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A MAKER–BREAKER GAME ON THE BOOLEAN HYPERCUBE WITH SUBCUBES AS WINNING SETS

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Abstract

We consider the Maker–Breaker positional game on the vertices of the n-dimensional hypercube $\{0, 1\}^n$ with k-dimensional subcubes as winning sets. In the general case, we give a lower bound on k which guarantees the existence of a winning strategy for Breaker, and an upper bound on k which guarantees the existence of a winning strategy for Maker. We also consider the problem of determining for which values of k can Breaker win with a pairing strategy. We prove that Breaker can win with a pairing strategy if $k = n-3$ and $n \geq 7$. We also use a graph theoretical approach to prove that Maker can occupy a 3-dimensional subcube if $n \geq 23$.

1. Introduction

A positional game is a generalization of Tic-Tac-Toe played on a hypergraph $(V, \mathcal{H})$ where the vertices can be considered the "board" on which the game is played, and the edges can be thought of as the "winning sets." A *positional game* on $(V, \mathcal{H})$ is a two-player game where at every turn each player alternately occupies a previously unoccupied vertex from V. In a *strong positional game*, the first player to occupy all vertices of some edge $A \in \mathcal{H}$ wins. If at the end of play no edge is completely occupied by either player, that play is declared a draw. Normal 3×3 Tic-Tac-Toe is a strong positional game where the vertices of the hypergraph are the nine positions and the edges are the eight winning lines. In a *Maker–Breaker* positional game, the first player, *Maker*, wins if she[1] occupies all vertices of some edge $A \in \mathcal{H}$, otherwise the second player, *Breaker*, wins. Therefore, by definition there are no draw plays in Maker–Breaker games. We say that a player P has a *winning strategy* if no matter

[1]In this paper, we will refer to Maker with feminine pronouns, such as "she" and "her," and we will refer to Breaker with masculine pronouns, such as "he."

how the other player plays, player P wins by following that winning strategy. It is well-known that in a finite Maker–Breaker game, exactly one player has a winning strategy. (For a nice introduction to positional games, please see [1], [2], and [5].)

We will consider a Maker–Breaker game on the n-dimensional boolean hypercube. Let Q_n be the set of vertices of the *n-dimensional boolean hypercube*, i.e., all n-tuples with entries from $\{0,1\}$. Thus, $Q_n = \{(x_1,\ldots,x_n) : x_i \in \{0,1\} \text{ for } 1 \le i \le n\}$. Clearly, $|Q_n| = 2^n$. A k-dimensional subcube of Q_n is formed by selecting $n-k$ coordinates to be fixed, choosing fixed values for each of those coordinates, then allowing the remaining k coordinates to take on all 2^k possible values. Thus, each k-dimensional subcube of Q_n has cardinality 2^k, and there are $\binom{n}{n-k}2^{n-k}$ distinct k-dimensional subcubes contained in Q_n. Let $\mathcal{Q}(n,k)$ denote the hypergraph whose vertex set is Q_n and whose edge set is the set of all k-dimensional subcubes of Q_n. For the Maker–Breaker positional game played on $\mathcal{Q}(n,k)$, we would like to determine for each n and k whether Maker will have a winning strategy, or Breaker will have a winning strategy. We do not expect to achieve that goal for every n and k, yet we obtain some partial results. (We note that throughout this paper, we use $\lg n$ for $\log_2 n$.)

Lemma 1. *If Breaker has a winning pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(n,k)$, then Breaker has a winning pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(n+1,k+1)$.*

Proposition 1. *For $n \ge 3$, Breaker has a winning pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(n,n-1)$.*

Proposition 2. *For $n \ge 4$, Breaker has a winning pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(n,n-2)$.*

Proposition 3. *For $n \ge 7$, Breaker has a winning pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(n,n-3)$.*

Proposition 4. *Maker has a winning strategy for the Maker–Breaker game played on $\mathcal{Q}(5,2)$.*

Proposition 5. *Maker has a winning strategy for the Maker–Breaker game played on $\mathcal{Q}(23,3)$.*

Proposition 6. *Breaker has a winning strategy for the Maker–Breaker game played on $\mathcal{Q}(6,3)$.*

Proposition 7. *If $k \ge \lg n+1$, then Breaker has a winning strategy for the Maker–Breaker game played on $\mathcal{Q}(n,k)$.*

Proposition 8. *If $k \le \lg\lg n - 1$, then Maker has a winning strategy for the Maker–Breaker game played on $\mathcal{Q}(n,k)$.*

The proofs of Propositions 1, 2, and 3 in Section 2 all use induction on n with Lemma 1 serving as the inductive step. Thus, the pairing strategy for the basis step becomes the challenge in each of the proofs, i.e., the challenge is to find a winning pairing strategy for Breaker for the $\mathcal{Q}(3,2)$, $\mathcal{Q}(4,2)$, and $\mathcal{Q}(7,4)$ games, respectively. The proof of Proposition 4 is a straightforward, almost brute force, approach for Maker. The proof we have for Proposition 5 makes use of some well-known facts from graph theory, while the proofs of Propositions 6 and 7 are straightforward applications of the Erdős–Selfridge theorem [4], which states that if $\mathcal{H}$ is an n-uniform, finite hypergraph and

$$|\mathcal{H}| + \mathit{MaxDeg}(\mathcal{H}) < 2^n \tag{1}$$

(where $\mathit{MaxDeg}(\mathcal{H}) = \max_{v\in V(\mathcal{H})} \#\{A \in \mathcal{H} : v \in A\}$), then Breaker has an explicit winning strategy for the Maker–Breaker game played on $\mathcal{H}$. The proof of Proposition 8 is similar to the proof of Proposition 7, except it relies on a Maker's win criterion given by Beck in [1].

It should be fairly obvious that if Maker has a winning strategy for $\mathcal{Q}(n,k)$, then for all $N \geq n$, Maker also has a winning strategy for $\mathcal{Q}(N,k)$. (Maker can simply restrict her moves to an n-dimensional subcube and use her strategy for $\mathcal{Q}(n,k)$.) Likewise, if Breaker has a winning strategy for $\mathcal{Q}(n,k)$, then for all $K \geq k$, Breaker has a winning strategy for $\mathcal{Q}(n,K)$. (Every K-dimensional winning set will contain a k-dimensional subcube of which Breaker occupies at least one vertex if Breaker uses his winning strategy for $\mathcal{Q}(n,k)$.) Thus, for a given value of n, it makes sense to speak of a threshold or breaking point $k(n)$ where Maker has a winning strategy for $\mathcal{Q}(n,k(n))$, but Breaker has a winning strategy for $\mathcal{Q}(n,k(n)+1)$. It is obvious that Maker has a winning strategy for $\mathcal{Q}(n,1)$ when $n \geq 2$, thus, $k(2) = 1$, i.e., Maker wins $\mathcal{Q}(2,1)$, but Breaker wins $\mathcal{Q}(2,2)$; moreover, Propositions 1 and 2 determine that $k(3) = k(4) = 1$, i.e., Maker wins $\mathcal{Q}(3,1)$ and $\mathcal{Q}(4,1)$, but Breaker wins $\mathcal{Q}(3,2)$ and $\mathcal{Q}(4,2)$. Propositions 2, 4, and 6 determine that $k(5) = k(6) = 2$, i.e., Maker wins $\mathcal{Q}(5,2)$ and $\mathcal{Q}(6,2)$, but Breaker wins $\mathcal{Q}(5,3)$ and $\mathcal{Q}(6,3)$. For $n \geq 16$, Propositions 7 and 8 provide us with a starting point for bounding the value of $k(n)$.

The remainder of the paper is organized as follows. In Section 2, we prove Lemma 1 and Propositions 1, 2, and 3. In Section 3, we provide alternate proofs for Propositions 1, 2, and 3 with the exception that these proofs require a larger value of n than is stated in the propositions, but use pairing strategies that requires less pairs than the inductive proofs in Section 2. In Section 4, we prove Propositions 4 and 5, and in Section 5, we prove Propositions 6, 7, and 8.

