TWO TREES ENUMERATING THE POSITIVE RATIONALS

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Abstract

We give two trees allowing us to represent all positive rational numbers. These trees can be seen as ternary and quinary analogues of the Calkin-Wilf tree. For each of these two trees, we give recurrence formulas allowing us to compute the rational number corresponding to the node n. These are analogues of the formulas given by Donald Knuth and Moshe Newman for the Calkin-Wilf tree. Finally, we show that the two sequences we have obtained, together with the Calkin-Wilf sequence, are the only ones which satisfy a relation analogous to Newman's relation and enumerate the positive rationals.

1. Introduction

It is well-known, since Cantor's first works on the theory of cardinality, that the rationals are countable. However, it is not so simple to give an explicit enumeration of all of them. Throughout this paper, we designate $\mathbb{N}$ as the set of nonnegative integers $\{0, 1, 2, 3, \ldots\}$ and $\mathbb{Q}_+$ as the set of nonnegative rationals. Most of the time (see [4]), one proves that $\mathbb{Q}_+$ is countable by constructing a bijection (or an injection) from $\mathbb{N}^2$ to $\mathbb{N}$, which yields an injection from $\mathbb{Q}_+$ to $\mathbb{N}$, and the conclusion follows from the Cantor-Bernstein theorem.

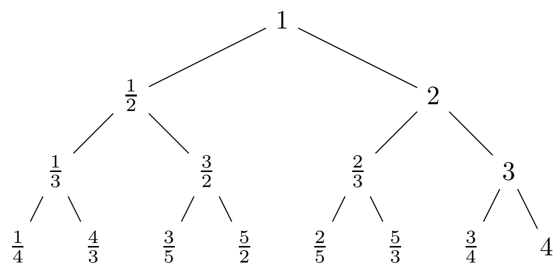
In 2000, N. Calkin and H. S. Wilf [7] described an elegant explicit enumeration of $\mathbb{Q}_+^*$ where $\mathbb{Q}_+^*$ denotes the set of positive rationals. Its first few terms are

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \frac{1}{5}, \frac{5}{4}, \frac{4}{7}, \ldots$$

This sequence, known as the Calkin-Wilf sequence, is defined by a binary tree in the following way:

- the top of the tree is $\frac{1}{1}$;
- the vertex labeled $\frac{a}{b}$ has two children: the left child labeled $\frac{a}{a+b}$ and the right child labeled $\frac{a+b}{b}$.

This leads to the Calkin-Wilf tree, whose first few rows are:



The Calkin-Wilf sequence is then obtained by reading the fraction $\frac{1}{1}$ on level 1, then the two fractions on level 2 from left to right, then the four fractions on level 3 from left to right, and so on. Besides the fact that every positive rational number appears once and only once in reduced form in the tree, this sequence has another remarkable property: the numerator of the term of rank $n+1$ is equal to the denominator of the term of rank n. In other words, there exists a sequence of positive integers (b_n) such that the term of rank n of the Calkin-Wilf sequence is equal to $\frac{b_n}{b_{n+1}}$. In fact, the sequence (b_n) was discovered as early as the mid 19th century, independently by the German mathematician M. Stern [18] and the French clockmaker A. Brocot [5] by considering the median fraction $\frac{a+b}{c+d}$ of two fractions $\frac{a}{b}$ and $\frac{c}{d}$. This procedure leads to another binary tree which enumerates the rationals, named the Stern-Brocot tree [8, pp. 116–123 and pp. 305–306] and closely connected to the Calkin-Wilf tree (see [11] and [2]). B. Reznick [17] notes that Stern proved in his 1858 paper that, for every pair of positive coprime integers (a, b), there exists one and only one integer n such that $b_n = a$ and $b_{n+1} = b$. In other words, Stern proved that $\mathbb{Q}_+^*$ is countable more than 15 years before Cantor's first papers on the subject. The sequence (b_n), which is now known as Stern's diatomic sequence, has been widely studied since that time and is known to be connected with many other subjects such as hyperbinary representations, Farey sequences, continued fractions, the Fibonacci sequence and the Minkowski ?-function (see [1, pp. 104–108] and [14]).

