COMPUTATIONAL REDISCOVERY OF RAMANUJAN'S TAU NUMBERS

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Abstract

According to a converse theorem of Hamburger type, Ramanujan's tau numbers are completely determined by the functional equation for Ramanujan's tau L-function. The paper presents a computational method for "extracting" the numbers from the equation.

1. The Result

The celebrated *Ramanujan's tau numbers* arise in many different areas of mathematics. For example, in the Online Encyclopedia of Integer Sequences (OEIS) [8], besides the main sequence

$$\begin{aligned}\text{A000594: } &\tau(1)=1,\ \tau(2)=-24,\ \tau(3)=252,\ \tau(4)=-1472,\ \tau(5)=4830,\\ &\tau(6)=-6048,\ \tau(7)=-16744,\ \tau(8)=84480,\ \tau(9)=-113643,\ \dots \qquad (1)\end{aligned}$$

one finds more than a hundred other related sequences. The tau numbers have many remarkable number-theoretical and combinatorial properties, and there is a great number of still unproved conjectures about them (see, for example, [2, Chapter 10] and [5, Chapter 2]).

Ramanujan's tau numbers can be defined in many diverse ways. One of the standard definitions is via the *ordinary generating function*:

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}. \qquad (2)$$

The right-hand side of (2) looks a bit mysterious: what is special about the exponent 24, and what is the role of the first factor q? But it turns out that with

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this definition Ramanujan's tau numbers have a great deal of remarkable number-theoretical properties. In particular, S. Ramanujan [7] made two striking discoveries:

- *if m and n are relatively prime, then*

$$\tau(mn) = \tau(m)\tau(n); \tag{3}$$

- *if p is prime and $k > 1$, then*

$$\tau(p^{k+1}) = \tau(p)\tau(p^k) - p^{11}\tau(p^{k-1}). \tag{4}$$

These two properties were proved later by L. J. Mordell [4].

Together with the Fundamental Theorem of Arithmetic, property (3) implies that all values of τ are uniquely determined by the values of this function at $n = 1$ and at all prime numbers. Analytically, this fact means that the *Dirichlet generating function* for Ramanujan's tau numbers can be expressed as a product over prime numbers:

$$L_\tau(s) = \sum_{n=1}^{\infty} \tau(n)n^{-s} = \prod_{p \text{ prime}} \sum_{k=0}^{\infty} \tau(p^k)p^{-ks}. \tag{5}$$

Property (4) allows one to find a closed expression for the the sum in the right-hand side of (5):

$$\sum_{k=0}^{\infty} \tau(p^k)p^{-ks} = \frac{1}{1 - \tau(p)p^{-s} + p^{11}p^{-2s}}. \tag{6}$$

Respectively,

$$L_\tau(s) = \prod_{p \text{ prime}} \frac{1}{1 - \tau(p)p^{-s} + p^{11}p^{-2s}}. \tag{7}$$

Expressions such as the right-hand side of (7) are usually called *Euler products* after the very first identity of this type, namely,

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}, \tag{8}$$

found by L. Euler.

Riemann's zeta function $\zeta(s)$ (defined by (8)) and *Ramanujan's tau L-function* $L_\tau(s)$ (defined by (5)) have many similar properties (see, for example, [2, Chapter 10]). While series and products in (5), (7) and (8) converge absolutely in half-planes only (for $\Re(s) > 1$ and for $\Re(s) > 13/2$ respectively), both functions can be analytically extended to the whole complex plane (except for the point $s = 1$ in the case of the zeta function).

It is expected that $L_\tau(s)$ satisfies a counterpart of the Riemann Hypothesis which was stated for the zeta function. Namely, B. Riemann predicted (this still remains unproved) that all non-real zeroes of the zeta function lie on the *critical line* $\Re(s) = 1/2$; for Ramanujan's tau L-function, the similar critical line is defined as $\Re(s) = 6$.

Both Riemann's zeta function and Ramanujan's tau L-function satisfy *functional equations*

$$g(s)\zeta(s) = g(1-s)\zeta(1-s) \tag{9}$$

and

$$g_\tau(s)L_\tau(s) = g_\tau(12-s)L_\tau(12-s) \tag{10}$$

respectively, where

$$g(s) = \pi^{-\frac{s}{2}}(s-1)\Gamma(\tfrac{s}{2}+1), \tag{11}$$

$$g_\tau(s) = (2\pi)^{-s}\,\Gamma(s). \tag{12}$$

H. Hamburger established that the functional equation (9) essentially uniquely (up to a constant factor) distinguishes the zeta function among all functions defined by Dirichlet series and satisfying certain mild extra restrictions. More precisely, he proved ([1], see also [6, Theorem 2.1]) the following.

