A BROWNIAN QUASI–HELIX IN $\mathbb{R}^4$, BUILT FROM AN AUTOMATIC SEQUENCE

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Abstract

This article revisits a 1981 paper of the author on helices and quasi-helices. It gives in particular a detailed exposition of quasi-helices in $\mathbb{R}^4$ involving automatic sequences.

– To Jeff Shallit on the occasion of his 60th birthday

1. History and Definitions

The history of Brownian motion has been described a number of times. In 1905, Einstein established his celebrated formula

$$\overline{\Delta X^2} = \frac{RT}{N}\frac{1}{3\pi\mu a}\Delta t$$

for spherical particles of radius a suspended in a liquid of viscosity μ at temperature T; the first member, $\overline{\Delta X^2}$, is the average of the squares of their displacements during an interval of time Δt; R is the constant of perfect gaz, and N the Avogadro number. In the following years Jean Perrin made a series of experiments leading to a new determination of the Avogadro number, and observed that the very irregular motion of particles resembled the nowhere differentiable functions of mathematicians. Norbert Wiener introduced what he called "the fundamental random function" as a mathematical model for the physical Brownian motion. It was called immediately "the Wiener process", and later on, following Paul Lévy, "the Brownian motion". Wiener gave several versions of the construction and derived a number of fundamental properties, Lévy developed the theory to a high point of

[1] This paper was written for the conference "Séminaire sur l'interface entre l'analyse harmonique et la théorie des nombres" held in Luminy in October 2005, and never published.

[2] Jean-Pierre Kahane did us the honor of accepting to submit his paper for this special issue on June 9, 2017. He accidentally passed away on June 21, 2017.

sophistication, and it is now a mathematical object of common use as well as a mine of interesting problems.

Here is the theory as it appears from the last exposition made by Norbert Wiener. The problem is to construct a random process $X(t, \omega)$, also denoted by $X(t)$ ($= X(t, \cdot)$), (t the time, $\omega \in \Omega$ the probability space) such that

1) for almost all ω, $X(t, \omega)$ is a continuous function of t

2) $X(t)$ is a Gaussian process, meaning that the distribution of any n–uple $(X(t_1), X(t_2), \dots X(t_n))$ is Gaussian

3) this Gaussian process has stationary increments, meaning that the distribution of $X(t) - X(s)$ depends on $t - s$ only

4) it satisfies a normalized Einstein equation; that is,

$$||X(t) - X(s)||_2^2 = |t - s|$$

where the norm is taken in $L^2(\Omega)$.
Here is such a construction. Let $\mathcal{H}$ be an infinite–dimensional subspace of $L^2(\Omega)$ consisting of Gaussian centered variables, and W an isometric linear mapping of $L^2(I)$ ($I = \mathbb{R}$, or $\mathbb{R}^+$, or $[0, 1]$) into $\mathcal{H}$. Let χ_t be the indicator function $1_{[0,t]}$. Then

$$X(t) = W(\chi_t)$$

satisfies all conditions 2) to 4). Moreover, given an orthonormal basis of $L^2(I)$, (u_n), its image by W is a normal sequence (sequence of independent Gaussian normalized random variables (ξ_n)), and expanding χ_t in the form

$$\chi_t = \Sigma\ a_n(t) u_n \qquad (\text{in } L^2(I))$$

results in an expansion of $X(t)$ as a random series of functions:

$$\chi(t) = \Sigma\ a_n(t) \xi_n \qquad (\text{in } L^2(\Omega))$$

or, more explicitly,

$$X(t, \omega) = \Sigma\ a_n(t) \xi_n(\omega)\,.$$

To prove condition 1), it is enough to establish that the series in the second member converges uniformly in t for almost all ω, and this is done rather easily when the u_n are classical orthonormal bases.

By definition, a helix is a curve in a Hilbert space, parametrized by $\mathbb{R}$, such that the distance between two points depends only on the distance of the parameters:

$$||X(t) - X(s)||_2^2 = \psi(t - s)\,,$$

and $\psi(\cdot)$ is called the helix function. A translation of the parameter results in a isometric motion of the curve onto itself. It is the abstract model for all Gaussian processes with independent increments.

When $\psi(t) = |t|$ we say that the curve is a Brownian helix. In contrast with the realizations of the Brownian motions (the functions $t \longrightarrow X(t, \omega)$ when ω is fixed), the Brownian helix is a very regular curve. However some basic properties of the Brownian motion can be read on the Brownian helix: its Hausdorff dimension is 2, its 2–dimensional Hausdorff measure is nothing but dt, and any three points on the curve are the vertices of a rightangle triangle: the increments starting from a point are orthogonal to the past (therefore, independent from the past).