The Calkin-Wilf sequence gives also the answer to a problem set by D. Knuth [9]: if $v_p(n)$ denotes the p-adic valuation of the positive integer n and $\mathbb{N}^*$ the set of positive integers $\{1, 2, 3, \ldots\}$, prove that the sequence (x_n) defined by

$$x_0 = 0 \text{ and, for every } n \in \mathbb{N}^*,\ x_n = \frac{1}{1 + 2v_2(n) - x_{n-1}} \tag{1}$$

enumerates the positive rationals. Various solutions to this problem were given in [10], among them C. P. Ruppert's solution, which associates to the sequence (x_n) a tree almost identical to Calkin-Wilf tree. The only difference is that the vertices are labeled, not by the rationals $\frac{a}{b}$, but by the pairs of coprime positive integers

(a, b), which is clearly the same. Hence Knuth's sequence (x_n) is exactly the same the Calkin-Wilf sequence.

The editors of [10] also quote an answer of Moshe Newman, who showed that the sequence (x_n) satisfies the recurrence relation:

$$\text{For every } n \in \mathbb{N}^*,\ x_n = \frac{1}{1 + 2\lfloor x_{n-1} \rfloor - x_{n-1}} \tag{2}$$

where $\lfloor x \rfloor$ denotes the integral part of the real number x. This implies, in particular, the striking result:

$$\text{For every } n \in \mathbb{N}^*,\ \ \lfloor x_{n-1} \rfloor = v_2(n). \tag{3}$$

Another way to formulate Newman's result consists in saying that the function f defined on $\mathbb{R}_+$ by

$$f : x \mapsto \frac{1}{1 + 2\lfloor x \rfloor - x} \tag{4}$$

generates all positive rationals by iteration starting from $x_0 = 0$.

The purpose of this paper is to construct two sequences (t_n) and (s_n) satisfying relations similar to (1) and (2). For doing this, we define two trees: a ternary tree associated to the sequence (t_n) and a quinary tree associated to the sequence (s_n). These two trees are not labeled by rationals or pairs of coprime integers, but by triples of integers. They can be considered as generalizations of the Calkin-Wilf tree, in the sense that they lead to sequences which enumerate the positive rationals and satisfy relations similar to (1), (2) and (3). However, these generalizations are quite different from those proposed by T. Mansour and M. Shattuck ([12] and [13]), by B. Bates and T. Mansour [3] and by S. H. Chan [6]. Finally, we show that the sequences (t_n) and (s_n) are, together with the Calkin-Wilf sequence, the only sequences (u_n) which enumerate the positive rationals and are defined by $u_0 = 0$ and a recurrence relation of the form

$$u_n := \frac{f(u_{n-1})}{k} \qquad (n = 1, 2, 3, \ldots) \tag{5}$$

where f is defined by (4) and $k \in \mathbb{N}^*$.

2. A Ternary Tree

2.1. Definition

We consider the ternary tree $\mathcal{A}_3$ whose vertices are labeled by triples of integers $(a\,; b\,; c)$ and such that:

- the top of the tree is $(1\,; 2\,; 0)$;

- the children of $(a\,;b\,;c)$ are defined by:
  - ▶ if b is odd:

$$(a\,;b\,;c)$$

$$(4(c+1)a-b\,;2a\,;0) \qquad (a\,;2a+b\,;c+1) \qquad (2a+b\,;2a+2b\,;0)$$

  - ▶ if b is even:

$$(a\,;b\,;c)$$

$$\left(2(c+1)a-\frac{b}{2}\,;a\,;0\right) \qquad (a\,;2a+b\,;c+1) \qquad \left(a+\frac{b}{2}\,;a+b\,;0\right)$$

Hence the first few levels of $\mathcal{A}_3$ are:

$$(1\,;2\,;0)$$

$$(1\,;1\,;0) \qquad\qquad (1\,;4\,;1) \qquad\qquad (2\,;3\,;0)$$

$$(3\,;2\,;0)\ (1\,;3\,;1)\ (3\,;4\,;0)\ (2\,;1\,;0)\ (1\,;6\,;2)\ (3\,;5\,;0)\ (5\,;4\,;0)\ (2\,;7\,;1)\ (7\,;10\,;0)$$

The three children of the vertex $N = (a\,;b\,;c)$ are called respectively the left, the middle and the right child of N and we say that N is the parent of these three children.

For every $n \in \mathbb{N}^*$, we denote by $N_n = (a_n\,;b_n\,;c_n)$ the vertex of index n of the tree $\mathcal{A}_3$ read from the top and, at each level, from left to right. Hence, $N_1 = (1\,;2\,;0)$, $N_2 = (1\,;1\,;0)$, $N_3 = (1\,;4\,;1)$, and so on. Observe that, by definition, for every $n \in \mathbb{N}^*$, the left, middle and right children of N_n are N_{3n-1}, N_{3n} and N_{3n+1}, respectively.