2. Pairing Strategies for Breaker

We begin with some notation. We will frequently denote a k-dimensional subcube of an n-dimensional hypercube by using a vector of length n where k of the coordinates are $*$'s and the other $n-k$ coordinates are 0's or 1's. For example, we will write $(1,*,*,0)$ for the 2-dimensional subcube $\{(1,0,0,0),(1,0,1,0),(1,1,0,0),(1,1,1,0)\}$ which is contained in the 4-dimensional hypercube. Notice that the $*$'s correspond to the coordinates which can be different for two distinct vectors from the subcube. Breaker will use a pairing strategy in the proofs, thus, for $\vec{v},\vec{w} \in Q_n$, we will write $\vec{v} \leftrightarrow \vec{w}$ if Breaker pairs together those two vectors in his strategy. Moreover, if $\vec{v} \leftrightarrow \vec{w}$, then whenever Maker occupies $\vec{v}$ (respectively, $\vec{w}$), Breaker immediately responds with $\vec{w}$ (respectively, $\vec{v}$) if that vector is unoccupied; otherwise Breaker occupies an arbitrary vector. Suppose S is a k-dimensional subcube. If $\vec{v} \leftrightarrow \vec{w}$ and $\{\vec{v},\vec{w}\} \subseteq S$, then we will say $\{\vec{v},\vec{w}\}$ *handles* S because Breaker will occupy one of $\vec{v}$ or $\vec{w}$ by the end of the game, thus, Maker cannot completely occupy all vectors in S. If $\vec{v} \in Q_n$, we will use $(\vec{v},c)$ to denote the $(n+1)$-dimensional vector whose first n coordinates are identical to the n coordinates of $\vec{v}$ and whose $(n+1)$th coordinate is c, where $c \in \{0,1\}$.

Proof of Lemma 1. Suppose Breaker has a winning pairing strategy for $\mathcal{Q}(n,k)$. We will obtain a pairing strategy for Breaker for $\mathcal{Q}(n+1,k+1)$ by doubling Breaker's pairing strategy for $\mathcal{Q}(n,k)$ as follows. If $\vec{v} \leftrightarrow \vec{w}$ in Breaker's pairing strategy for $\mathcal{Q}(n,k)$, then $(\vec{v},c) \leftrightarrow (\vec{w},c)$ for $c \in \{0,1\}$ in Breaker's pairing strategy for $\mathcal{Q}(n+1,k+1)$. We must show that Breaker's pairing strategy for $\mathcal{Q}(n+1,k+1)$ is a winning strategy.

Let S be an arbitrary $(k+1)$-dimensional subcube of Q_{n+1}. Let $i_1 < \cdots < i_{n-k}$ be the $n-k$ coordinates whose entries remain constant across all vectors in S, and let $j_1 < \cdots < j_{k+1}$ be the $k+1$ coordinates whose entries can change as we vary the vectors in S.

Suppose that $i_{n-k} < n+1$, thus, $j_{k+1} = n+1$ and the entry in the last coordinate does not remain constant across all vectors in S. Let S' be formed by taking each vector in S and truncating its $(n+1)$th coordinate; thus, $S' \subset Q_n$. Since $i_{n-k} < n+1$, then $i_1,\ldots,i_{n-k}$ are exactly the $n-k$ coordinates whose entries remain constant across all vectors in S' and $j_1,\ldots,j_k$ are the k coordinates whose entries can change as we vary the vectors in S'. Notice that S' is a k-dimensional subcube of Q_n. Thus, there exists a pair of vectors $\{\vec{v},\vec{w}\} \subset Q_n$ such that $\vec{v} \leftrightarrow \vec{w}$ in Breaker's winning pairing strategy for $\mathcal{Q}(n,k)$ and $\{\vec{v},\vec{w}\} \subset S'$. Since $\vec{v} \leftrightarrow \vec{w}$, then $(\vec{v},0) \leftrightarrow (\vec{w},0)$. Notice that $\{(\vec{v},0),(\vec{w},0)\} \subset S$, thus, $\{(\vec{v},0),(\vec{w},0)\}$ handles S.

Now suppose that $i_{n-k} = n+1$. Let c_{n-k} be the value of coordinate $i_{n-k} = n+1$ for each vector S. As before, let S' be formed by taking each vector in S and truncating its $(n+1)$th coordinate. However, since $i_{n-k} = n+1$, then $i_1,\ldots,i_{n-k-1}$ are exactly the $n-k-1$ coordinates whose entries remain constant across all vectors in S' and $j_1, \ldots, j_{k+1}$ are the $k+1$ coordinates whose entries can change as we vary the vectors in S'. Notice that S' is a $(k+1)$-dimensional subcube of Q_n. Nonetheless, we can create a k-dimensional subcube S'' of S', as follows. For each vector in S' set the value of coordinate j_{k+1} equal to, say, 0 and leave all of the other coordinates of the vector alone. Thus, $i_1, \ldots, i_{n-k-1}$ and j_{k+1} are exactly the $n-k$ coordinates whose entries remain constant across all vectors in S'' and $j_1, \ldots, j_k$ are the k coordinates whose entries can change as we vary the vectors in S''. Since S'' is a k-dimensional subcube of Q_n, then there exists a pair of vectors $\{\vec{v}, \vec{w}\} \subset Q_n$ such that $\vec{v} \leftrightarrow \vec{w}$ in Breaker's winning pairing strategy for $\mathcal{Q}(n,k)$ and $\{\vec{v}, \vec{w}\} \subset S''$. Since $\vec{v} \leftrightarrow \vec{w}$, then $(\vec{v}, c_{n-k}) \leftrightarrow (\vec{w}, c_{n-k})$. Notice that $\{(\vec{v}, c_{n-k}), (\vec{w}, c_{n-k})\} \subset S$, and thus, $\{(\vec{v}, c_{n-k}), (\vec{w}, c_{n-k})\}$ handles S. □

Proof of Proposition 1. Our proof proceeds by induction on n with $n = 3$ as the base case. We begin by proving that Breaker can win $\mathcal{Q}(3,2)$ by using a pairing strategy. Breaker uses the following pairing strategy to win $\mathcal{Q}(3,2)$:

$$(0,0,0) \leftrightarrow (1,0,0),$$
$$(1,1,0) \leftrightarrow (1,1,1),$$
$$(0,1,1) \leftrightarrow (0,0,1).$$

The pair $\{(0,0,0),(1,0,0)\}$ handles the subcubes $(*,*,0)$ and $(*,0,*)$; the pair $\{(1,1,0),(1,1,1)\}$ handles the subcubes $(*,1,*)$ and $(1,*,*)$; and the pair $\{(0,1,1),$ $(0,0,1)\}$ handles the subcubes $(*,*,1)$ and $(0,*,*)$. Since all 2-dimensional subcubes are handled by some pair, this is a winning pairing strategy for Breaker for $\mathcal{Q}(3,2)$. Therefore, by Lemma 1 and induction, Breaker has a winning pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(n,n-1)$ for $n \geq 3$. □

We note that we created the pairing strategy for $\mathcal{Q}(3,2)$ by first building a length 6 cycle in Q_3 that begins at $(0,0,0)$ and proceeds by changing the first coordinate, then the second coordinate, then the third, then repeats that same sequence of coordinate changes whereupon we return to $(0,0,0)$. Once we have this 6-cycle, we pair together the first and second vectors, the third and fourth vectors, and the fifth and sixth vectors.