Simple examples of helices are:

1) the line ($\psi(t) = a^2t^2$)
2) the circle ($\psi(t) = r^2 \ \sin^2 \omega t$)
3) the three–dimensional helices ($\psi(t) = a^2t^2 + r^2 \ \sin^2 \omega t$)
4) generalizations of those, with

$$\psi(t) = a^2t^2 + \int \ \sin^2 \omega t \ \mu(d\omega)\,,$$

where μ is a positive measure on $\mathbb{R}^+$ such that the integral is finite. Actually this is the general form of a helix function.

Except when μ is carried on a finite set, the helix cannot be imbedded in a finite dimensional Euclidean space.

At the end of the 1970s, Patrice Assouad developed a theory of Lipschitz embeddings of a metric space into another [2]. He introduced and built quasi–helices in Euclidean spaces, meaning that

$$0 < a < \frac{||X(t) - X(s)||_2^2}{\psi(t-s)} < b < \infty$$

for some a and b, and all t and s. When $\psi(t) = |t|$ we call them Brownian quasi–helices. Assouad constructed Brownian quasi–helices in Euclidean $\mathbb{R}^n$ for $n \geq 3$, and this gives a new way to prove that the realizations of Brownian motion are continuous almost surely. He asked whether a and b can be taken near 1 when n is large, that is, whether the Brownian helix can be approximated (in this sense) by Brownian quasi–helices. We gave a positive answer with an explicit construction that was published in our paper on helices and quasi–helices [3].

2. A Construction of Brownian Quasi–helices by Means of Walsh Matrices

Let us consider $\mathbb{R}^{2^n}$ $(n \geq 1)$ as a Euclidean space. Let $N = 2^n$. If we want to construct a function $X : \mathbb{N} \longrightarrow \mathbb{R}^N$ such that $X(0) = 0$ and $||X(t) - X(s)||^2 = |t-s|$ when $|t-s| \leq N$, we have to choose an orthonormal basis $u_0, u_1, \ldots u_{N-1}$, define

$u_{N+j} = u_j$, and write

$$X(t) = \sum_{0 \le j \le t-1} \pm u_j \, .$$

At this stage there is no restriction on the signs $\pm$, and we may choose $+$ when $0 \le j \le N-1$. If we try to obtain $||X(2t)-X(2s)||^2 = 2|t-s|$, $||X(4t)-X(4s)||^2 = 4|t-s|$ etc when $|t-s| \le N$, we have more and more conditions on the $\pm$ and we are led to the following construction.

We define the Walsh matrix of order N as the $N \times N$ matrix obtained as the n^{th} tensor power of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, that is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad (n \text{ times})\, .$$

For example, the Walsh matrix of order 4 is

$$M = M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & +1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

and the matrix M_{n+1} of order 2^{n+1} is obtained from M_n as

$$M_{n+1} = \begin{pmatrix} M_n & M_n \\ M_n & -M_n \end{pmatrix}$$

The first N^2 signs $\pm$ are those of the entries of the Walsh matrix, read line by line. In order to obtain the following signs, we extend the Walsh matrix by a series of vertical translations and change of signs of some lines according to the following rule: the first row is nothing but the whole sequence of entries, written from line to line and from left to right.

With this procedure we define $X(t)$ when t is an integer and we can extend the construction to all $t > 0$, then to all real t. It is proved in [3] that we obtain a quasi–helix with a and b close to 1 when n is large enough: it is the answer to the question of Assouad.

However, it was not proved that the construction provides a quasi–helix when $n = 2$ (it was remarked that it gives a Peano curve in the plane when $n = 1$). The aim of the present paper is to give a detailed exposition of the case $n = 2$ (most of it could be copied for $n > 2$) and to prove that we obtain a quasi–helix. Instead of $t \in \mathbb{R}$ we shall consider only $t \in \mathbb{R}^+$ and a curve starting from 0 ($X(0) = 0$). We shall investigate the geometric properties of the curve, some of them leading to open questions of a combinatorial or arithmetical nature.

The sequences that we construct are automatic in the sense of [1].

3. Description of the Sequence

3.1 It is a sequence of $+1$ and -1 as described before, in case $N = 4$. We write it as a succession of $+$ and $-$:

$$++++\quad +-+-\quad ++--\quad +--+\quad ++++\ \cdots$$

The gaps between the blocks of four letters have no meaning, except a help to understand the construction. The construction proceeds as follows: given the initial word of length 4^j, we divide it into four words A, B, C, D of equal length 4^{j-1} and write it $ABCD$; then

$$A\ B\ C\ D\ A\ (-B)\ C\ (-D)\ A\ B\ (-C)(-D)\ A(-B)(-C)\ D$$

is the initial word of length 4^{j+1}. We shall give several equivalent definitions, using substitutions, explicit expressions, or generating functions.