Theorem (H. Hamburger). *Let $F(s)$ be a Dirichlet series, absolutely convergent for $\sigma > 1$, such that $(s-1)^m F(s)$ is entire of finite order for some integer m. Moreover, suppose that $F(s)$ satisfies the functional equation*

$$g(s)F(s) = g(1-s)F(1-s)$$

where $g(s)$ is defined by (11). Then $F(s) = c\zeta(s)$ for some $c \in \mathbb{C}$.

Later, similar *converse theorems* were established for many other functional equations (for a recent survey of converse theorems of Hamburger type, see [6]). In particular, from general results of E. Hecke [3] it follows that function $L_\tau(s)$ is in a similar sense determined by the functional equation

$$g_\tau(s)D(s) = g_\tau(12-s)D(12-s) \tag{13}$$

where $g_\tau(s)$ is defined by (12) and

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{14}$$

is a Dirichlet series.

Ramanujan's tau numbers are just the coefficients of their Dirichlet generating function $L_\tau(s)$, so together with an extra normalizing condition

$$a_1 = 1 \tag{15}$$

these numbers are also uniquely determined by the functional equation (13).

The author was interested in the computational aspect of theorems of Hamburger type: *a functional equation is given, how could one find the coefficients of the corresponding Dirichlet series?* For example, how such a simple factor (12) from (13) produces the sequence (1) with such a rich structure? Moreover, how such a rediscovery of Ramanujan's tau numbers could be made by a computer, that is, by pure calculation without any knowledge from the proof of the Hamburger theorem? This paper presents a possible way to do it.

The main tool in our computations will be as follows. We will consider a two-parameter family of finite Dirichlet series

$$D_{M,N}(s) = \sum_{n=1}^{N} a_{M,N,n} n^{-s} \tag{16}$$

with growing M and $N > M$. These finite Dirichlet series will mimic the infinite sum (14) in the following sense.

The gamma function satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s) \tag{17}$$

or, more generally, for a natural number k

$$\Gamma(s+k) = (s+k-1)^{\underline{k}}\,\Gamma(s) \tag{18}$$

where

$$m^{\underline{k}} = m(m-1)\dots(m-k+1) \tag{19}$$

denotes the *falling factorial*. Respectively, if s is greater than 6 and is an integer or a half-integer, then the functional equation (13) simplifies to

$$(s-1)^{\underline{2s-12}} D(s) = (2\pi)^{2s-12} D(12-s). \tag{20}$$

We will require that $D_{M,N}(s)$ should satisfy the formal counterpart of this equation, that is,

$$(s-1)^{\underline{2s-12}} D_{M,N}(s) = (2\pi)^{2s-12} D_{M,N}(12-s). \tag{21}$$

This goal will be achieved in two steps. At first, on the basis of previous calculations of $D_{M',N}(s)$ for $M' < M$, certain integer values will be assigned to the coefficients $a_{M,N,1}, \dots, a_{M,N,M}$. After that the values of the remaining coefficients,

$$a_{M,N,M+1}, \dots, a_{M,N,N}, \tag{22}$$

can be determined by solving the system consisting of $N-M$ linear equations (21) for $s = 6.5,\ \dots,\ 6 + (N-M)/2$.