Proof of Proposition 2. Our proof proceeds by induction on n with $n = 4$ as the base case. We begin by proving that Breaker can win $\mathcal{Q}(4,2)$ by using a pairing strategy. Breaker uses the following pairing strategy to win $\mathcal{Q}(4,2)$:

$$\begin{aligned}
(0,0,0,0) &\leftrightarrow (1,0,0,0),\\
(1,1,0,0) &\leftrightarrow (1,1,1,0),\\
(1,1,1,1) &\leftrightarrow (0,1,1,1),\\
(0,0,1,1) &\leftrightarrow (0,0,0,1),\\
(0,0,1,0) &\leftrightarrow (0,1,1,0),\\
(0,1,0,0) &\leftrightarrow (0,1,0,1),\\
(1,1,0,1) &\leftrightarrow (1,0,0,1),\\
(1,0,1,1) &\leftrightarrow (1,0,1,0).
\end{aligned}$$

The pair $\{(0,0,0,0),(1,0,0,0)\}$ handles $(*,*,0,0),(*,0,*,0),(*,0,0,*)$.
The pair $\{(1,1,0,0),(1,1,1,0)\}$ handles $(*,1,*,0),(1,*,*,0),(1,1,*,*)$.
The pair $\{(1,1,1,1),(0,1,1,1)\}$ handles $(*,*,1,1),(*,1,*,1),(*,1,1,*)$.
The pair $\{(0,0,1,1),(0,0,0,1)\}$ handles $(*,0,*,1),(0,*,*,1),(0,0,*,*)$.
The pair $\{(0,0,1,0),(0,1,1,0)\}$ handles $(*,*,1,0),(0,*,*,0),(0,*,1,*)$.
The pair $\{(0,1,0,0),(0,1,0,1)\}$ handles $(*,1,0,*),(0,*,0,*),(0,1,*,*)$.
The pair $\{(1,1,0,1),(1,0,0,1)\}$ handles $(*,*,0,1),(1,*,*,1),(1,*,0,*)$.
The pair $\{(1,0,1,1),(1,0,1,0)\}$ handles $(*,0,1,*),(1,*,1,*),(1,0,*,*)$.

Since all 2-dimensional subcubes are handled by some pair, this is a winning pairing strategy for Breaker for $\mathcal{Q}(4,2)$. Therefore, by Lemma 1 and induction, Breaker has a winning pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(n,n-2)$ for $n \geq 4$. □

We note that we created the pairing strategy for $\mathcal{Q}(4,2)$ by building two length 8 cycles in Q_4. The first 8-cycle can be constructed by starting at the vector $(0,0,0,0)$, then changing the first coordinate, then the second, then the third, then the fourth, then repeating this same sequence of coordinate changes whereupon we return to $(0,0,0,0)$. Once we have this 8-cycle (in this order), we pair together the first and second vectors, the third and fourth vectors, the fifth and sixth vectors, and the seventh and eighth vectors. The second 8-cycle can be constructed by adding the vector $(1,0,1,0)$ to each vector in the first 8-cycle, where addition is performed mod 2. Thus, we obtain the cycle $((1,0,1,0),(0,0,1,0),(0,1,1,0),\ldots,(1,0,1,1))$. However, for the pairing strategy to work, (using this ordering of the second 8-cycle) we pair together the second and third vectors, the fourth and fifth vectors, the sixth and seventh vectors, and the eighth and first vectors.

Proof of Proposition 3. Our proof proceeds by induction on n with $n = 7$ as the base case. We need to prove that Breaker can win $\mathcal{Q}(7,4)$ by using a pairing strategy.

The pairing strategy that we discovered contains 39 pairs, thus, we will simply list the pairs in five groups of 7, where each group of 7 corresponds to a 14-cycle and there is another group of four extra pairs listed together. Breaker uses the following pairing strategy to win $\mathcal{Q}(7,4)$:

$(0,0,0,0,0,0,0) \leftrightarrow (1,0,0,0,0,0,0)$,
$(1,1,0,0,0,0,0) \leftrightarrow (1,1,1,0,0,0,0)$,
$(1,1,1,1,0,0,0) \leftrightarrow (1,1,1,1,1,0,0)$,
$(1,1,1,1,1,1,0) \leftrightarrow (1,1,1,1,1,1,1)$,
$(0,1,1,1,1,1,1) \leftrightarrow (0,0,1,1,1,1,1)$,
$(0,0,0,1,1,1,1) \leftrightarrow (0,0,0,0,1,1,1)$,
$(0,0,0,0,0,1,1) \leftrightarrow (0,0,0,0,0,0,1)$,

$(0,1,1,1,0,1,0) \leftrightarrow (1,1,1,1,0,1,0)$,
$(1,0,1,1,0,1,0) \leftrightarrow (1,0,0,1,0,1,0)$,
$(1,0,0,0,0,1,0) \leftrightarrow (1,0,0,0,1,1,0)$,
$(1,0,0,0,1,0,0) \leftrightarrow (1,0,0,0,1,0,1)$,
$(0,0,0,0,1,0,1) \leftrightarrow (0,1,0,0,1,0,1)$,
$(0,1,1,0,1,0,1) \leftrightarrow (0,1,1,1,1,0,1)$,
$(0,1,1,1,0,0,1) \leftrightarrow (0,1,1,1,0,1,1)$,

$(0,1,0,1,1,1,0) \leftrightarrow (1,1,0,1,1,1,0)$,
$(1,0,0,1,1,1,0) \leftrightarrow (1,0,1,1,1,1,0)$,
$(1,0,1,0,1,1,0) \leftrightarrow (1,0,1,0,0,1,0)$,
$(1,0,1,0,0,0,0) \leftrightarrow (1,0,1,0,0,0,1)$,
$(0,0,1,0,0,0,1) \leftrightarrow (0,1,1,0,0,0,1)$,
$(0,1,0,0,0,0,1) \leftrightarrow (0,1,0,1,0,0,1)$,
$(0,1,0,1,1,0,1) \leftrightarrow (0,1,0,1,1,1,1)$,

$(1,1,0,1,0,1,0) \leftrightarrow (0,1,0,1,0,1,0)$,
$(0,0,0,1,0,1,0) \leftrightarrow (0,0,1,1,0,1,0)$,
$(0,0,1,0,0,1,0) \leftrightarrow (0,0,1,0,1,1,0)$,
$(0,0,1,0,1,0,0) \leftrightarrow (0,0,1,0,1,0,1)$,
$(1,0,1,0,1,0,1) \leftrightarrow (1,1,1,0,1,0,1)$,
$(1,1,0,0,1,0,1) \leftrightarrow (1,1,0,1,1,0,1)$,
$(1,1,0,1,0,0,1) \leftrightarrow (1,1,0,1,0,1,1)$,

$(1,0,1,1,1,0,0) \leftrightarrow (0,0,1,1,1,0,0)$,
$(0,1,1,1,1,0,0) \leftrightarrow (0,1,0,1,1,0,0)$,
$(0,1,0,0,1,0,0) \leftrightarrow (0,1,0,0,0,0,0)$,
$(0,1,0,0,0,1,0) \leftrightarrow (0,1,0,0,0,1,1)$,
$(1,1,0,0,0,1,1) \leftrightarrow (1,0,0,0,0,1,1)$,
$(1,0,1,0,0,1,1) \leftrightarrow (1,0,1,1,0,1,1)$,
$(1,0,1,1,1,1,1) \leftrightarrow (1,0,1,1,1,0,1)$,

$(0,1,1,0,1,1,1) \leftrightarrow (0,1,0,0,1,1,1)$,
$(0,0,0,1,0,0,0) \leftrightarrow (0,0,0,1,0,0,1)$,
$(0,0,1,1,0,0,1) \leftrightarrow (0,0,1,1,1,0,1)$,
$(0,1,1,0,1,1,0) \leftrightarrow (0,1,1,0,1,0,0)$.

