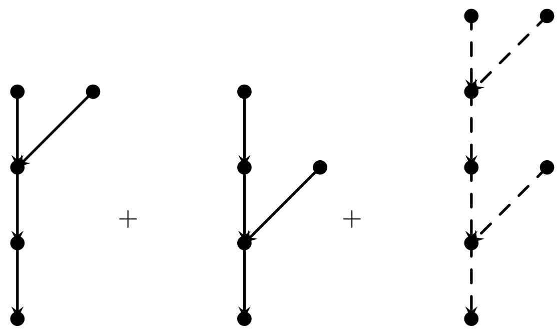

Figure 4: Sum of BLUE CORDONS and RED CORDONS positions, where → is blue, ⇢ is red.

5. $[001^{2k}] \equiv_I \{k+1 \mid k\}, k \geq 1$

6. $[01] = 1$, *and in general we have* $[01^{2k}] = [01^{2k+1}] = k+1, k \geq 1$.

Proof. From the position $[\alpha\gamma]$, a move to $[\alpha'\gamma]$ is, in our notation, a move in α that modifies, but does not completely eliminate, the string α. For example, let $\alpha = 0000$. Then in $[\alpha 0]$ Right has the options [000], [00] both of which would be combined into one symbol $[\alpha' 0]$. The option to [0] (and to [ ]) will be listed separately. β indicates a non-empty string after a move has occurred (could be different from α'). The α' and β strings do not affect the values and so we use them as generic string representations.

If G is the empty cordon, $G = 0$ since neither player has a move. If G is [0] there is only one move, for Left to move to [ ]. So $G = \{0 \mid \} = 1$. If G is [00] then Left has a move to [0] and Right has a move to [ ]. Hence $G = \{1 \mid 0\}$. Our template for the typical argument, where we have included the reasons for each step, is

$$\begin{aligned} [000] &= \{[00] \mid [0], [\,]\}\text{—options} \\ &= \{\{1 \mid 0\} \mid 1, 0\}\text{—values} \\ &\equiv_I 0 \quad \text{—reduced canonical form.} \end{aligned}$$

For the rest of the analysis we consider the moves in the order: 1) in α; 2) any moves on the stalk (not in α); then 3) any leaf arc moves. Moves in 1) where at least one such move exists are noted by †. A move which may or may not exist for a player is noted by $*$.

Case 1:

$$\begin{aligned}
[\alpha 10] &= \{[\alpha' 10]^{\dagger}, [000]^{\dagger}, [0], [\alpha 00] \mid [\alpha' 10]^{*}, [000]^{*}, [\,]\} \\
&\equiv_I \{\{1 \mid 0\}^{\dagger}, 0^{\dagger}, 1, 0 \mid \{1 \mid 0\}^{*}, 0^{*}, 0\} \\
&\equiv_I \{1 \mid 0\}, \quad \text{by Corollary 2.9.}
\end{aligned}$$

Similarly,

$$\begin{aligned}
[\alpha 00] &= \{[\alpha' 00]^{\dagger}, [00]^{\dagger} \mid [\alpha' 00]^{*}, [00]^{*}, [0], [\,]\} \\
&\equiv_I \{0^{\dagger}, \{1 \mid 0\}^{\dagger} \mid 0^{*}, \{1 \mid 0\}^{*}, 1, 0\} \\
&\equiv_I 0, \quad \text{by Lemma 2.7.}
\end{aligned}$$

Case 2:

$$\begin{aligned}
[\alpha 01^{2k+1}] &= \{[\alpha' 01^{2k+1}]^{\dagger}, [01^{2k+1}]^{\dagger}, [001^{2j+1}]_{j=0}^{k-1}, [001^{2j}]_{j=1}^{k-1}, [00], [\,], [\alpha 001^{2k}], \\
&\quad [\beta 01^{2j+1}]_{j=0}^{k-1}, [\beta 101^{2j}]_{j=1}^{k-1}, [\beta 10] \\
&\mid \quad [\alpha' 01^{2k+1}]^{*}, [01^{2k+1}]^{*}, [001^{2k}]\} \\
&\equiv_I \{k+1^{\dagger}, k+1^{\dagger}, \{j+1\}_{j=0}^{k-1}, \{j+1 \mid j\}_{j=1}^{k-1}, \{1 \mid 0\}, 0, k, \\
&\quad \{j+1\}_{j=0}^{k-1}, \{j+1 \mid j\}_{j=1}^{k-1}, \{1 \mid 0\} \\
&\mid \quad k+1^{*}, k+1^{*}, \{k+1 \mid k\}\} \\
&\equiv_I k+1, \quad \text{by Lemma 2.7.}
\end{aligned}$$