Beforehand let us write the sequence in a tabular form as in the previous section:

$+$	$+$	$+$	$+$		$a_0\ a_1\ a_2\ a_3$
$+$	$-$	$+$	$-$	A	$a_4\ a_5$ ....
$+$	$+$	$-$	$-$		....
$+$	$-$	$-$	$+$		.... a_{15}
$+$	$+$	$+$	$+$		a_{16} ....
$-$	$+$	$-$	$+$	B	....
$+$	$+$	$-$	$-$		....
$-$	$+$	$+$	$-$		.... a_{31}
$+$	$+$	$+$	$+$		a_{32} ....
$+$	$-$	$+$	$-$	C	....
$-$	$-$	$+$	$+$		....
$-$	$+$	$+$	$-$		.... a_{47}
$+$	$+$	$+$	$+$		a_{48} ....
$-$	$+$	$-$	$+$	D	....
$-$	$-$	$+$	$+$		....
$+$	$-$	$-$	$+$		.... a_{63}
					a_{64} ....
				A	....
					....
					.... a_{79}
					a_{80} ....
				$-B$	....
					....
					....

3.2 Let us give an explicit expression for a_n. Writing

$$n = n_0 + 4n_1 + \cdots + 4^\nu n_\nu \qquad (n_\nu = 1, 2, 3;\ n_j = 0, 1, 2, 3 \text{ if } j < \nu)$$

the construction shows that

$$a_n = a_{n_0+4n_1}\ a_m\,, \quad m = n_1 + 4n_2 + \cdots + 4^{\nu-1}n_\nu\,,$$

that is

$$a_n = a_{n_0+4n_1}\ a_{n_1+4n_2} \cdots a_{n_{\nu-1}+4n_\nu}\,.$$

In the second member we find a_j's with $j \leq 15$. Their value is -1 when $j = 5, 7, 10, 11, 13$ and 14 and $+1$ otherwise. Now let us express a_n as a function of n written in the 4–adic system of numeration. We obtain the formula

$$a_n = (-1)^{A_n}$$

where A_n is the number of $11, 13, 22, 23, 31, 32$ in the 4–adic expansion of n. For example, if $n = 1\ 3\ 2\ 0\ 0\ 1\ 1\ 1\ 0\ 2\ 3\ 1\ 1\ 1\ 2\ 2$ the significant links are

1_3_2 0 0 1_1_1 0 2_3_1_1_1 2_2

A_n is nine and $a_n = -1$.

3.3 Let us describe the sequence by means of a substitution rule.

We start from an alphabet made of eight letters: $+a, +b, +c, +d, -a, -b, -c, -d$. The substitution rule is

$$(S_0) \qquad \begin{array}{lcl} +a & \longrightarrow & +a+b+c+d \\ +b & & +a-b+c-d \\ +c & & +a+b-c-d \\ +d & & +a-b-c+d \\ -a & & -a-b-c-d \\ -b & & -a+b-c+d \\ -c & & -a-b+c+d \\ -d & & -a+b+c-d \end{array}$$

The infinite word beginning with $+a$ and invariant under the substitution is

$$W = +a+b+c+d+a-b+c-d+a+b-c-d+a-b-c+d+\cdots$$

Replacing a, b, c, d by 1 (or, in a graphic way, in suppressing them), we obtain our sequence of ± 1 (or $\pm$).

3.4 Actually there is a simpler substitution rule leading to the same result, namely

$$(S_1) \qquad \begin{array}{lcl} +a & \longrightarrow & +a+b \\ +b & & +c+d \\ +c & & +a-b \\ +d & & +c-d \\ -a & & -a-b \\ -b & & -c-d \\ -c & & -a+b \\ -d & & -c+d \end{array}$$

It can be checked immediately that $(S_1)(S_1) = (S_0)$.

3.5 The generating function of the sequence (a_n) is

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

It can be defined using partial sums of order 4^n. Let us introduce the matrix

$$M(z) = \begin{pmatrix} 1 & z & z^2 & z^3 \\ 1 & -z & z^2 & -z^3 \\ 1 & z & -z^2 & -z^3 \\ 1 & -z & -z^2 & z^3 \end{pmatrix}$$

and define four sequences of polynomials by the formulas

$$\begin{pmatrix} P_0 \\ Q_0 \\ R_0 \\ T_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} P_{n+1} \\ Q_{n+1} \\ R_{n+1} \\ T_{n+1} \end{pmatrix} = M(z^{4^n}) \begin{pmatrix} P_n \\ Q_n \\ R_n \\ T_n \end{pmatrix}$$

$(n = 0, 1, \ldots)$. Then

$$\begin{pmatrix} P_n \\ Q_n \\ R_n \\ T_n \end{pmatrix} = M(z^{4^{n-1}}) M(z^{4^{n-2}}) \cdots M(z^4) M(z) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