We used a computer algebra system to check that these 39 pairs do, indeed, handle all of the 4-dimensional subcubes of Q_7. Thus, this is a winning pairing strategy for Breaker for the Maker–Breaker game played on $\mathcal{Q}(7,4)$. Therefore, by Lemma 1 and induction, Breaker has a winning pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(n,n-3)$ for $n \geq 7$. □

3. Alternate Pairing Strategies for Breaker

Alternate Proof of Proposition 1. Let $n \geq 3$ be arbitrary. For $1 \leq k \leq n$, let $\vec{e_j}$ be the n-dimensional vector whose jth coordinate is 1 and whose other $n-1$ coordinates are 0. Let $\vec{0}$ be the n-dimensional vector whose coordinates are all 0, and let $\vec{1}$ be the n-dimensional vector whose coordinates are all 1. Let $\vec{v_1} = \vec{0}$ and let $\vec{w_1} = \vec{e_1}$. Let $\vec{v_2} = \sum_{1 \leq j \leq n-1} \vec{e_j}$ and let $\vec{w_2} = \vec{1}$. Let $\vec{v_3} = \sum_{2 \leq j \leq n} \vec{e_j}$ and let $\vec{w_3} = \sum_{3 \leq j \leq n} \vec{e_j}$. Breaker pairs together $\vec{v_i}$ and $\vec{w_i}$ for $1 \leq i \leq 3$. Pair $\{\vec{v_1}, \vec{w_1}\}$ handles the $n-1$ subcubes whose entry in the jth coordinate is fixed at 0 for $2 \leq j \leq n$, i.e., $(*, 0, *, *, \ldots, *), (*, *, 0, *, \ldots, *), \ldots, (*, *, \ldots, *, 0)$. Pair $\{\vec{v_2}, \vec{w_2}\}$ handles the $n-1$ subcubes whose entry in the jth coordinate is fixed at 1 for $1 \leq j \leq n-1$, i.e., $(1, *, *, \ldots, *), (*, 1, *, \ldots, *), \ldots, (*, *, \ldots, *, 1, *)$. Pair $\{\vec{v_3}, \vec{w_3}\}$ handles the subcube whose whose entry in the 1st coordinate is fixed at 0, i.e., $(0, *, *, \ldots, *)$ and the subcube whose whose entry in the nth coordinate is fixed at 1, i.e., $(*, *, \ldots, *, 1)$. Since all $(n-1)$-dimensional subcubes are handled by some pair, this is a winning pairing strategy for Breaker for the Maker–Breaker game played on $\mathcal{Q}(n, n-1)$ for $n \geq 3$. $\square$

Alternate Proof of Proposition 2. We will assume $n \geq 6$. Breaker will use a pairing strategy in which if two vectors are paired together, then they differ in exactly one coordinate. We will describe the pairing that Breaker uses by drawing representations of the vectors that are paired together. When the "$\emptyset$" symbol appears in a coordinate, that represents the single coordinate in which the two vectors of the pair differ. We use the notation "$(0 \overset{k}{\rule{2em}{0.4pt}} 0)$," or simply "$(0 \rule{2em}{0.4pt} 0)$," when we mean a block of k coordinates that are all 0's, and similarly for a block of 1's.

Below are the first two groups of pairs in Breaker's strategy, with $n-1$ pairs in each group. The first entry in Group 1 is "$(\emptyset, 0 \overset{n-1}{\rule{2em}{0.4pt}} 0)$," which means Breaker pairs $(0, \overbrace{0, \ldots, 0}^{n-1})$ with $(1, \overbrace{0, \ldots, 0}^{n-1})$. Note that each pair in Group 2 interchanged the 0's and 1's in the corresponding pair in Group 1.

Group 1	Group 2
$(\emptyset, 0 \overset{n-1}{\rule{2em}{0.4pt}} 0)$	$(\emptyset, 1 \overset{n-1}{\rule{2em}{0.4pt}} 1)$
$(\emptyset, 1, 0 \overset{n-2}{\rule{2em}{0.4pt}} 0)$	$(\emptyset, 0, 1 \overset{n-2}{\rule{2em}{0.4pt}} 1)$
$\vdots$	$\vdots$
$(\emptyset, 1 \overset{k}{\rule{2em}{0.4pt}} 1, 0 \overset{n-k-1}{\rule{2em}{0.4pt}} 0)$	$(\emptyset, 0 \overset{k}{\rule{2em}{0.4pt}} 0, 1 \overset{n-k-1}{\rule{2em}{0.4pt}} 1)$
$\vdots$	$\vdots$
$(\emptyset, 1 \overset{n-2}{\rule{2em}{0.4pt}} 1, 0)$	$(\emptyset, 0 \overset{n-2}{\rule{2em}{0.4pt}} 0, 1)$

There are also 8 extra pairs:

	Extra Pairs
1	$(0, \emptyset, 0, 1, 0 \overset{n-4}{\text{———}} 0)$
2	$(1, \emptyset, 0, 1, 0 \overset{n-4}{\text{———}} 0)$
3	$(0, \emptyset, 0, 1 \overset{n-4}{\text{———}} 1, 0)$
4	$(1, \emptyset, 0, 1 \overset{n-4}{\text{———}} 1, 0)$
5	$(0, 0, 1, 0, \emptyset, 0 \overset{n-5}{\text{———}} 0)$
6	$(1, 0, 1, 0, \emptyset, 0 \overset{n-5}{\text{———}} 0)$
7	$(0, 1, 1, \emptyset, 0 \overset{n-5}{\text{———}} 0, 1)$
8	$(1, 1, 1, \emptyset, 0 \overset{n-5}{\text{———}} 0, 1)$

Notice that there are $n-1$ pairs in Group 1 and $n-1$ pairs in Group 2. Thus, this pairing strategy uses a total of $2n+6$ pairs. In the four tables below we indicate how each type of subcube in $\mathcal{Q}(n, n-2)$ is handled by this pairing strategy.

Subcubes of form $(*, \ldots, *, \underset{i}{0}, *, \ldots, *, \underset{j}{0}, *, \ldots, *)$	
index locations	handled by
$i \geq 2,\ j > i$	Group 1
$i = 1, j > i$	Extra Pairs 1 and 5

Subcubes of form $(*, \ldots, *, \underset{i}{0}, *, \ldots, *, \underset{j}{1}, *, \ldots, *)$	
index locations	handled by
$i \geq 2,\ j > i$	Group 2
$i = 1, j > i$	Extra Pairs 3 and 7

Subcubes of form $(*, \ldots, *, \underset{i}{1}, *, \ldots, *, \underset{j}{0}, *, \ldots, *)$	
index locations	handled by
$i \geq 2,\ j > i$	Group 1
$i = 1, j > i$	Extra Pairs 2 and 6

Subcubes of form $(*, \ldots, *, \underset{i}{1}, *, \ldots, *, \underset{j}{1}, *, \ldots, *)$	
index locations	handled by
$i \geq 2,\ j > i$	Group 2
$i = 1, j > i$	Extra Pairs 4 and 8

$\square$

Alternate Proof of Proposition 3. We will assume $n \geq 10$. As in the Alternate Proof of Proposition 2, Breaker will use a pairing strategy in which if two vectors are paired together, then the single coordinate in which they differ will be represented with the "$\emptyset$" symbol. In the Alternate Proof of Proposition 2, we used the notation "$(0 \overset{k}{\text{———}} 0)$" when referring to a block of k coordinates that are all 0's. In this proof, we will not include the number indicating the size of the blocks of 0's and 1's, as the size of the blocks will be implied by the pair itself. Instead, we simply use the notation "(0——0)."

Below are the ten groups of pairs, along with the 16 extra pairs, in Breaker's strategy.

Group 1	Group 2
$(\emptyset, 0\text{——}0)$	$(\emptyset, 1\text{——}1)$
$(\emptyset, 1, 0\text{——}0)$	$(\emptyset, 0, 1\text{——}1)$
$\vdots$	$\vdots$
$(\emptyset, 1\text{—}1, 0\text{—}0)$	$(\emptyset, 0\text{—}0, 1\text{—}1)$
$\vdots$	$\vdots$
$(\emptyset, 1\text{——}1, 0)$	$(\emptyset, 0\text{——}0, 1)$

Group 3	Group 4
$(\emptyset, 0, 1, 0\text{——}0)$	$(\emptyset, 1, 0, 1\text{——}1)$
$\vdots$	$\vdots$
$(\emptyset, 0\text{—}0, 1, 0\text{—}0)$	$(\emptyset, 1\text{—}1, 0, 1\text{—}1)$
$\vdots$	$\vdots$
$(\emptyset, 0\text{——}0, 1, 0)$	$(\emptyset, 1\text{——}1, 0, 1)$

Group 5	Group 6
$(0, \emptyset, 1, 0, 0, 1\text{——}1)$	$(1, \emptyset, 0, 1, 1, 0\text{——}0)$
$\vdots$	$\vdots$
$(0, \emptyset, 1, 0\text{—}0, 1\text{—}1)$	$(1, \emptyset, 0, 1\text{—}1, 0\text{—}0)$
$\vdots$	$\vdots$
$(0, \emptyset, 1, 0\text{——}0, 1)$	$(1, \emptyset, 0, 1\text{——}1, 0)$