Case 3:

$$\begin{aligned}
[\alpha 001^{2k}] &= \{[\alpha' 001^{2k}]^{\dagger}, [001^{2k}]^{\dagger}, [001^{2j}]_{j=1}^{k-1}, [001^{2j+1}]_{j=0}^{k-2}, [00], [\,], \\
&\quad [\alpha 0001^{2k-1}], [\beta 101^{2j}]_{j=1}^{k-1}, [\beta 01^{2j+1}]_{j=0}^{k-1}, [\alpha 001^{2k-1}0] \\
&\mid \quad [\alpha' 001^{2k}]^{*}, [001^{2k}]^{*}, [01^{2k}], [001^{2k-1}]\} \\
&\equiv_I \{k^{\dagger}, \{k+1 \mid k\}^{\dagger}, \{j+1 \mid j\}_{j=1}^{k-1}, \{j+1\}_{j=0}^{k-2}, \{1 \mid 0\}, 0, \\
&\quad k-1, \{j+1 \mid j\}_{j=1}^{k-1}, \{j+1\}_{j=0}^{k-1}, \{1 \mid 0\} \\
&\mid \quad k^{*}, \{k+1 | k\}^{*}, k+1, k\} \\
&\equiv_I k, \quad \text{by Lemma 2.7.}
\end{aligned}$$

Case 4:

$$\begin{aligned}[\alpha 101^{2k}] &= \{[\alpha' 101^{2k}]^{\dagger}, [0001^{2k}]^{\dagger}, [01^{2k}], [001^{2j+1}]_{j=0}^{k-2}, [001^{2j}]_{j=1}^{k-1}, [00], [\,],\\ &\quad [\alpha 001^{2k}], [\alpha 1001^{2k-1}], [\beta 101^{2j}]_{j=1}^{k-1}, [\beta 01^{2j+1}]_{j=0}^{k-2}, [\beta 10]\\ &\quad \mid \quad [\alpha' 101^{2k}]^{*}, [0001^{2k}]^{*}, [001^{2k-1}]\}\\ &\equiv_I \{\{k+1 \mid k\}^{\dagger}, k^{\dagger}, k+1, \{j+1\}_{j=0}^{k-2}, \{j+1 \mid j\}_{j=1}^{k-1}, \{1 \mid 0\}, 0,\\ &\quad k, k-1, \{j+1 \mid j\}_{j=1}^{k-1}, \{j+1\}_{j=0}^{k-2}, \{1 \mid 0\}\\ &\quad \mid \quad \{k+1 | k\}^{*}, k^{*}, k\}\\ &\equiv_I \{k+1 \mid k\}, \quad \text{by Corollary 2.9.}\end{aligned}$$

Case 5:

$$\begin{aligned}[001^{2k}] &= \{[01^{2k}], [001^{2j}]_{j=1}^{k-1}, [001^{2j+1}]_{j=0}^{k-2}, [00], [\,],\\ &\quad [0001^{2k-1}], [\beta 101^{2j}]_{j=1}^{k-1}, [\beta 01^{2j+1}]_{j=0}^{k-2}, [\beta 10]\\ &\quad \mid \quad [001^{2k-1}]\}\\ &\equiv_I \{k+1, \{j+1 \mid j\}_{j=1}^{k-1}, \{j+1\}_{j=0}^{k-2}, \{1 \mid 0\}, 0,\\ &\quad k, \{j+1 \mid j\}_{j=1}^{k-1}, \{j+1\}_{j=0}^{k-2}, \{1 \mid 0\}\\ &\quad \mid \quad k\}\\ &\equiv_I \{k+1 \mid k\}, \quad \text{by Corollary 2.9.}\end{aligned}$$

In the next case, we claim that the canonical form is simple.