When $|z| = 1$, we have $M(z)M(\overline{z}) = 4I$, therefore the matrix $\frac{1}{2}M(z)$ is unitary, hence

$$\begin{aligned} |P_n|^2 + |Q_n|^2 + |R_n|^2 + |T_n|^2 &= 4(|P_{n-1}|^2 + |Q_{n-1}|^2 + |R_{n-1}|^2 + |T_{n-1}|^2) \\ &= 4^n(1+1+1+1) = 4^{n+1} \end{aligned}$$

We obtain the generating function as

$$f(z) = \lim_{n\to\infty} P_n(z).$$

We can write the generating function in a more interesting form:

$$f(z) = f_0(z^4) + z f_1(z^4) + z^2 f_2(z^4) + z^3 f_3(z^4)\,,$$

where the coefficients of the power series f_0, f_1, f_2, f_3 are the columns of the table in 3.0. In order to obtain these coefficients, we can start from W in 3.2 and replace a, b, c, d by $1, 1, 1, 1$ (for f_0), $1, -1, 1, -1$ (for f_1), $1, 1, -1, -1$ (for f_2) and $1, -1, -1, 1$ (for f_3). Then $f_0 = f$. Writing

$$F(z) = \begin{pmatrix} f_0(z) \\ f_1(z) \\ f_2(z) \\ f_3(z) \end{pmatrix},$$

we see that the functional equation of the generating functions of the columns is $F(z) = M(z)F(z^4)\,.$

4. Description of the Curve

4.1 Let u_0, u_1, u_2, u_3 be an orthonormal basis of the Euclidean space $\mathbb{R}^4$, and define $u_{j+4} = u_j$ $(j = 0, 1, \ldots)$. The partial sums of the series

$$a_0 u_0 + a_1 u_1 + a_2 u_2 + \cdots$$

(that can be obtained from W in 3.2 by replacing a, b, c, d by u_0, u_1, u_2, u_3) will be denoted by $S(n)$. Then $S(n) = a_0 u_0 + a_1 u_1 + \cdots + a_{n-1} u_{n-1} \in \mathbb{Z}^4$. It is easy to check in the table in 3.0 that $S(16n) = 4S(n)$ $(n \in \mathbb{N})$. Moreover it is not difficult to see (we shall be more specific later) that $||S(n) - S(m)||^2 \leq b\ |n - m|$ for some $b < \infty$ and all n and m. This allows, first, to define $S(t)$ when t is a binary number via a formula $S(16^\nu t) = 4^\nu S(t)$, then to check that

$$S(16t) = 4S(t) \quad \text{and} \quad ||S(t) - S(s)||^2 \leq b|t - s|$$

for such numbers, then to extend $S(\cdot)$ by continuity on $\mathbb{R}^+$, and check the above formulas for all $t \geq 0$ and $s \geq 0$.

The curve we consider is the image of $\mathbb{R}^+$ by $S(\cdot)$.

Clearly (changing t into $16t$) the curve is invariant under a homothety of center 0 and ratio 4. Our main purpose is to prove that it is a Brownian quasi–helix. We shall point out some geometric properties first.

4.2 The matrix M transforms u_0, u_1, u_2, u_3 into u'_0, u'_1, u'_2, u'_3:

$$(u'_0, u'_1, u'_2, u'_3) = M(u_0, u_1, u_2, u_3)$$

and the partial sums of order n of the series $\Sigma a_j u'_j$ are the partial sums of order $4n$ of the series $\Sigma a_j u_j$. Therefore the equation

$$S(4t) = M\ S(t)$$

holds true when $t = n \in \mathbb{N}$ and by extension for all $t \in \mathbb{R}^+$.

It is easy to check that the eigenvalues of M are 2 and -2, and that

$$M \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 2 & 0 \\ -2 & 2 \end{pmatrix}, \qquad M \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ -1 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 2 & -2 \\ 2 & 2 \\ 2 & 0 \end{pmatrix}.$$

Let

$$\begin{aligned} v_0 &= \tfrac{1}{2}(u_0 + u_1 + u_2 - u_3)\,, \quad & v_1 &= \tfrac{\sqrt{2}}{2}(u_0 + u_3) \\ v_2 &= \tfrac{1}{2}(u_0 - u_1 - u_2 - u_3)\,, \quad & v_3 &= \tfrac{\sqrt{2}}{2}(u_1 - u_2)\,. \end{aligned}$$

They constitute an orthonormal system. The vectors v_0 and v_1 generate a plane, P, which is the eigenspace of the eigenvalue 2, and v_2 and v_3 a plane, Q, corresponding to the eigenvalue -2. Expressed via the orthonormal basis v_0, v_1, v_2, v_3, the operator M takes the form

$$M' = U\ M\ U^{-1} = 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where U is the unitary matrix carrying u_0, u_1, u_2, u_3 onto v_0, v_1, v_2, v_3. It means that the transformation $S(t) \longrightarrow S(4t)$ is the product of a homothety of center 0 and ratio 2 and an orthogonal symmetry with respect to the plane Q.