Group 7	Group 8
$(0, \emptyset, 1, 0, 1, 0\text{——}0)$	$(1, \emptyset, 0, 1, 0, 1\text{——}1)$
$\vdots$	$\vdots$
$(0, \emptyset, 1, 0, 1\text{—}1, 0\text{—}0)$	$(1, \emptyset, 0, 1, 0\text{—}0, 1\text{—}1)$
$\vdots$	$\vdots$
$(0, \emptyset, 1, 0, 1\text{——}1, 0)$	$(1, \emptyset, 0, 1, 0\text{——}0, 1)$

Group 9	Group 10
$(0, \emptyset, 0, 1, 0, 1\text{——}1)$	$(1, \emptyset, 1, 0, 1, 0\text{——}0)$
$\vdots$	$\vdots$
$(0, \emptyset, 0, 1, 0\text{—}0, 1\text{—}1)$	$(1, \emptyset, 1, 0, 1\text{—}1, 0\text{—}0)$
$\vdots$	$\vdots$
$(0, \emptyset, 0, 1, 0\text{——}0, 1)$	$(1, \emptyset, 1, 0, 1\text{——}1, 0)$

	New Extra Pairs
1	$(0,0,0,0,0,\emptyset,1,0,1,0$———$0)$
2	$(1,1,1,1,1,\emptyset,0,1,0,1$———$1)$
3	$(0,0,0,0,0,\emptyset,1,0,1$————$1)$
4	$(1,1,1,1,1,\emptyset,0,1,0$————$0)$
5	$(1,0,0,0,0,\emptyset,1,0,1,0$———$0)$
6	$(0,1,1,1,1,\emptyset,0,1,0,1$———$1)$
7	$(0,1,0,1,\emptyset,0$————————$0)$
8	$(1,0,1,0,\emptyset,1$————————$1)$
9	$(0,0,1,1,1,0,0,\emptyset,0$————$0)$
10	$(1,1,0,0,0,1,1,\emptyset,1$————$1)$
11	$(1,0,1,1,0,0,0,\emptyset,0$————$0)$
12	$(0,1,0,0,1,1,1,\emptyset,1$————$1)$
13	$(0,1,\emptyset,0,0,1,0$———————$0)$
14	$(1,0,\emptyset,1,1,0,1$———————$1)$
15	$(0,0,0,1,1,1,\emptyset,1,0$————$0)$
16	$(1,1,1,0,0,0,\emptyset,0,1$————$1)$

Besides the 16 extra pairs, there are $n-1$ pairs in each of Groups 1 and 2, $n-3$ pairs in each of Groups 3 and 4, and $n-5$ pairs in the remaining 6 groups. Thus, this pairing strategy uses a total of $10n-22$ pairs. In the table below, we indicate which groups and extra pairs handle each type of subcube in $\mathcal{Q}(n,n-3)$ in this pairing strategy:

Subcubes of form $(*,\ldots,*,b_i,*,\ldots,*,b_j,*,\ldots,*,b_k,*,\ldots,*)$		
(b_i,b_j,b_k)	Groups	Extra Pairs
$(0,0,0)$	$1,5,7,9$	$1,9$
$(1,1,1)$	$2,6,8,10$	$2,10$
$(0,0,1)$	$2,5,7,9$	$3,9,15$
$(1,1,0)$	$1,6,8,10$	$4,10,16$
$(1,0,0)$	$1,6,10$	$5,11$
$(0,1,1)$	$2,5,9$	$6,12$
$(0,1,0)$	$3,5,7,9$	$7,13$
$(1,0,1)$	$4,6,8,10$	$8,14$

□

In the appendix, we provide alternate tables which more precisely indicate which Groups and Extra Pairs handle which subcubes.

4. Building a 2-dimensional and a 3-dimensional Subcube

Before we describe Maker's strategies, we start with some definitions. Let us say that a vector is in *level* j or is *a level* j *vector* if exactly j of its coordinates are 1. Let us call a vector $\vec{v}$ a *Maker-vector* if Maker occupies $\vec{v}$ (respectively, a *Breaker-vector* if Breaker occupies $\vec{v}$). It will also be convenient to refer to a vector by the coordinates which equal 1 in the vector. For example, we would refer to the vector $(1, 0, 1, 1, 0)$ as $\{1, 3, 4\}$ for short. Let $\vec{v}, \vec{w} \in Q_n$, and let $S_{\vec{v}}$ and $S_{\vec{w}}$ be the subsets of $\{1, \ldots, n\}$ which correspond to $\vec{v}$ and $\vec{w}$, respectively. We will say $S_{\vec{v}}$ is *below* $S_{\vec{w}}$ (or $\vec{v}$ is *below* $\vec{w}$) if and only if $S_{\vec{v}} \subseteq S_{\vec{w}}$.

Proof of Proposition 4. We will prove that Maker can occupy a 2-dimensional subcube of Q_5. We will use the subset notation for the vectors in Q_5. Maker's first move is $\emptyset$. For $1 \leq j \leq 3$, let B_j be Breaker's jth move. Without loss of generality, $5 \in B_1$. (Simply relabel the coordinates to make this true.) Maker's second move is $\{1\}$. Without loss of generality, $4 \in B_2$. (Again, relabel the coordinates, if necessary.) Maker's third move is $\{2\}$. Breaker is now forced to occupy $\{1, 2\}$, or lose the game next turn, thus, $B_3 = \{1, 2\}$. Maker's fourth move is $\{3\}$. Maker wins in turn 5 by occupying either $\{1, 3\}$ or $\{2, 3\}$ since Breaker cannot occupy both vectors during turn 4. $\square$

We now describe how Maker can occupy a 3-dimensional subcube. In the strategy we describe for Maker, she will restrict herself to occupying a 3-dimensional subcube that contains vector $\vec{0}$. Thus, we will make some assumptions about Breaker's moves which are summarized and justified in Lemma 2 and its proof. We begin by dividing Maker's strategy into stages, where *the first move of every stage is made by Maker*, and *for each of Stage 1, 2, and 3, the last move is made by Breaker*.

Stage 1: Maker occupies the level 0 vector $\vec{0} = (0, \ldots, 0)$. (We assume Breaker occupies a level 1 vector during this stage.)

Stage 2: Maker occupies as many level 1 vectors as possible during this stage, until all level 1 vectors are occupied by either Maker or Breaker. (We assume that Breaker only occupies level 1 or level 2 vectors during this stage.)

Stage 3: At the beginning of this stage, Maker identifies the largest set I of level 1 Maker-vectors such that there are no level 2 Breaker-vectors that are adjacent to two vectors from I, i.e., every level 2 vector which is adjacent to two vectors from I is currently unoccupied. Let A_I be the set of level 2 vectors that are adjacent to two vectors from I, i.e., $A_I = \{\{a, b\} : \{a\}, \{b\} \in I\}$. Without loss of generality, $\{1\} \in I$. Maker occupies as many level 2 vectors of the form $\{1, x\}$ (where $\{1, x\} \in A_I$) as possible during this stage, until all level 2 vectors of the form $\{1, x\} \in A_I$ are occupied by either Maker or Breaker. (We assume that Breaker occupies only elements of A_I during this stage.)

Stage 4: At the beginning of this stage, Maker identifies the largest set I_2 of Maker-vectors of the form $\{1, x\} \in A_I$ such that every level 2 vector $\{x, y\}$ where $\{1, x\}, \{1, y\} \in I_2$ is currently unoccupied. If $|I_2| \geq 3$, then it is easy to show that Maker can win. Indeed, suppose $\{\{1, x\}, \{1, y\}, \{1, z\}\} \subseteq I_2$. Then Maker occupies $\{x, y\}$, which forces Breaker to occupy $\{1, x, y\}$ (or lose next turn). Then Maker occupies $\{x, z\}$, which forces Breaker to occupy $\{1, x, z\}$ (or lose next turn). Then Maker occupies $\{y, z\}$. Maker wins on the next turn by occupying either $\{1, y, z\}$ or $\{x, y, z\}$, since Breaker cannot occupy both vectors on his last turn.