Case 6: We claim that $[01^{2k}] = k+1$. Consider first the base case: $k = 1$,

$$\begin{aligned}[011] &= \{[001], [00], [\,], [010] \mid \cdot\}\\ &= \{\{1 \mid \{1 \mid 0\}\}, \{1 \mid 0\}, 0, \{1 \mid 0\} \mid \cdot\}\\ &= 2\end{aligned}$$

We need to show that $[01^{2k}] - k - 1 = 0$. If Right moves first, he only has one move which is to $[01^{2k}] - k$. Left's best move is to $[001^{2k-1}] - k$ (removing the top leaf arc). From here, Right has two possibilities. He could move to $[001^{2k-1}] - k + 1$ and Left moves to $[01^{2k-1}] - k + 1$ but $[01^{2k-1}] = k$ (by induction) and so $[01^{2k-1}] - k + 1 = k - k + 1 = 1 > 0$. Otherwise, Right moves to $[001^{2k-2}] - k$ and Left moves to $[01^{2k-2}] - k = 0$ (by induction). So Right loses going first.

Next we check Left moving first in $[01^{2k}] - k - 1$. Left can move to $[\,] - k - 1 < 0$ by taking $\overrightarrow{v_1 v_0}$, and loses. Left could move to $[00] - k - 1$ and Right wins by moving to $-k-1$ (deleting $\overrightarrow{v_1 v_0}$). She could remove a stalk arc which results in a cordon with an even number of leaves: $[001^{2j}] - k - 1 \leq_I \{k | k-1\} - k - 1 \leq_I \{-1 | -2\} < 0$ $(j \leq k-1)$ and loses (by case 3). Or she could remove a stalk arc which results in a cordon with an odd number of leaves, $[001^{2j-1}] - k - 1 \leq_I k - k - 1 \leq_I -1 < 0$ (case 2), and Left loses. What remains to check are the options where Left removes a leaf arc:

1) $[\beta 01^{2j-1}] - k - 1 \leq_I k - k - 1 \leq_I -1 < 0$ ($j \leq k-2$); (case 2)
2) $[\beta 101^{2j}] - k - 1 \leq_I \{k|k-1\} - k - 1 < 0$; (case 4)
3) $[\beta 10] - k - 1 \equiv_I \{1|0\} - k - 1 \leq \{-k| - k - 1\} < 0$, $k \geq 2$. (case 1)

As all three options result in a negative value, Left loses moving first when removing a leaf arc. Hence $G = [01^{2k}] = k + 1$.

Let $k \geq 1$. We claim $[01^{2k+1}] = k + 1$. Consider first the base case: $k = 1$,

$$\begin{aligned}
[0111] &= \{[0011], [001], [00], [\,], [0101], [0110] \mid \cdot\} \\
&= \{\{2 \mid \{1 \mid \{1|0\}\}\}, \{1 \mid \{1|0\}\}, \{1|0\}, 0, \{1 \mid \{1|0\}\}, \{1, \{1|0\} \mid 0\} \mid \cdot\} \\
&= 2
\end{aligned}$$

We need to show that $[01^{2k+1}] - k - 1 = 0$. If Right moves first in $[01^{2k+1}] - k - 1$, he only has one move which is to $[01^{2k+1}] - k$ and Left responds to $[001^{2k}] - k$. Right moves to one of the following positions: 1) $[001^{2k-1}] - k$ and Left responds to $[01^{2k-1}] - k = 0$ by induction; 2) $[001^{2k-1}] - k + 1 \equiv_I k - k + 1 > 0$ (case 2). Hence, in both cases Right loses moving first.

Now we consider Left moving first in $[01^{2k+1}] - k - 1$. All moves except $\overrightarrow{v_n v_{n-1}}$ and $\overrightarrow{v_{n-1} v_{n-2}}$ are considered in the previous paragraph. The two extra cases are:

1) $[001^{2k}] - k - 1$ and Right responds to $[001^{2k-1}] - k - 1 \equiv_I k - k - 1 < 0$;
2) $[001^{2k-1}] - k - 1 \equiv_I k - k - 1 < 0$,

both of which follow from case 2.

In all situations, Left loses moving first, so $[01^{2k+1}] = k + 1$. □

In a RED-BLUE STALKS position, the top arc trivially has even in-degree (0) and all other (interior) arcs have odd in-degree. Therefore, in a RED-BLUE STALKS position Left can remove a blue top arc or any red interior arc and Right can remove a red top arc or any blue interior arc.

The notation we use for RED-BLUE STALKS places emphasis on who can move (Left (L) or Right (R)) rather than the color of the arc. For example, in Figure 5, the positions listed from top to bottom, in order from left to right, are RLLR + RRLR + LRLL. Note that removing an arc changes the symbol immediately below (to the right).