4.3 Clearly $M^2 = 4I$ (I being the identity matrix), In turn, M is the square of another simple matrix, as it can be guessed from 3.2 and 3.3. Let us define

$$T = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Then $M = T^2$.

The eigenvalues of T are $\sqrt{2}$, $-\sqrt{2}$, $i\sqrt{2}$ and $-i\sqrt{2}$. The vectors

$$w_0 = \frac{\sqrt{2}}{2}(v_0 + v_1), \qquad w_1 = \frac{\sqrt{2}}{2}(v_0 - v_1)$$

are eigenvectors corresponding to $\sqrt{2}$ and $-\sqrt{2}$. Defining w_2 and w_3 in such a way that w_0, w_1, w_2, w_3 is a direct orthonormal basis, and W being the unitary matrix carrying u_0, u_1, u_2, u_3 onto w_0, w_1, w_2, w_3, we can write

$$WTW^{-1} = T' = \sqrt{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It means that T' is decomposed into:

1) a homothety of center 0 and ratio $\sqrt{2}$

2) a rotation of $\frac{\pi}{2}$ of the orthogonal projection on Q

3) a symmetry with respect to w_1 of the orthogonal projection on P.

In the same way as we obtained the equation $S(4t) = MS(t)$, we now have $S(2t) = T\,S(t)$ and we have just given the interpretation of the transformation $S(t) \longrightarrow S(2t)$ as a product of simple transformations.

4.4 We have investigated the properties of the transformations $S(t) \longrightarrow S(16t)$, $S(t) \longrightarrow S(4t)$, $S(t) \longrightarrow S(2t)$ as products of homotheties and isometries. Now we shall look at the effect of a translation of t by an integer. We are interested in differences $S(t) - S(s)$.

Let us begin with integers $m < n < 16^k$. Let us divide the series $a_0 u_0 + a_1 u_1 + \cdots$ into consecutive blocks of length 16^k, so that the series reads

$$+A + B + C + D + A - B + C - D + \cdots$$

If $j = j_0 + 4j_1$ ($j_0 = 0, 1, 2, 3, j_1 \in \mathbb{N}$), the j–th term is of type A, B, C, D according to the value of j_0 and its sign is a_j. Therefore

$$S(n + j \cdot 16^k) - S(m + j \cdot 16^k) = a_j (S(n + j_0 16^k) - S(m + j_0 16^k)) \,.$$

If $0 < s < t < 1$, we can approximate s and t by $m\,16^{-k}$ and $n\,16^{-k}$ and we obtain

$$S(t + j) - S(s + j) = a_j (S(t + j_0) - S(s + j_0))$$

($j_0 = 1, 2, 3, 4,\ j_0 = j$ modulo 4).

This expresses that all arcs $\mathcal{A}_j = S([j, j+1])$ are isometric (actually translates or symmetric according to the value of a_j) of one of the arcs $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ (according to the value of j_0). Using 4.2, this holds true when we replace the $\mathcal{A}_j$s by $\mathcal{A}_j^\nu = S([j2^\nu, (j+1)2^\nu])$ whatever $\nu \in \mathbb{Z}$.

5. It is a Brownian Helix

5.1 What we have to prove is that, writing

$$a = \inf_{0<s<t} \frac{||S(t) - S(s)||}{\sqrt{|t - s|}} \leq \sup_{0<s<t} \frac{||S(t) - S(s)||}{\sqrt{|t - s|}} = b \,,$$

we have $a > 0$ and $b < \infty$. We can also write

$$a = \inf_{m<n} \frac{||S(n) - S(m)||}{\sqrt{|n - m|}} \,, \quad b = \sup_{m<n} \frac{||S(n) - S(m)||}{\sqrt{|n - m|}}$$

5.2 The easy part is $b < \infty$. Let us first assume $[m, n] = [j2^k, (j + 1)2^k]$. Then, according to 4.3, $||S(n) - S(m)|| = 2^{k/2}$. In the general case, let us decompose $[m, n]$ into such intervals in a minimal way, so that there are at most two intervals of the same length in the decomposition. If the largest length is 2^k, we obtain

$$||S(n) - S(m)|| \leq 2(2^{k/2} + 2^{(k-1)/2} + \cdots) \leq 2(1 + \sqrt{2})2^{k/2}$$

and therefore $||S(n) - S(m)|| \leq 2(1 + \sqrt{2})|n - m|^{1/2}$. This gives $b \leq 2(1 + \sqrt{2})$.

5.3 To prove $a > 0$ is more tricky. We shall use two lemmas.

Lemma 1. *There exists* $\alpha > 0$ *such that* $||S(n+h) - S(n)|| \leq 1 - \alpha$ *for all* $n \in \mathbb{N}$ *and* $h \in [-\frac{1}{2}, \frac{1}{2}]$ *(with* $n + h \geq 0$*).*

Lemma 2. *There exists an integer* A *such that* $||S(n) - S(m)|| \geq 2$ *for all integers* m *and* n *such that* $n - m \geq A$.