Lemma 2. *If Breaker occupies a vector from level j where $j \geq 3$ during Stages 1–3, then we may either ignore Breaker's move or substitute a level 2 vector for Breaker's move.*

Proof of Lemma 2. Maker's strategy requires she win by occupying a 3-dimensional subcube containing the vector $\vec{0}$. Thus, any moves by Breaker that are in level j where $j \geq 4$ may be ignored. Suppose that during Stage 1, 2, or 3, Breaker occupies a level 3 vector $\{x, y, z\}$. Since there is exactly one 3-dimensional subcube which contains $\vec{0}$ and $\{x, y, z\}$, then if Breaker already occupied some vector below $\{x, y, z\}$, e.g., $\{x\}$ or $\{x, y\}$ or $\{y, z\}$, then we may ignore Breaker's move $\{x, y, z\}$ since Breaker previously killed the unique 3-dimensional subcube which contains $\vec{0}$ and $\{x, y, z\}$. If Breaker had not occupied any vector below $\{x, y, z\}$ before he occupied $\{x, y, z\}$, then we will substitute Breaker's move $\{x, y, z\}$ with $\{x, y\}$ so that Breaker's move is now in level 2. (Technically, Maker pretends that Breaker occupies $\{x, y\}$ and uses her winning strategy for the case when Breaker occupies $\{x, y\}$.) By occupying $\{x, y\}$, Breaker still kills the 3-dimensional subcube which contains $\vec{0}$ and $\{x, y, z\}$ along with all other 3-dimensional subcubes which contain $\{x, y\}$. Thus, if Maker can win by using the strategy which defeats Breaker when Breaker takes $\{x, y\}$ instead of $\{x, y, z\}$, then the 3-dimensional subcube that Maker occupies does not contain $\{x, y\}$, thus, it is does not contain $\{x, y, z\}$, therefore, Maker can use the same strategy to win the game where Breaker takes $\{x, y, z\}$. □

Proof of Proposition 5. Suppose we play the Maker–Breaker positional game on $\mathcal{Q}(d + 1, 3)$. During Stage 1, Maker occupies the level 0 vector, and for reasons similar to those in the proof of Lemma 2 (and w.l.o.g.), we may assume that Breaker occupies the level 1 vector $\{d+1\}$. At the beginning of Stage 2, the available level 1 vectors are $\{1\}, \ldots, \{d\}$. Let k be the number of level 1 vectors that Breaker occupies during Stage 2, let e be the number of level 2 vectors $\{x, y\}$ that Breaker occupies during Stage 2 where $\{x\}$ and $\{y\}$ are Maker-vectors at the end of Stage 2, and

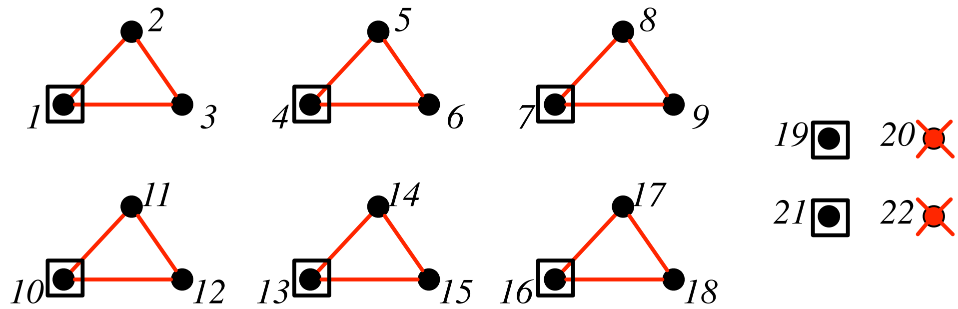

Figure 1: A possible example of $G = (V, E)$. In this picture, $V = \{1, \ldots, 18, 19, 21\}$. Breaker occupied the level 2 vectors corresponding to the edges shown and the two level 1 vectors $\{20\}$ and $\{22\}$. The independent set $I = \{1, 4, 7, 10, 13, 16, 19, 21\}$.

let n be the number of level 1 vectors that Maker occupies during Stage 2. Notice that $k + e = n$ since we are assuming that Breaker only occupies level 1 and level 2 vectors during Stage 2, and since Maker and Breaker occupy the same number of vectors during Stage 2. (Technically, we also assume that Breaker only occupies level 2 vectors $\{x, y\}$ such that $\{x\}$ and $\{y\}$ are both Maker-vectors at the end of Stage 2, since those are the only level 2 vectors that Maker can use to occupy a 3-dimensional subcube containing $\vec{0}$.) Also, since Stage 2 ends when all level 1 vertices are occupied, we have $d = k + n$. Substituting for n, we get $d = 2k + e$. We now consider the graph $G = (V, E)$ where V is the set of level 1 Maker-vectors and E is the set of level 2 Breaker-vectors $\{x, y\}$ such that $\{x\}, \{y\} \in V$. At the beginning of Stage 3, Maker selects a maximum independent set I in G. (See Figure 1.) Since I is an independent set in G, for every $\{x\}, \{y\} \in I$, the level 2 vector $\{x, y\}$ is unoccupied at the beginning of Stage 3. Let $\alpha(G)$ be the independence number of G and let t be the average degree in G. A well-known result in graph theory (a consequence of Turán's Theorem) states that $\alpha(G) \geq \frac{n}{(t+1)}$. Since $t = \frac{2e}{n}$, we have $|I| \geq \frac{n^2}{2e+n}$. After making the substitutions $n = d - k$ and $e = d - 2k$, we obtain

$$|I| \geq \frac{(d-k)^2}{2(d-2k) + (d-k)} = \frac{(d-k)^2}{3d-5k}. \tag{2}$$

We let $f(k) = \frac{(d-k)^2}{3d-5k}$, and since $f(k)$ is a lower bound for $|I|$, we wish to minimize $f(k)$ over the interval $[0, \frac{d}{2}]$. We use $\frac{d}{2}$ as our right endpoint because we know $e = d - 2k$ and $e \geq 0$. We determine that $f'(k) = \frac{(d-k)(5k-d)}{(3d-5k)^2}$, thus, the only critical point of $f(k)$ in $[0, \frac{d}{2}]$ occurs at $k = \frac{d}{5}$. It is easy to check that the minimum value of $f(k)$ on $[0, \frac{d}{2}]$ occurs at $k = \frac{d}{5}$. Therefore, $|I| \geq f(\frac{d}{5}) = \frac{8d}{25}$.

Without loss of generality, the vector $\{1\} \in I$. (Simply relabel the vectors if this is not the case.) During Stage 3, Maker occupies as many level 2 vectors of the form $\{1, x\}$, where $\{x\} \in I$, as possible. We will assume that Breaker only occupies vectors of the form $\{x, y\}$ where both $\{x\}, \{y\} \in I$. Similar to above, we consider a graph $G_2 = (V_2, E_2)$. This time V_2 is the set of level 2 Maker-vectors at the end of Stage 3 (thus, every vector in V_2 has the form $\{1, x\}$) and E_2 is the set of level 2 vectors $\{x, y\}$ such that $\{1, x\}, \{1, y\} \in V_2$ and $\{x, y\}$ is a Breaker-vector at the end of Stage 3. Let k_2 be the number of level 2 vectors of the form $\{1, x\}$ that Breaker occupies during Stage 3, and let $e_2 = |E_2|$ and $n_2 = |V_2|$. Let $d_2 = |I| - 1$, i.e., the number of level 1 Maker-vectors in I that are not the vector $\{1\}$. Then, similar to above, we may assume that $k_2 + e_2 = n_2$ and $d_2 = k_2 + n_2 = 2k_2 + e_2$. Using the same reasoning as above, we obtain an inequality analogous to inequality (2), and we can conclude that Maker can select an independent set I_2 in G_2 such that $|I_2| \geq \frac{8d_2}{25}$. (See Figure 2.)