Theorem 4.2. RED-BLUE STALKS *has the following classification:*

1. *If* $G \in \{LR^k, \alpha LLR^k, RL^k, \alpha RRL^k, k \geq 2\}$, *then* $G \equiv_I 0$.
2. (a) *If* $G \in \{LR, \alpha LLR, \alpha LRL^k, k \geq 3\}$, *then* $G \equiv_I \{1|0\}$.

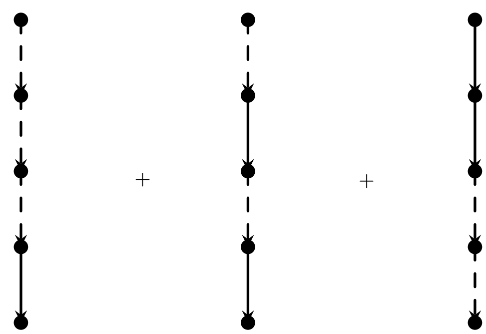

Figure 5: RED-BLUE STALKS, where → is blue, ⇢ is red.

(b) *If* $G \in \left\{RL, \alpha RRL, \alpha RLR^k, k \geq 3\right\}$, *then* $G \equiv_I \{0|-1\}$.

3. (a) *If* $G = L^k$, $k \geq 1$, *then* $G = 1$.

   (b) *If* $G = R^k$, $k \geq 1$, *then* $G = -1$.

4. *If* $G \in \{\alpha LRL, \alpha RLR\}$, *then* $G \equiv_I \{1|-1\}$.

5. (a) *If* $G = \alpha RLRR$, *then* $G \equiv_I \{\{1|0\}\,|-1\}$.

   (b) *If* $G = \alpha LRLL$, *then* $G \equiv_I \{1|\,\{0|-1\}\}$.

There are no restrictions on α.

Proof. For cases 1 and 4 (cases which encompass a position and its negative), we only show the proof for some of the positions and leave the proof of negatives to the reader. We also show cases 2a, 3a, and 5a. Cases which are not directly shown, namely 2b, 3b, 5b, are negatives of cases 2a, 3a, and 5a, respectively. For the rest of the analysis we consider the moves in the order: 1) in α; and 2) any moves on the stalk (not in α). Options are marked by † if, among all such marked options for a player, at least one such option exists for that player. A move which may or may not exist for a player is marked by $*$.

`Case 1:`

$$\begin{aligned} LR^k, k\geq 2 &= \{LR^{k-1\dagger}, LR^{\dagger} \quad \mid \quad \left\{LR^j\right\}_{j=2}^{k-2*}, LR^*, L, \emptyset\} \\ &\equiv_I \{0^{\dagger}, \{1\mid 0\}^{\dagger} \quad \mid \quad 0^*, \{1\mid 0\}^*, 1, 0\} \\ &\equiv_I 0, \quad \text{by Lemma 2.7.} \end{aligned}$$

$$\begin{aligned} \alpha LLR^k, k\geq 3 &= \{\alpha' LLR^{k*}, RLR^{k*}, R^{k+1}, LR^{k-1} \\ &\qquad \mid \quad \alpha' LLR^{k*}, RLR^{k*}, \left\{LR^j\right\}_{j=2}^{k-2*}, LR, L, \emptyset\} \\ &\equiv_I \{0^*, \{0\mid -1\}^*, -1, 0 \\ &\qquad \mid \quad 0^*, \{0\mid -1\}^*, 0^*, \{1\mid 0\}, 1, 0\} \\ &\equiv_I 0, \quad \text{by Lemma 2.7.} \end{aligned}$$

$$\begin{aligned} \alpha LLRR &= \{\alpha' LLRR^*, RLRR^*, RRR, LR \quad \mid \quad \alpha' LLRR^*, RLRR^*, L, \emptyset\} \\ &\equiv_I \{0^*, \{\{1\mid 0\}\mid -1\}^*, -1, \{1\mid 0\} \quad \mid \quad 0^*, \{\{1\mid 0\}\mid -1\}^*, 1, 0\} \\ &\equiv_I 0, \quad \text{by Lemma 2.7.} \end{aligned}$$

`Case 2a:` Note that when $\alpha = \emptyset$ we have equality:

$$LR = \{L\mid \emptyset\} = \{1\mid 0\} \text{ and } LLR = \{RR, L\mid \emptyset\} = \{-1, 1\mid 0\} = \{1\mid 0\}$$

