A GENERALIZATION OF AN IDENTITY DUE TO KIMURA AND RUEHR

J.-P. Allouche
CNRS, Institut de Mathématiques de Jussieu-PRG, Université Pierre et Marie Curie, Paris, France
jean-paul.allouche@imj-prg.fr

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Abstract

An identity stated by Kimura and proved by Ruehr, Kimura and others stipulates that for any continuous function f on $[-\frac{1}{2}, \frac{3}{2}]$ one has

$$\int_{-1/2}^{3/2} f(3x^2 - 2x^3)dx = 2\int_0^1 f(3x^2 - 2x^3)dx.$$

We prove that this equality is not an isolated example by providing a family of polynomials, related to the Tchebychev polynomials and of which $(3x^2 - 2x^3)$ is a particular case, giving rise to similar identities.

– To Jeff Shallit on the occasion of his 60th birthday

1. Introduction

In this text, we address an identity that we call the Kimura–Ruehr identity: this was a question posed by Kimura and answered by Ruehr, but also by the proposer as well as by nine other contributors; see [3]. It reads

Let f be a real function that is continuous on $[-\frac{1}{2}, \frac{3}{2}]$. Then

$$\int_{-1/2}^{3/2} f(3x^2 - 2x^3)dx = 2\int_0^1 f(3x^2 - 2x^3)dx. \tag{1}$$

In his proof [3], Ruehr notes that the identity is equivalent to the identities obtained for $f(x) = x^n$ for all nonnegative integers. In particular, he points out the identities

$$\sum_{0 \le j \le n} 3^j \binom{3n-j}{2n} = \sum_{0 \le j \le 2n} (-3)^j \binom{3n-j}{n} \tag{2}$$

and
$$\sum_{0\le j\le n} 2^j \binom{3n+1}{n-j} = \sum_{0\le j\le 2n} (-4)^j \binom{3n+1}{n+1+j}. \tag{3}$$

Equality (2) is the corrected version of the corresponding one given in [3], as indicated in [4] (also see [1]).

A way to generalize these Identities (2) and (3) is to introduce polynomials whose values at some point coincide with the quantities above: this was done in [1], and, with two extra parameters, in [2].

Now another question that quickly comes to mind when looking at Equality (1) is whether this equality is "isolated", or whether it is an instance in a general family of identities. Here we give a countable family of equalities that generalize Equality (1): they are somehow based on trigonometry (actually on the use of Tchebychev polynomials), in relation to the spirit of Ruehr's original proof.

2. Definitions

Recall that the Tchebychev polynomials of the first kind, $T_n(X)$, are defined by $T_0(X) = 1$, $T_1(X) = X$, and for all $n \ge 0$, $T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X)$. They have the property that, for all $\theta \in \mathbb{R}$, the relation $T_n(\cos\theta) = \cos n\theta$ holds.

In the rest of the paper we will use the following quantities.

Definition 1.

- For each integer $n > 1$, a_n and b_n are defined by
$$\begin{aligned} a_n &:= \cos^2\frac{\pi}{n} - \cos^2\frac{\pi}{2n} = \frac{1}{2}\left(\cos\frac{2\pi}{n} - \cos\frac{\pi}{n}\right), \\ b_n &:= \cos^2\frac{\pi}{2n} = \frac{1}{2}\left(\cos\frac{\pi}{n} + 1\right). \end{aligned}$$

- Furthermore, let f be a continuous function on $[0,1]$. For $n > 1$, we let $A_n(f)$ and $B_n(f)$ denote the two quantities
$$A_n(f) := \frac{1}{a_n}\int_0^{\pi/2n} f(\cos^2 nu)\sin 2u \; du,$$
$$B_n(f) := \frac{1}{a_n}\int_0^{\pi/2n} f(\cos^2 nu)\cos 2u \; du.$$

Remark 1. Note that $a_n < 0$.

Definition 2. We define the polynomials $V_n(X)$ and $W_n(X)$ by

$$V_n(X^2) := T_n^2(X) \quad \text{and} \quad W_n(X) := V_n(a_nX + b_n).$$

Remark 2. It is clear from the recurrence property of the Tchebychev polynomials given above that $T_n(X)$ is even (resp. odd) if n is even (resp. odd). Thus $T_n^2(X)$ is always an even polynomial, so that the polynomial V_n is well defined.