If $d_2 = 7$, then $|I_2| \geq 2.24$, and since $|I_2|$ is integral, then $|I_2| \geq 3$. If $d = 22$, then $|I| \geq 7.04$, and since $|I|$ is integral, then $|I| \geq 8$; thus, $d_2 \geq 7$. Therefore, Maker can win $\mathcal{Q}(23, 3)$. □

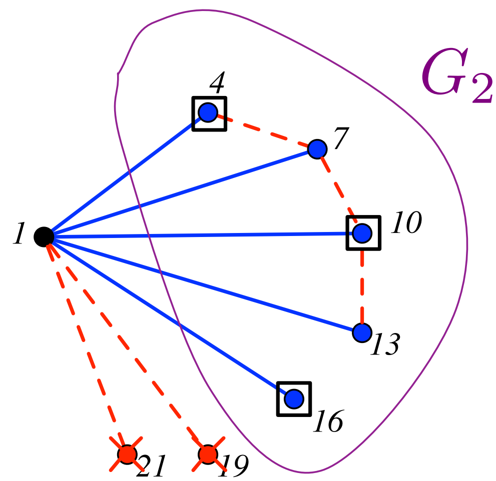

Figure 2: In this picture of Stage 3, Maker occupied the level 2 vectors shown with solid blue lines and Breaker occupied the level 2 vectors shown with dotted red lines. While G_2 is technically defined differently, we can imagine that $V_2 = \{4, 7, 10, 13, 16\}$, $E_2 = \{\{4, 7\}, \{7, 10\}, \{10, 13\}\}$, and $I_2 = \{4, 10, 16\}$.

5. Proofs of Propositions 6, 7, and 8

Proof of Proposition 6. The Erdős–Selfridge theorem states that if $\mathcal{H}$ is an m-uniform, finite hypergraph and

$$|\mathcal{H}| + MaxDeg(\mathcal{H}) < 2^m \tag{3}$$

(where $MaxDeg(\mathcal{H}) = \max_{v \in V(\mathcal{H})} \#\{A \in \mathcal{H} : v \in A\}$), then Breaker has an explicit winning strategy for the Maker–Breaker game played on $\mathcal{H}$. In the game played on $\mathcal{Q}(6,3)$, the hypergraph $\mathcal{H}$ is 8-uniform, $|\mathcal{H}| = \binom{6}{3}2^3 = 160$, and $\mathcal{H}$ is 20-regular. Thus, $MaxDeg(\mathcal{H}) = 20$. Since $180 < 256$, by the Erdős–Selfridge theorem, Breaker has an explicit winning strategy for the Maker–Breaker game played on $\mathcal{Q}(6,3)$. $\square$

Proof of Proposition 7. We will show that if $k \geq \lg n + 1$, then Breaker has a winning strategy for the Maker–Breaker positional game on $\mathcal{Q}(n,k)$ by applying the Erdős–Selfridge theorem. Since $MaxDeg(\mathcal{H}) \leq |\mathcal{H}|$, then we can use a slightly weaker form of the Erdős–Selfridge theorem which states that if $\mathcal{H}$ is an m-uniform hypergraph, then Breaker has an explicit winning strategy if

$$|\mathcal{H}| < 2^{m-1}. \tag{4}$$

In the game $\mathcal{Q}(n,k)$, we have that the hypergraph $\mathcal{H}$ is m-uniform where $m = 2^k$ and we have $|\mathcal{H}| = \binom{n}{k}2^{n-k}$. Thus, it is enough to show that if $k \geq \lg n + 1$, then

$$\binom{n}{k}2^{n-k} < 2^{2^k - 1}. \tag{5}$$

Since $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$, it is enough to show that

$$\left(\frac{en}{k}\right)^k 2^{n-k} < 2^{2^k - 1}.$$

By taking the binary logarithm of both sides, we obtain

$$k \lg\left(\frac{en}{k}\right) + n - k < 2^k - 1,$$

or equivalently,

$$k \lg e + k \lg n - k \lg k + n - k + 1 < 2^k.$$

If we factor out n, we get

$$n\left[1 + \frac{k}{n}(\lg e - 1) + \frac{k}{n}\lg n - \frac{k}{n}\lg k + \frac{1}{n}\right] < 2^k.$$

Again, we take the binary logarithm of both sides to obtain

$$\lg n + \lg\left[1 + \frac{k}{n}(\lg e - 1) + \frac{k}{n}\lg n - \frac{k}{n}\lg k + \frac{1}{n}\right] < k.$$

Since we are looking for the smallest value of k for which this inequality holds, let us assume that $k \leq 2\lg n$. We also observe that $k \geq \lg n$. Thus, it is enough to show that

$$\lg n + \lg\left[1 + \frac{2\lg n}{n}(\lg e - 1) + \frac{2\lg n}{n}\lg n - \frac{\lg n}{n}\lg\lg n + \frac{1}{n}\right] < k. \tag{6}$$

Thus, if $k \geq \lg n + 1$ and $n \geq 62$, then k will satisfy inequality (6), moreover, k will satisfy inequality (5). When $1 \leq n \leq 61$ and $k = \lceil \lg n \rceil + 1$, then k satisfies inequality (5). (This can be verified on a computer algebra system.) □

Proof of Proposition 8. We will show that if $k = \lg\lg n - 1$, then Maker has a winning strategy for the Maker–Breaker game played on $\mathcal{Q}(n,k)$. Theorem 1.2 from [1] states that if $(V, \mathcal{H})$ is an m-uniform hypergraph and

$$\frac{|\mathcal{H}|}{|V|} > 2^{m-3}\Delta_2(\mathcal{H}), \tag{7}$$

where $\Delta_2(\mathcal{H}) = \max\{\deg(x,y) : x, y \in V, x \neq y\}$ and $\deg(x,y) = |\{A \in \mathcal{H} : x, y \in A\}|$, then Maker has a winning strategy in the Maker–Breaker game on $(V, \mathcal{H})$.

In the game $\mathcal{Q}(n,k)$, we have $m = 2^k$, $|\mathcal{H}| = \binom{n}{k}2^{n-k}$, and $|V| = 2^n$. We observe that if $\vec{v}$ and $\vec{w}$ are two vectors that differ in exactly j coordinates, then $\deg(\vec{v}, \vec{w}) = \binom{n-j}{k-j}$. Thus, $\Delta_2(\mathcal{H}) = \binom{n-1}{k-1}$, since $j \geq 1$ when $\vec{v} \neq \vec{w}$, and $\binom{n-j}{k-j}$ is a nonincreasing function in j when $n \geq k$. Thus, it is enough to show that if $k = \lg\lg n - 1$, then

$$\binom{n}{k}2^{n-k}2^{-n} > 2^{2^k-3}\binom{n-1}{k-1}, \tag{8}$$

which is equivalent to

$$\frac{n}{k} > 2^{2^k-3+k}.$$

After taking the binary logarithm, rearranging terms, and taking the binary logarithm again, we arrive at the equivalent inequality

$$\lg\lg n > k + \lg\left(1 + \frac{k}{2^k} + \frac{\lg k}{2^k} - \frac{3}{2^k}\right). \tag{9}$$

Thus, if $k \leq \lg\lg n - 1$, then k will satisfy inequality (9), moreover, k will satisfy inequality (8). □

6. Conclusion

If $k(n)$ is the largest value of k such that Maker wins the positional game played on $\mathcal{Q}(n,k)$, then Propositions 7 and 8 demonstrate that

$$\lg\lg n - 1 \leq k(n) \leq \lg n. \tag{10}$$

It would be interesting to determine the order of magnitude of $k(n)$ or improve upon the bounds in inequalities (10).

If we let $ps(n)$ be the smallest value of k such that Breaker wins the positional game on $\mathcal{Q}(n,k)$ by *using a pairing strategy*, then Proposition 3 shows that $ps(n) \leq n-3$. By using Proposition 9 in [3], we can conclude that $ps(n) > \ln(n)$. (Indeed, the maximum pair degree of $\mathcal{H}$ is $\Delta_2(\mathcal{H}) = \binom{n-1}{k-1}$, $|V(\mathcal{H})| = 2^n$, $|\mathcal{H}| = \binom{n}{k}2^{n-k}$, and when $k = \lfloor \ln(n) \rfloor$, we have $\Delta_2(\mathcal{H})|V(\mathcal{H})|/2 < |\mathcal{H}|$.) Thus, we have proven that $\ln(n) < ps(n) \leq n-3$. It is an intriguing problem to describe the asymptotic behavior of $ps(n)$ or improve upon the current bounds on $ps(n)$.