When $\alpha \neq \emptyset$ we do not have equality:

$$\begin{aligned} \alpha LLR &= \{\alpha' LLR^*, RLR^*, RR, L \quad \mid \quad \alpha' LLR^*, RLR^*, \emptyset\} \\ &\equiv_I \{\{1\mid 0\}^*, \{1\mid -1\}^*, -1, 1 \quad \mid \quad \{1\mid 0\}^*, \{1\mid -1\}^*, 0\} \\ &\equiv_I \{1\mid 0\}, \quad \text{by Corollary 2.9.} \end{aligned}$$

$$\begin{aligned} \alpha LRL^k, k\geq 3 &= \{\alpha' LRL^{k*}, RRL^{k*}, L^{k+1}, \left\{RL^j\right\}_{j=2}^{k-2}, RL, R, \emptyset \\ &\qquad \mid \alpha' LRL^{k*}, RRL^{k*}, RL^{k-1}\} \\ &\equiv_I \{\{1\mid 0\}^*, 0^*, 1, 0, \{0\mid -1\}, -1, 0 \\ &\qquad \mid \{1\mid 0\}^*, 0^*, 0\} \\ &\equiv_I \{1\mid 0\}, \quad \text{by Corollary 2.9.} \end{aligned}$$

`Case 3a:` Claim: $L^k - 1$ is a second player win, for $k \geq 1$.

Proof. Left moving first takes the top arc from L^k leaving $RL^{k-2} - 1$ (if she takes anything lower, she is only eliminating moves for herself). Right responds by playing in RL^{k-2} to $RL^{k-3} - 1$. As long as Left doesn't play $\overrightarrow{v_1v_0}$, Right will always have a move in RL^j, for $j < k-2$ and hence can run Left out of moves and saving -1 for his last move, and wins. If Left plays $\overrightarrow{v_1v_0}$ (the bottom arc), Right responds in -1, and wins the game.

Right moving first, he only has one option which is to move in -1 to 0. Then Left takes the bottom arc on L^k leaving 0, and hence Left wins. □

Case 4:

$$\begin{aligned}\alpha LRL &= \{\alpha' LRL^*, RRL^*, LL, \emptyset \;\mid\; \alpha' LRL^*, RRL^*, R\}\\ &\equiv_I \{\{1 \mid -1\}^*, \{0 \mid -1\}^*, 1, 0 \;\mid\; \{1 \mid -1\}^*, \{0 \mid -1\}^*, -1\}\\ &\equiv_I \{1 \mid -1\}, \quad \text{by Corollary 2.9.}\end{aligned}$$

Case 5a:

$$\begin{aligned}\alpha RLRR &= \{\alpha' RLRR^*, LLRR^*, LR \;\mid\; \alpha' RLRR^*, LLRR^*, RRR, L, \emptyset\}\\ &\equiv_I \{\{\{1 \mid 0\} \mid -1\}^*, 0^*, \{1 \mid 0\} \;\mid\; \{\{1 \mid 0\} \mid -1\}^*, 0^*, -1, 1, 0\}\\ &\equiv_I \{\{1 \mid 0\} \mid -1\}, \quad \text{by Corollary 2.9.}\end{aligned}$$

□

4.1. Temperature

Let $T(n)$ be the THINNING THICKETS cordon of height $n+1$ where all internal vertices have leaves, and every arc is blue, except for $\overrightarrow{v_1v_0}$ which is red. Right has only one move, he can only delete $\overrightarrow{v_1v_0}$ which deletes the whole cordon. Left has many moves but we only need to consider one.

Let $S(n)$ be the THINNING THICKETS cordon $T(n)$ without a leaf arc on v_1. Let $U(n)$ be defined as $S(n)$ without a leaf arc on v_n. It follows that $S(n) \geq \{U(n) \mid \cdot\}$ since Right has no move and Left can move to $U(n)$. In $U(n)$, Right's only move is to remove the arc $\overrightarrow{v_nv_{n-1}}$ and Left can move to $S(n-1)$ by removing $\overrightarrow{v_{n+1}v_n}$. Therefore, $U(n) \geq \{S(n-1) \mid U(n-1)\}$. THINNING THICKETS positions $T(n)$, $S(n)$, and $U(n)$ are pictured in Figure 6.

Lemma 4.3. *For $n \geq 1$, both $S(2n)$ and $S(2n+1)$ are greater than or equal to n.*

Proof. Our proof is by induction on the height of the cordon. First, both $S(1), S(2) \geq 1$ since Right has no move and Left can move to 0.