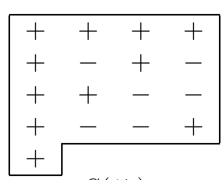
Assuming that this is correct, the result is at hand: given t and s such that $t - s \geq A + 1$, we can write $s = m + h$ and $t = n + h'$ with $h,\ h' \in [-\frac{1}{2}, \frac{1}{2}]$ and $m - n \geq A$, therefore $||S(t) - S(s)|| \geq 2\alpha$. Whenever $(A+1)2^k < t - s \leq (A+1)2^{k+1}$ ($k \in \mathbb{Z}$), we have

$$||S(t) - S(s)|| \geq 2\alpha 2^{k/2} \geq \frac{2\alpha}{\sqrt{2(A+1)}}\ |t - s|^{1/2}$$

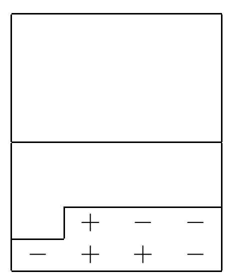
and therefore

$$a \geq \frac{2\alpha}{\sqrt{2(A+1)}}\,.$$

5.4 *Proof of Lemma 1.* From now on it may be useful to represent $S(n)$ on the table of 3.0, and also the differences $S(n) - S(m)$, as figures consisting of consecutive lines plus or minus part of a line above and below, in such a way that each column in the figure has a sum equal to the corresponding coordinate of $S(n)$ or $S(n) - S(m)$.



$S(17)$



$S(32) - S(25)$

Let us consider $||S(16n+m) - S(16n)||^2$ for $n \in \mathbb{N}$ and $m = 0, \pm 1, \pm 2, \pm 3, \ldots, \pm 8$. It is sufficient to consider the four cases $n = 0, 1, 2, 3$, and to look at the figures (depending on m) in each case. The result is

$$\begin{array}{ll} ||S(16n+m) - S(16n)||^2 \leq 8 & (n \text{ odd}) \\ ||S(16n+m) - S(16n)||^2 \leq 9 & (n \text{ even}) \end{array}$$

with equality only when m is odd (as for $S(32) - S(25)$). Therefore, going one step further,

$$||S\big(16n + m + \tfrac{P}{16}\big) - S(16n)|| \quad \leq \sqrt{8} + \tfrac{1}{4}\sqrt{9} \ \ (n \text{ odd})$$

$$||S\big(16n + m + \tfrac{P}{16}\big) - S(16n)|| \quad \leq \sqrt{9} + \tfrac{1}{4}\sqrt{8} \ \ (n \text{ even})$$

when $p = 0, \pm 1, \pm 2, \ldots, \pm 8$. Proceeding that way we finally obtain

$$||S(16(n+t)) - S(16n)|| \leq 4(1-\alpha) \qquad \big(-\tfrac{1}{2} \leq t \leq \tfrac{1}{2}\big)$$

where

$$4(1-\alpha) = \big(\sqrt{9} + \frac{1}{4}\sqrt{8}\big)\big(1 + \frac{1}{16} + \frac{1}{16^2} + \cdots\big)$$

as the second member is less than 4 and that proves Lemma 1.

5.5 *Proof of Lemma 2.* Here again we look at the table. We can compute $||S(n) - S(m)||$ when $n - m$ is in a given interval, and moreover give the couples (m, n) for which the infimum is attained, and the expression of $S(n) - S(m)$ (that is, the coordinates with respect to u_0, u_1, u_2, u_3).

1) $4 < n - m \leq 16$. It suffices to consider the first 16 lines of the table (fig. 1), since adding to m and n a multiple of 64 does not change $||S(n) - S(m)||$. We obtain $\inf ||S(n) - S(m)||^2 = 2$ realized for $(5, 11)$, $(23, 29)$, $(35, 41)$ and $(53, 59)$:

$$\begin{array}{ll} S(11) - S(5) = u_1 - u_4\,, & S(29) - S(23) = u_2 - u_3 \\ S(41) - S(35) = -u_2 + u_3\,, & S(59) - S(53) = -u_1 + u_4. \end{array}$$