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Appendix

We provide the following tables which give a more detailed account of how the Groups and Extra Pairs from the Alternate Proof of Proposition 3, found in Section 3, allow Breaker to block all subcubes in the $\mathcal{Q}(n,n-3)$ game when $n \geq 10$.

Subcubes of form $(*,\ldots,*,\underset{i}{0},*,\ldots,*,\underset{j}{0},*,\ldots,*,\underset{k}{0},*,\ldots,*)$			
i	j	k	handled by
$i \geq 2$	$j > i$	$k > j$	Group 1
$i = 1$	$j = 2$	$\{3,4,5,8\} \cup \{10,\ldots,n\}$, $\{6,7\} \cup \{9,\ldots,n\}$	Extra Pair 1, Extra Pair 9
$i = 1$	$j = 3$	$\{5,\ldots,n-1\}$, $\{4,5,8\} \cup \{10,\ldots,n\}$	Group 9, Extra Pair 1
$i = 1$	$j = 4$	$\{5,\ldots,n-1\}$, $\{6,\ldots,n\}$	Group 5, Group 7
$i = 1$	$j = 5$	$\{6,\ldots,n-1\}$, $\{8\} \cup \{10,\ldots,n\}$	Group 5, Extra Pair 1
$i = 1$	$j \geq 6$	$k > j$	Group 7

Subcubes of form $(*,\ldots,*,\underset{i}{1},*,\ldots,*,\underset{j}{1},*,\ldots,*,\underset{k}{1},*,\ldots,*)$			
i	j	k	handled by
$i \geq 2$	$j > i$	$k > j$	Group 2
$i = 1$	$j = 2$	$\{3,4,5,8\} \cup \{10,\ldots,n\}$, $\{6,7\} \cup \{9,\ldots,n\}$	Extra Pair 2, Extra Pair 10
$i = 1$	$j = 3$	$\{5,\ldots,n-1\}$, $\{4,5,8\} \cup \{10,\ldots,n\}$	Group 10, Extra Pair 2
$i = 1$	$j = 4$	$\{5,\ldots,n-1\}$, $\{6,\ldots,n\}$	Group 6, Group 8
$i = 1$	$j = 5$	$\{6,\ldots,n-1\}$, $\{8\} \cup \{10,\ldots,n\}$	Group 6, Extra Pair 2
$i = 1$	$j \geq 6$	$k > j$	Group 8

Subcubes of form $(*,\ldots,*,\underset{i}{0},*,\ldots,*,\underset{j}{0},*,\ldots,*,\underset{k}{1},*,\ldots,*)$			
i	j	k	handled by
$i \geq 2$	$j > i$	$k > j$	Group 2
$i = 1$	$j = 2$	$\{7\} \cup \{9,\ldots,n\}$, $\{3,4,5\}$, $\{4,5,6,8\}$	Extra Pair 3, Extra Pair 9, Extra Pair 15
$i = 1$	$j = 3$	$\{4\} \cup \{6,\ldots,n\}$, $\{4,5,6,8\}$	Group 9, Extra Pair 15
$i = 1$	$j \geq 4$	$k > j$	Groups 5, 7

Subcubes of form $(*,\dots,*,\underset{i}{1},*,\dots,*,\underset{j}{1},*,\dots,*,\underset{k}{0},*,\dots,*)$			
i	j	k	handled by
$i \geq 2$	$j > i$	$k > j$	Group 1
$i = 1$	$j = 2$	$\{7\} \cup \{9,\dots,n\}$, $\{3,4,5\}$, $\{4,5,6,8\}$	Extra Pair 4, Extra Pair 10, Extra Pair 16
$i = 1$	$j = 3$	$\{4\} \cup \{6,\dots,n\}$, $\{4,5,6,8\}$	Group 10, Extra Pair 16
$i = 1$	$j \geq 4$	$k > j$	Groups 6, 8

Subcubes of form $(*,\dots,*,\underset{i}{1},*,\dots,*,\underset{j}{0},*,\dots,*,\underset{k}{0},*,\dots,*)$			
i	j	k	handled by
$i \geq 2$	$j > i$	$k > j$	Group 1
$i = 1$	$j = 2$	$\{3,4,5,8\} \cup \{10,\dots,n\}$, $\{5,6,7\} \cup \{9,\dots,n\}$	Extra Pair 5, Extra Pair 11
$i = 1$	$j = 3$	$\{6,\dots,n\}$, $\{4,5,8\} \cup \{10,\dots,n\}$	Group 6, Extra Pair 5
$i = 1$	$j = 4$	$\{6,\dots,n\}$, $\{5,8\} \cup \{10,\dots,n\}$	Group 10, Extra Pair 5
$i = 1$	$j = 5$	$\{8\} \cup \{10,\dots,n\}$, $\{6,7\} \cup \{9,\dots,n\}$	Extra Pair 5, Extra Pair 11
$i = 1$	$j \geq 6$	$k > j$	Group 6

Subcubes of form $(*,\dots,*,\underset{i}{0},*,\dots,*,\underset{j}{1},*,\dots,*,\underset{k}{1},*,\dots,*)$			
i	j	k	handled by
$i \geq 2$	$j > i$	$k > j$	Group 2
$i = 1$	$j = 2$	$\{3,4,5,8\} \cup \{10,\dots,n\}$, $\{5,6,7\} \cup \{9,\dots,n\}$	Extra Pair 6, Extra Pair 12
$i = 1$	$j = 3$	$\{6,\dots,n\}$, $\{4,5,8\} \cup \{10,\dots,n\}$	Group 5, Extra Pair 6
$i = 1$	$j = 4$	$\{6,\dots,n\}$, $\{5,8\} \cup \{10,\dots,n\}$	Group 9, Extra Pair 6
$i = 1$	$j = 5$	$\{8\} \cup \{10,\dots,n\}$, $\{6,7\} \cup \{9,\dots,n\}$	Extra Pair 6, Extra Pair 12
$i = 1$	$j \geq 6$	$k > j$	Group 5

Subcubes of form $(*,\ldots,*,\underset{i}{0},*,\ldots,*,\underset{j}{1},*,\ldots,*,\underset{k}{0},*,\ldots,*)$			
i	j	k	handled by
$i \geq 2$	$j > i$	$k > j$	Group 3
$i = 1$	$j = 2$	$\{3\} \cup \{6,\ldots,n\}$, $\{4,5\} \cup \{7,\ldots,n\}$	Extra Pair 7, Extra Pair 13
$i = 1$	$j = 3$	$\{4,\ldots,n-1\}$, $\{4\} \cup \{6,\ldots,n\}$	Group 5, Group 7
$i = 1$	$j = 4$	$\{5,\ldots,n-1\}$, $\{6,\ldots,n\}$	Group 9, Extra Pair 7
$i = 1$	$j \geq 5$	$k > j$	Group 7

Subcubes of form $(*,\ldots,*,\underset{i}{1},*,\ldots,*,\underset{j}{0},*,\ldots,*,\underset{k}{1},*,\ldots,*)$			
i	j	k	handled by
$i \geq 2$	$j > i$	$k > j$	Group 4
$i = 1$	$j = 2$	$\{3\} \cup \{6,\ldots,n\}$, $\{4,5\} \cup \{7,\ldots,n\}$	Extra Pair 8, Extra Pair 14
$i = 1$	$j = 3$	$\{4,\ldots,n-1\}$, $\{4\} \cup \{6,\ldots,n\}$	Group 6, Group 8
$i = 1$	$j = 4$	$\{5,\ldots,n-1\}$, $\{6,\ldots,n\}$	Group 10, Extra Pair 8
$i = 1$	$j \geq 5$	$k > j$	Group 8