Consider $S(2n+1) - n$, $n > 0$. Right only has one move; that is to $S(2n+1) - (n-1)$. From here, Left responds to $U(2n) - (n-1)$. Right has two options: (i) $U(2n) - (n-2)$ or (ii) $U(2n-1) - (n-1)$. From (i), Left moves to $S(2n-1) - (n-2) \geq$

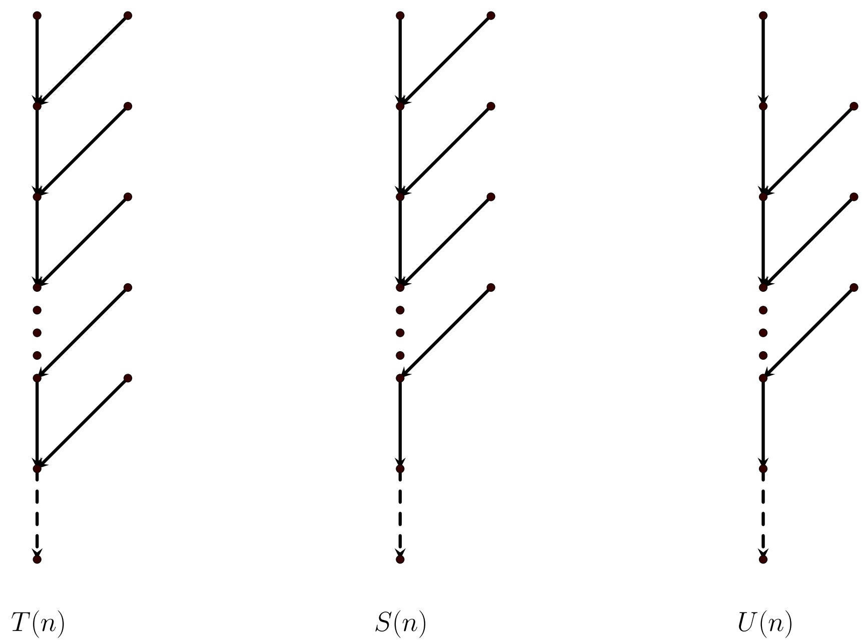

Figure 6: Cordons $T(n)$, $S(n)$ and $U(n)$, where $\rightarrow$ is blue, $\dashrightarrow$ is red.

0, by induction. From (ii), Left responds to $S(2n-2)-(n-1) \geq 0$, by induction. Hence $S(2n+1)-n \geq 0$.

In $S(2n)-n$, Right moves to $S(2n)-(n-1)$ and Left responds to $U(2n)-(n-1)$. From here, either (i) Right moves to $U(2n-1)-(n-1)$ and Left responds to $S(2n-2)-(n-1) \geq 0$, by induction, or (ii) Right moves to $U(2n)-(n-2)$ and Left moves to $S(2n-1)-(n-2) \geq 0$, by induction. □

The next theorem follows from Lemma 4.3. For more information on Left and Right scaffolds see [2].

Theorem 4.4. *The temperature of $T(2n)$ is at least n.*

Proof. The Left options of $T(2n)$ include $S(2n)$ and the only Right option is 0. Since $S(2n) \geq n$ then the Right stop of $S(2n) \geq n$. It now follows that the Left scaffold of $T(2n)$ is at least $S(2n)_t - t \geq 2n - t$. The Right scaffold is $0 + t$. The temperature of $T(2n)$ is at least the value of t when these two lines intersect, namely n. □

Corollary 4.5. *For any positive integer n there is a* THINNING THICKETS *position with temperature greater than n.*

5. Open Problems

Since the nim-dimension is infinite, we know there are positions with value $*n$ for all n.

Using CGSuite [15], we were able to find all the positions up to height 10 but no clear patterns emerged. Using similar techniques, to those in Section 3.2, we can show:

- A GREEN CORDONS position with one leaf arc has nim-value 2 if and only if $n-a(1)$ is even.
- A GREEN CORDONS position with two leaves has nim-value 2 if and only if $n-a(2)$ is odd and $a(2)-a(1)$ is even.
- A GREEN CORDONS position with two leaves has nim-value 3 if and only if $n-a(2)$ is even.

Problem 5.1. Is there a characterization, similar to that for nim-values 0 and 1, for GREEN CORDONS with nim-value n for $n \geq 2$?

Problem 5.2. Are there THINNING THICKETS positions with values $\{0|\,\{0|-n\}\}$ for all n?

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