In accordance with (15), we start by putting

$$a_{M,N,1} = 1 \tag{23}$$

n	$a_{1,40,n}$	$\frac{a_{1,40,n}}{\tau(n)}$
1	**1**	
2	-23.9999998088...	0.99999999203...
3	251.9999296657...	0.99999972089...
4	-1471.9896684994...	0.99999298131...
5	4829.1878260000...	0.99983184803...
6	-6008.7075108132...	0.99350322599...
7	-18021.2327635150...	1.07628002648...
8	114131.4715225206...	1.35098806252...
9	-627764.9680649609...	5.52400911683...
10	677139.33398...$\cdot 10^1$	-5.84143662855...$\cdot 10^1$
11	-726508.78159...$\cdot 10^2$	-1.35894589271...$\cdot 10^2$
12	629469.61495...$\cdot 10^3$	-1.69693974011...$\cdot 10^3$
13	-446455.39726...$\cdot 10^4$	7.72764466361...$\cdot 10^3$
14	264116.58937...$\cdot 10^5$	6.57241871152...$\cdot 10^4$
15	-131902.30896...$\cdot 10^6$	-1.08368915312...$\cdot 10^5$
16	561177.40571...$\cdot 10^6$	5.68490467083...$\cdot 10^5$
17	-204930.92070...$\cdot 10^7$	2.96746132682...$\cdot 10^5$
18	646393.69446...$\cdot 10^7$	2.36997180670...$\cdot 10^6$
19	-177017.07880...$\cdot 10^8$	-1.66035179936...$\cdot 10^6$
20	422662.28121...$\cdot 10^8$	-5.94481784499...$\cdot 10^6$
21	-882873.93521...$\cdot 10^8$	2.09237219116...$\cdot 10^7$
22	161757.14922...$\cdot 10^9$	-1.26070518766...$\cdot 10^7$
23	-260436.65773...$\cdot 10^9$	-1.39694715462...$\cdot 10^7$
24	368904.12765...$\cdot 10^9$	1.73284241061...$\cdot 10^7$
25	-459911.72578...$\cdot 10^9$	1.80363021143...$\cdot 10^7$
26	504456.23327...$\cdot 10^9$	3.63815600146...$\cdot 10^7$
27	-486221.38176...$\cdot 10^9$	6.63520041136...$\cdot 10^6$
28	410936.63100...$\cdot 10^9$	1.66727727502...$\cdot 10^7$
29	-303565.31021...$\cdot 10^9$	-2.36409374047...$\cdot 10^6$
30	195131.34182...$\cdot 10^9$	-6.67987164868...$\cdot 10^6$
31	-108491.66351...$\cdot 10^9$	2.05308779963...$\cdot 10^6$
32	517628.62548...$\cdot 10^8$	-2.63147959653...$\cdot 10^5$
33	-209725.08751...$\cdot 10^8$	-1.55672227852...$\cdot 10^5$
34	711610.74909...$\cdot 10^7$	4.29347397164...$\cdot 10^4$
35	-198413.79513...$\cdot 10^7$	2.45338393998...$\cdot 10^4$
36	442674.42823...$\cdot 10^6$	2.64626867015...$\cdot 10^3$
37	-759775.20238...$\cdot 10^5$	4.16970190434...$\cdot 10^2$
38	941751.85236...$\cdot 10^4$	-3.68052853327...$\cdot 10^1$
39	-750306.74459...$\cdot 10^3$	5.15356046621...
40	288516.93819...$\cdot 10^2$	0.07070828093...

Table 1: Coefficients of $D_{1,40}(s)$ (the value in bold font was assumed)

n	$a_{2,60,n}$
1	**1**
2	**-24**
3	251.9999996376...
4	-1471.9998178947...
5	4829.9632177702...
6	-6043.9015870799...
7	-17034.7222907912...
8	98781.1308120727...
9	-630524.2005730589...
10	141967.69945...$\cdot 10^2$
11	-312893.27864...$\cdot 10^3$
12	556396.60637...$\cdot 10^4$
13	-816921.70473...$\cdot 10^5$
14	100802.31458...$\cdot 10^7$
15	-105955.66037...$\cdot 10^8$

n	$a_{3,60,n}$
1	**1**
2	**-24**
3	**252**
4	-1471.9999996964...
5	4829.9999074631...
6	-6047.9911420556...
7	-16744.1087033100...
8	84437.2081240563...
9	-109486.8742145104...
10	-328560.5125312068...
11	790626.86817...$\cdot 10^1$
12	-190170.19334...$\cdot 10^3$
13	381571.88089...$\cdot 10^4$
14	-619271.65233...$\cdot 10^5$
15	830436.31394...$\cdot 10^6$

n	$a_{4,65,n}$
1	**1**
2	**-24**
3	**252**
4	**-1472**
5	4829.9999969634...
6	-6047.9990590632...
7	-16744.1075358648...
8	84485.2350828571...
9	-113639.5272105033...
10	-131544.3590314542...
11	159828.55559...$\cdot 10^1$
12	-428780.81586...$\cdot 10^2$
13	121315.56362...$\cdot 10^4$
14	-267038.27692...$\cdot 10^5$
15	472032.53535...$\cdot 10^6$

n	$a_{5,70,n}$
1	**1**
2	**-24**
3	**252**
4	**-1472**
5	**4830**
6	-6048.0000068875...
7	-16743.9949504933...
8	84478.7635385720...
9	-113484.7770769369...
10	-128610.3182084142...
11	123950.48150...$\cdot 10^1$
12	-292647.49132...$\cdot 10^2$
13	912888.84350...$\cdot 10^3$
14	-230093.88974...$\cdot 10^5$
15	473484.93436...$\cdot 10^6$