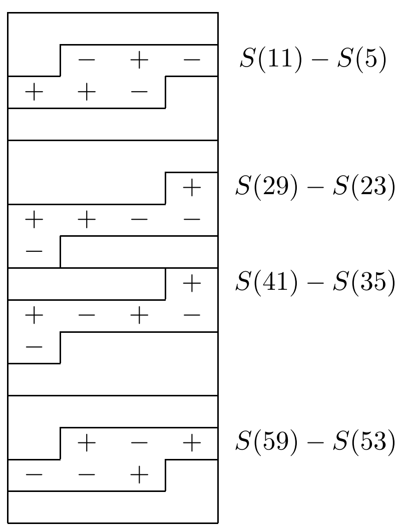


Figure 1

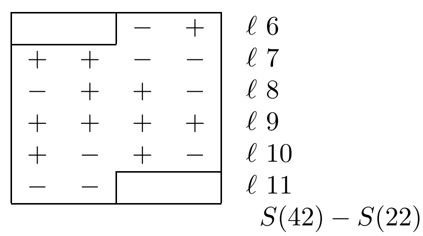


Figure 2

2) $16 \leq n - m \leq 64$. It suffices now to consider the first 256 terms (the first 64 lines of the table). The idea in order to pick the infimum is to start from $S(4 \times 11) - S(4 \times 5)$ and the analogues, and modify the figure in order to diminish $||S(n) - S(m)||^2$ (fig. 2). As a first example, $S(44) - S(20) = 2u_2 + 2u_3$ (obtained

by replacing $u_1 + u_2 + u_3 + u_4$ and u_4 by $u_1 - u_2 - u_3 + u_4$ in the expression of $S(11) - S(5)$), and the modification provides $S(42) - S(22) = u_1 + u_2 + u_3 - u_4$. The result is

$$\inf ||S(n) - S(m)||^2 = 4$$

realized for $(22, 42)$ and $(214, 234)$, with

$$\begin{aligned} S(42) - S(22) &= u_1 + u_2 - u_3 - u_4 \,, \\ S(234) - S(214) &= -u_1 - u_2 - u_3 + u_4 \,. \end{aligned}$$

This proves Lemma 2 with $A = 16$.

Actually the proof can be given in a more concentrated form. It is enough to show that $||S(n) - S(m)||^2 \leq 3$ is impossible when $n - m \geq 16$. Let us assume *ab absurdo* that $||S(n) - S(m)||^2 \leq 3$. Let us add or remove the minimal number of terms in order to transform $S(n) - S(m)$ into a difference of the form $S(4n') - S(4m')$ (that is, to transform the figure $S(n) - S(m)$ into a rectangle). In general, this minimal number is ≤ 4 and has the same parity as $||(S(n) - S(m)||^2$; here it is ≤ 3 and the resulting $S(4n') - S(4m')$ has its squared norm ≤ 2, and therefore $||S(n') - S(m')||^2 \leq 3$ and the process goes on until we reach $S(n^\times) - S(m^\times)$ with $n^\times \leq 64$. Then we know the possible pairs $(m^\times, n^\times)$, namely (5,11), (23,29), (35,41) and (53,59), and the reverse process never gives a squared norm ≤ 3.

5.6 Remarks and questions

The estimates we gave for b and a are quite rough. We can ask for better estimates and conjectures. The actual problem, of a combinatorial or arithmetical nature, is to compute these numbers exactly.

We were interested in estimating b from above and a from below. Examples provide estimates in the opposite direction:

$$\begin{aligned} b &\geq \tfrac{||S(17)||}{\sqrt{17}} = \tfrac{5}{\sqrt{17}} \geq 1.21 \\ a &\leq \tfrac{||S(42)-S(22)||}{\sqrt{20}} = \tfrac{2}{\sqrt{20}} = \tfrac{1}{\sqrt{5}} \leq 0.45. \end{aligned}$$

It seems not impossible that the estimate for α is precise; that is, $a = \frac{1}{\sqrt{5}}$. A careful investigation of the table would confirm or disprove this conjecture. It would lead also to a better estimate for b.

6. Projections of the Curve

6.1 The direction of u_0 is special: all first coordinates of the $S(n)$ are ≥ 0. That means that the partial sums $S_0(n)$ of the original series described in 3.0 are positive.

A simple way to see it is to use Lemma 1 (of 5.2). Since $S_0(n) \geq 1$ for $n = 1, 2, 3, 4, 5, 6, 7, 8$, we have $S_0(t) > 0$ for $\frac{1}{2} \leq t \leq 8$, and therefore (changing t into $16^k t$), $S_0(t) > 0$ for all $t > 0$.

6.2 We are mainly interested in the three–dimensional projections of the curve. It seems likely that all parallel projections of the curve on a three–dimensional subspace of $\mathbb{R}^4$ have an infinity of double points. The question can be formulated in the equivalent forms:

1) is every direction in $\mathbb{R}^4$ the direction of some $S(t) - S(s)$?
2) are the $\frac{S(n)-S(m)}{\sqrt{n-m}}$ $(n > m)$ dense on the sphere S^3?

6.3 Let us project the curve from 0 on the sphere S^3; that is, consider

$$\mathcal{S}(t) = \frac{S(t)}{||S(t)||} \qquad (t > 0)\,.$$

We obtain a closed $\mathcal{C}$, the image of any interval $[a, 16a]$ by $\mathcal{S}(\cdot)$.