3. Three Lemmas

Lemma 1. *Let f be a continuous function on $[0, 1]$. Then*

$$\int_0^1 f(W_n(x))\ \mathrm{d}x = A_n(f)\cos\frac{2\pi}{n} - B_n(f)\sin\frac{2\pi}{n}\cdot$$

Proof. We make the change of variables $a_nx + b_n = \cos^2 t$. Thus $t \in [\frac{\pi}{2n}, \frac{\pi}{n}]$, and $a_n\ \mathrm{d}x = -2\sin t\cos t\ \mathrm{d}t = -\sin 2t\ \mathrm{d}t$. Hence

$$\begin{aligned} a_n\int_0^1 f(W_n(x))\ \mathrm{d}x = a_n\int_0^1 f(V_n(a_nx + b_n))\ \mathrm{d}x &= -\int_{\frac{\pi}{2n}}^{\frac{\pi}{n}} f(V_n(\cos^2 t))\sin 2t\ \mathrm{d}t \\ = -\int_{\frac{\pi}{2n}}^{\frac{\pi}{n}} f(T_n^2(\cos t))\sin 2t\ \mathrm{d}t &= -\int_{\frac{\pi}{2n}}^{\frac{\pi}{n}} f(\cos^2 nt)\sin 2t\ \mathrm{d}t. \end{aligned}$$

Putting $t = \frac{\pi}{n} - u$ in the last integral yields

$$a_n\int_0^1 f(W_n(x))\ \mathrm{d}x = -\int_0^{\frac{\pi}{2n}} f(\cos^2 nu)\sin\left(\frac{2\pi}{n} - 2u\right)\ \mathrm{d}u,$$

which gives the result by expanding $\sin(\frac{2\pi}{n} - 2u)$. □

Lemma 2. *Let f be a continuous function on $[0, 1]$. Then*

$$\int_{\frac{1-b_n}{a_n}}^0 f(W_n(x))\ \mathrm{d}x = -A_n(f).$$

Proof. We make the same change of variables as in Lemma 1, obtaining

$$a_n\int_{\frac{1-b_n}{a_n}}^0 f(W_n(x))\ \mathrm{d}x = -\int_0^{\frac{\pi}{2n}} f(V_n(\cos^2 t))\sin 2t\ \mathrm{d}t = -\int_0^{\frac{\pi}{2n}} f(\cos^2 nt)\sin 2t\ \mathrm{d}t.$$

□

Lemma 3. *Let f be a continuous function on* $[0,1]$. *Then*

$$\int_{\frac{1-b_n}{a_n}}^{-\frac{b_n}{a_n}} f(W_n(x))\ \mathrm{d}x = \begin{cases} -A_n(f) - B_n(f)\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}, & \text{if } n \text{ is odd;} \\ -2A_n(f) - 2B_n(f)\frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Making once more the change of variable used in Lemma 1, we obtain

$$a_n \int_{\frac{1-b_n}{a_n}}^{-\frac{b_n}{a_n}} f(W_n(x))\ \mathrm{d}x = -\int_0^{\frac{\pi}{2}} f(\cos^2 nt)\sin 2t\ \mathrm{d}t = -\sum_{k=0}^{n-1} I_{k,n},$$

where

$$I_{k,n} = \int_{\frac{k\pi}{2n}}^{\frac{(k+1)\pi}{2n}} f(\cos^2 nt)\sin 2t\ \mathrm{d}t.$$

Now we will give another expression for $I_{k,n}$ according to the parity of k.

- If k is odd, we make in $I_{k,n}$ the change of variable $t = \frac{(k+1)\pi}{2n} - u$. This yields

$$I_{k,n} = \int_0^{\frac{\pi}{2n}} f(\cos^2(\tfrac{(k+1)\pi}{2} - nu))\sin(\tfrac{(k+1)\pi}{n} - 2u)\ \mathrm{d}u.$$

  But k is odd, hence $\frac{(k+1)}{2}$ is an integer. Thus, expanding the sine, we obtain

$$I_{k,n} = -a_n A_n \cos\left(\frac{(k+1)\pi}{n}\right) + a_n B_n \sin\left(\frac{(k+1)\pi}{n}\right).$$

- If k is even, we make in $I_{k,n}$ the change of variable $t = \frac{k\pi}{2n} + u$. This yields

$$I_{k,n} = \int_0^{\frac{\pi}{2n}} f(\cos^2(\tfrac{k\pi}{2} + nu))\sin(\tfrac{k\pi}{n} + 2u)\ \mathrm{d}u.$$

  But k is even, hence $\frac{k}{2}$ is an integer. Thus, expanding the sine, we obtain

$$I_{k,n} = a_n A_n \cos\left(\frac{k\pi}{n}\right) + a_n B_n \sin\left(\frac{k\pi}{n}\right).$$

We thus have

$$a_n \int_{\frac{1-b_n}{a_n}}^{-\frac{b_n}{a_n}} f(W_n(x))\ \mathrm{d}x = -\sum_{k=0}^{n-1} I_{k,n} = \Sigma_1(n) + \Sigma_2(n),$$

where

$$\Sigma_1(n) = a_n A_n \sum_{\substack{0\le k\le n-1 \\ k \text{ odd}}} \cos\left(\frac{(k+1)\pi}{n}\right) - a_n B_n \sum_{\substack{0\le k\le n-1 \\ k \text{ odd}}} \sin\left(\frac{(k+1)\pi}{n}\right)$$

and

$$\Sigma_2(n) = -a_n A_n \sum_{\substack{0\le k\le n-1\\ k \text{ even}}} \cos\left(\frac{k\pi}{n}\right) - a_n B_n \sum_{\substack{0\le k\le n-1\\ k \text{ even}}} \sin\left(\frac{k\pi}{n}\right).$$