Table 2: Initial coefficients of $D_{2,60}(s)$, $D_{3,60}(s)$, $D_{4,65}(s)$, and $D_{5,70}(s)$ (the values in bold font were assumed)

s	$\left\|\frac{D_{1,40}(s)}{L_\tau(s)} - 1\right\|$	s	$\left\|\frac{D_{1,40}(s)}{L_\tau(s)} - 1\right\|$
-10.5	$3.11427\ldots\cdot 10^{-14}$	$6+5i$	$2.40639\ldots\cdot 10^{-10}$
$-8+2i$	$1.00329\ldots\cdot 10^{-12}$	$6+10i$	$1.85987\ldots\cdot 10^{-8}$
$-6+4i$	$7.31581\ldots\cdot 10^{-13}$	$8+10i$	$4.76871\ldots\cdot 10^{-9}$
$-4+6i$	$3.66512\ldots\cdot 10^{-12}$	$10+10i$	$1.01404\ldots\cdot 10^{-9}$
$-2+8i$	$1.78973\ldots\cdot 10^{-11}$	$12+12i$	$4.78240\ldots\cdot 10^{-10}$
$10i$	$2.03252\ldots\cdot 10^{-10}$	$14+12i$	$5.76541\ldots\cdot 10^{-11}$
$2+10i$	$1.00476\ldots\cdot 10^{-9}$	$16+12i$	$6.58641\ldots\cdot 10^{-12}$
$4+10i$	$4.77546\ldots\cdot 10^{-9}$	$18+12i$	$9.05989\ldots\cdot 10^{-13}$
6	$7.44535\ldots\cdot 10^{-11}$	$20+12i$	$1.85557\ldots\cdot 10^{-13}$

Table 3: The accuracy of approximation $L_\tau(s)$ by $D_{1,40}(s)$ for various values of the argument

for all M and N. After that we proceed as follows.

Table 1 presents the coefficients of $D_{1,40}(s)$ defined in the above described way. We observe that the value of $a_{1,40,2}$ is very close to an integer and from now on we will assume that

$$a_{M,N,2} = -24 \tag{24}$$

for $N > M > 2$.

Table 2 presents (a part of) the coefficients of $D_{2,60}(s)$. We observe that the value of $a_{2,60,3}$ is very close to an integer and from now on we will assume that

$$a_{M,N,3} = 252 \tag{25}$$

for $N > M > 3$.

Continuing in this style with the other data from Table 2, we'll come to the assumptions that for $N > M > 5$

$$a_{M,N,4} = -1472,\ a_{M,N,5} = 4830,\ a_{N,M,6} = -6048. \tag{26}$$

The six values, 1, -24, 252, -1472, 4830, -6048 are sufficient for the OEIS [9] to recognize them as the beginning of A000594, the sequence of Ramanujan's tau numbers. In other words, starting from the functional equation (13), calculating some real numbers and rounding them to rather close integers, we are able to surmise that Ramanujan's tau L-function should give a solution of this functional equation.

Remarks. Our introduction of the functional equation (21) was quite formal, by the mere syntactical resemblance with the functional equations (10) and (13). It cannot be justified reasonably because the series in right-hand side of (10) does not converge for the range of values of s used by us (that is, for $s \geq 6.5$). So it is not surprising that most of the coefficients of our finite Dirichlet series $D_{M,N}(s)$ differ, and considerably, from the corresponding coefficients of $L_\tau(s)$ (see, for example, the last column in Table 1). On the contrary, it was unsuspected that a few initial coefficients of $D_{M,N}(s)$ still turned out to be rather close to the initial tau numbers.

Even more startling is the following observation. In spite of the fact that the coefficients of the finite Dirichlet series $D_{M,N}(s)$ are so different from the corresponding coefficients of the infinite Dirichlet series (5), the values of the both series are very close each other for a large range of the values of s, including those where the infinite series diverges – see Table 3.

For getting approximate values of the coefficients of $L_\tau(s)$ and its values we used (21) for a discrete set of values of s (for integers and half-integers only). Does it indicate that the converse theorem for $L_\tau(s)$ can be improved by demanding the validity of the functional equation (10) just for these values of s?

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[8] Online Encyclopedia of Integer Sequences: `https://oeis.org`.

[9] Online Encyclopedia of Integer Sequences, recognizing sequence 1, −24, 252, −1472, 4830, −6048: `https://oeis.org/search?q=1%2C-24%2C252%2C-1472%2C4830%2C-6048`.