We know that $\mathcal{C}$ is invariant under the isometries of $\mathbb{R}^4$ defined by $\frac{1}{2}M$ and $\frac{1}{\sqrt{2}}T$ (see 4.1 and 4.2). The first takes the form

$$\frac{1}{2}M' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

with respect to the orthonormal basis (v_0, v_1, v_2, v_3) and the second

$$\frac{1}{\sqrt{2}}T' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with respect to the orthonormal basis (w_0, w_1, w_2, w_3). The vectors v_0 and v_1 (w_0 and w_1 as well) generate a plane P such that the mapping $\mathcal{S}(t) \longrightarrow \mathcal{S}(4t)$ is an orthogonal symmetry with respect to P. For the projection of $\mathcal{C}$ on P, the change of t into $2t$ means a symmetry with respect to the line generated by w_0.

We claim that $\mathcal{C}$ has a double point at $t = \frac{1}{3}$, $t' = \frac{4}{3}$: $\mathcal{T}(\frac{4}{3}) = \mathcal{T}(\frac{1}{3})$. In order to prove this claim, we expand t and t' in base 4 (we underline the expansion)

$$t = \underline{0.1111}\cdots \qquad t' = \underline{1.1111}\cdots\,.$$

Using base 4 again, we easily obtain the figures and the values of $S(\underline{1})$, $S(\underline{11})$, $S(\underline{111})$

and so on: $S(1) = u_0 = (1\ 0\ 0\ 0)$

$$\begin{aligned}
S(\underline{11}) &= S(\underline{10}) + (S(\underline{11}) - S(\underline{10})) = (1\ 1\ 1\ 1) + (1\ 0\ 0\ 0)\\
S(\underline{111}) &= S(\underline{100}) + (S(\underline{110}) - S(\underline{100})) + S(\underline{111}) - (S(\underline{110})\\
&= (4\ 0\ 0\ 0) + (1\ 1\ 1\ 1) - (1\ 0\ 0\ 0)\\
S(\underline{1111}) &= S(\underline{1000}) + (S(\underline{1100}) - S(\underline{1000})) + (S(\underline{1110}) - S(\underline{1100}))\\
&\qquad + (S(\underline{1111}) - S(\underline{1110}))\\
&= (4\ 4\ 4\ 4) + (4\ 0\ 0\ 0) - (1\ 1\ 1\ 1) + (1\ 0\ 0\ 0)\\
S(\underline{11111}) &= (16\ 0\ 0\ 0) + (4\ 4\ 4\ 4) - (4\ 0\ 0\ 0) + (1\ 1\ 1\ 1) - (1\ 0\ 0\ 0)\\
S(\underline{111111}) &= (16\ 16\ 16\ 16) + (16\ 0\ 0\ 0) - (4\ 4\ 4\ 4) + (4\ 0\ 0\ 0)\\
&\qquad - (1\ 1\ 1\ 1) + (1\ 0\ 0\ 0)
\end{aligned}$$

The ratio between two consecutive vectors tends to 2 (meaning that the ratios of coordinates tend to 2), and hence

$$\mathcal{S}(t') = \mathcal{T}(t) \qquad \big(t = \frac{1}{3}\big)\,.$$

By isometry we also have

$$\mathcal{T}(2t') = \mathcal{T}(2t)\,.$$

These double points are contained in the plane P, and they are symmetric with respect to the line generated by w_0.

We believe, but did not prove, that these are the only multiple points of the curve $\mathcal{C}$. In that case, $\mathcal{C}$ is a Brownian quasi–helix (actually, a Brownian quasi–circle) on some 4–covering of the sphere S^3.

6.4 One can see the curve $\mathcal{C}$ in two other ways.

First, taking into account that the first coordinate $S_0(t)$ is always positive, we can consider

$$\mathcal{R}(t) = \frac{S(t)}{S_0(t)}$$

and the curve $\mathcal{C}'$ described by $\mathcal{R}(\cdot)$, projection of the original curve with a source at 0 and a screen at the hyperplane $x_0 = 1$.

Symmetries and double points can be studied on this model as well.

Secondly, we obtain a projective model of $\mathcal{C}$, say, $\mathcal{C}''$, on choosing four points A_0, A_1, A_2, A_3 in $\mathbb{R}^4$, defining $A_{j+4} = A_j$ $(j = 0, 1, \ldots)$, starting with a point $M_0 = A_0$ and defining the sequence of points

$$M_{n+1} = \frac{1}{a_0 + a_1 + \cdots a_n}((a_0 + a_1 + \cdots a_{n-1})M_n + a_n A_n)\,.$$

Some real figures would help. If a reader is willing to draw figures of the above curves, I'll appreciate to see them.

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