Rearranging $\Sigma_1(n)+\Sigma_2(n)$ and simplifying by a_n finally gives

$$\int_{\frac{1-b_n}{a_n}}^{-\frac{b_n}{a_n}} f(W_n(x))\ \mathrm{d}x = \begin{cases} -A_n - 2B_n \displaystyle\sum_{\substack{1\le k\le n-1\\ k \text{ even}}} \sin\left(\frac{k\pi}{n}\right), & \text{if } n \text{ is odd;}\\ -2A_n - 2B_n \displaystyle\sum_{\substack{1\le k\le n-1\\ k \text{ even}}} \sin\left(\frac{k\pi}{n}\right), & \text{if } n \text{ is even.}\end{cases}$$

To finish the proof of the lemma it suffices to recall the classical computation (where $\Im(z)$ is the imaginary part of the complex number z)

$$\sum_{\substack{1\le k\le n-1\\ k \text{ even}}} \sin\left(\frac{k\pi}{n}\right) = \Im\left(\sum_{\substack{1\le k\le n-1\\ k \text{ even}}} e^{\frac{ik\pi}{n}}\right) = \begin{cases} \dfrac{\cos\frac{\pi}{2n}}{2\sin\frac{\pi}{2n}} & \text{if } n \text{ is odd,}\\ \dfrac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}} & \text{if } n \text{ is even.}\end{cases}$$

$\square$

4. The Main Result

Now we are ready to prove our main result (recall the definitions of a_n, b_n, $A_n(f)$, $B_n(f)$ and $W_n(X)$ given in Section 2).

Theorem 1. *Let $n > 2$ be an integer, and let f be a continuous function on $[0,1]$.*

- *For odd n, we have*

$$\int_0^1 f(W_n(x))\ \mathrm{d}x = \\ -\cos\frac{2\pi}{n}\int_{\frac{1-b_n}{a_n}}^{\frac{-b_n}{a_n}} f(W_n(x))\ \mathrm{d}x + \left(2\cos\frac{\pi}{n} - 1\right)\int_0^{\frac{-b_n}{a_n}} f(W_n(x))\ \mathrm{d}x.$$

- *For even n, we have*

$$\int_{\frac{1-b_n}{a_n}}^1 f(W_n(x))\ \mathrm{d}x = \sin^2\frac{\pi}{n}\int_{\frac{1-b_n}{a_n}}^{\frac{-b_n}{a_n}} f(W_n(x))\ \mathrm{d}x.$$

Proof. First we eliminate $B_n(f)$ between the relations given in Lemmas 1 and 3. Then we use Lemma 2 to get rid of $A_n(f)$. Finally, for even n, we multiply by $\sin\frac{\pi}{n}$, divide by $2\cos\frac{\pi}{n}$, and we combine two integrals, obtaining the statement above; for odd n, we multiply by $\sin\frac{\pi}{2n}$ and we divide by $\cos\frac{\pi}{2n}$, obtaining the equality

$$\int_0^1 f(W_n(x))\,\mathrm{d}x = \\ 4\cos\frac{\pi}{n}\sin^2\frac{\pi}{2n}\int_{\frac{1-b_n}{a_n}}^{\frac{-b_n}{a_n}} f(W_n(x))\,\mathrm{d}x + \left(1-2\cos\frac{\pi}{n}\right)\int_{\frac{1-b_n}{a_n}}^{0} f(W_n(x))\,\mathrm{d}x.$$

The theorem follows by writing $\int_{\frac{1-b_n}{a_n}}^{0} = \int_{\frac{1-b_n}{a_n}}^{\frac{-b_n}{a_n}} - \int_0^{\frac{-b_n}{a_n}}$ and rearranging. ∎

Remark 3.

- Theorem 1 above is still true, but trivial, for $n = 1$ and $n = 2$.
- For $n = 3$, Theorem 1 gives
$$\int_0^1 f(3x^2-2x^3)\,\mathrm{d}x = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{3}{2}} f(3x^2-2x^3)\,\mathrm{d}x,$$
which is exactly Ruehr's identity.
- If $n = 4$, then $a_4 = -\frac{\sqrt{2}}{4}$, and $b_4 = \frac{2+\sqrt{2}}{4}$. Thus, Theorem 1 gives
$$\int_{1-\sqrt{2}}^{1} f((x^2-2x)^2)\,\mathrm{d}x = \frac{1}{2}\int_{1-\sqrt{2}}^{1+\sqrt{2}} f((x^2-2x)^2)\,\mathrm{d}x.$$

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