Lemma 1. *For every $n \in \mathbb{N}^*$, $c_n = v_3(n)$.*

Proof. For $n = 1$, the statement is true since $c_1 = 0 = v_3(1)$. Assume that $c_n = v_3(n)$ for a given $n \in \mathbb{N}^*$. Then the left child of N_n is N_{3n-1}, whence by definition

$c_{3n-1} = 0 = v_3(3n-1)$. Similarly, N_{3n+1} is the right child of N_n and $c_{3n+1} = 0 = v_3(3n+1)$. Finally, as N_{3n} is the middle child of N_n, $c_{3n} = c_n + 1 = v_3(n) + 1 = v_3(3n)$, and Lemma 1 is proved by induction. □

Lemma 2. *For every $n \in \mathbb{N}^*$, a_n and b_n are positive coprime integers and, moreover, $2a_n \geqslant b_n - 4a_nc_n$.*

Proof. For $n = 1$, the statement is true since $a_1 = 1$, $b_1 = 2$ and $c_1 = 0$. Assume that, for a given $n \in \mathbb{N}^*$, a_n and b_n are positive coprime integers satisfying $2a_n \geqslant b_n - 4a_nc_n$.

Assume that b_n is odd. Then the three children of N_n are

$$N_{3n-1} = (4(c_n+1)a_n - b_n\,;2a_n\,;0)$$
$$N_{3n} = (a_n\,;2a_n + b_n\,;c_n+1)$$
$$N_{3n+1} = (2a_n + b_n\,;2a_n + 2b_n\,;0)$$

As a_n and b_n are positive integers, it is clear that $b_{3n-1} = 2a_n$, $a_{3n} = a_n$, $b_{3n} = a_{3n+1} = 2a_n + b_n$ and $b_{3n+1} = 2a_n + 2b_n$ are positive integers. Moreover, since $2a_n \geqslant b_n - 4a_nc_n$,

$$a_{3n-1} = 4(c_n+1)a_n - b_n = 4a_n - (b_n - 4a_nc_n) \geqslant 2a_n \tag{6}$$

and a_{3n-1} is also a positive integer.

Let $d = \gcd(a_{3n-1}, b_{3n-1})$. Then d divides $b_{3n-1} = 2a_n$ and $2(c_n+1)b_{3n-1} - a_{3n-1} = b_n$. Hence d is odd since b_n is odd and therefore d divides a_n. As a_n and b_n are coprime, we have $d = 1$, which means that a_{3n-1} and b_{3n-1} are coprime. Similarly we obtain $\gcd(a_{3n}, b_{3n}) = \gcd(a_{3n+1}, b_{3n+1}) = 1$.

Finally, for N_{3n-1} we have, by using (6),

$$b_{3n-1} - 4a_{3n-1}c_{3n-1} = 2a_n \leqslant a_{3n-1} \leqslant 2a_{3n-1}.$$

For N_{3n}, by using (6),

$$b_{3n} - 4a_{3n}c_{3n} = b_n - 4a_nc_n - 2a_n \leqslant 2a_n - 2a_n = 0 \leqslant 2a_{3n},$$

and for N_{3n+1},

$$b_{3n+1} - 4a_{3n+1}c_{3n+1} = 2a_n + 2b_n \leqslant 4a_n + 2b_n = 2a_{3n+1}.$$

In the case where b_n is even, the proof is similar. One only has to replace (6) by

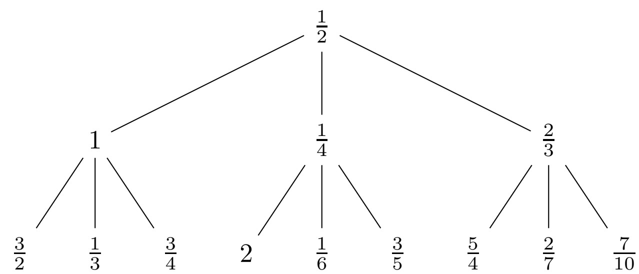
$$a_{3n-1} = 2(c_n+1)a_n - \frac{b_n}{2} = 2a_n - \frac{b_n - 4a_nc_n}{2} \geqslant a_n. \tag{7}$$

Hence Lemma 2 is proved by induction. □

Now, for every $n \in \mathbb{N}^*$, we put

$$t_n = \frac{a_n}{b_n}.$$

By Lemma 2, $(t_n)_{n \in \mathbb{N}^*}$ is a sequence of positive reduced rationals. The first few terms of this sequence are:



We remark that, for every $k \in \mathbb{N}^*$,

$$t_{3k-1} = \frac{4(c_k+1)a_k - b_k}{2a_k} = 2(v_3(k)+1) - \frac{1}{2t_k} = 2v_3(3k) - \frac{1}{2t_k}, \tag{8}$$

$$t_{3k} = \frac{a_k}{2a_k + b_k} = \frac{t_k}{2t_k + 1}, \tag{9}$$

$$t_{3k+1} = \frac{2a_k + b_k}{2a_k + 2b_k} = \frac{2t_k + 1}{2t_k + 2}. \tag{10}$$

We extend this sequence to $\mathbb{N}$ by putting

$$t_0 = 0.$$

We will show that $(t_n)_{n \in \mathbb{N}}$ enumerates the nonnegative rationals, i.e. that $n \mapsto t_n$ is a bijection from $\mathbb{N}$ to $\mathbb{Q}_+$. Before this, we will give two recurrence relations satisfied by the sequence (t_n).

2.2. Two Recurrence Relations

First we prove that the sequence (t_n) satisfies a recurrence relation similar to (1).

Proposition 1. *For every $n \in \mathbb{N}^*$, $t_n = \dfrac{1}{2(1 + 2v_3(n) - t_{n-1})}$.*

Proof. The statement is true for $n = 1$ and $n = 2$, since

$$\frac{1}{2(1+2v_3(1) - t_0)} = \frac{1}{2} = t_1 \qquad \text{and} \qquad \frac{1}{2(1+2v_3(2) - t_1)} = 1 = t_2.$$

Now assume that, for a given integer $n \geqslant 3$, the property is true for every positive integer $j \leqslant n-1$. Denote by N_k ($k \in \mathbb{N}^*$) the parent of N_n.

1st case. — If N_n is the left child of N_k, then $n = 3k-1$ and N_{n-1} is the right child of N_{k-1}. As the property is true when $n = k$, we have by using (8)

$$t_n = 2(v_3(k)+1) - \frac{1}{2t_k} = 2(v_3(k)+1) - (1 + 2v_3(k) - t_{k-1}) = 1 + t_{k-1}. \tag{11}$$

Moreover, since N_{n-1} is the right child of N_{k-1}, $t_{n-1} = \frac{2t_{k-1}+1}{2t_{k-1}+2}$ by (10). But $v_3(n) = v_3(3k-1) = 0$, whence

$$\frac{1}{2(1+2v_3(n)-t_{n-1})} = \frac{1}{2\left(1 - \frac{2t_{k-1}+1}{2t_{k-1}+2}\right)} = 1 + t_{k-1} = t_n.$$

2nd case. — If N_n is the middle child of N_k, then $n = 3k$ and N_{n-1} is the left child of N_k. By using (8), we have

$$t_{n-1} = t_{3k-1} = 2v_3(3k) - \frac{1}{2t_k} = 2v_3(n) - \frac{1}{2t_k}.$$

Therefore by using (9) we obtain

$$\frac{1}{2(1+2v_3(n)-t_{n-1})} = \frac{1}{2+\frac{1}{t_k}} = \frac{t_k}{2t_k+1} = t_n.$$

3rd case. — If N_n is the right child of N_k, then $n = 3k+1$ and N_{n-1} is the middle child of N_k. Hence, by (9) and (10),

$$t_{n-1} = \frac{t_k}{2t_k+1} \qquad \text{and} \qquad t_n = \frac{2t_k+1}{2t_k+2}.$$

Since $v_3(n) = v_3(3k+1) = 0$, we have

$$\frac{1}{2(1+2v_3(n)-t_{n-1})} = \frac{1}{2\left(1 - \frac{t_k}{2t_k+1}\right)} = \frac{2t_k+1}{2t_k+2} = t_n.$$

The proof by induction is now complete. □

Corollary 1. *For every $k \in \mathbb{N}^*$, $t_{3k-1} = 1 + t_{k-1}$.*

Proof. This is exactly the equality (11). □

Corollary 2. *For every $k \in \mathbb{N}^*$, $t_{3k} \in (0, \frac{1}{2})$, $t_{3k+1} \in (\frac{1}{2}, 1)$ and $t_{3k+2} \in (1, +\infty)$.*

Proof. Let $k \in \mathbb{N}^*$. Since $t_k > 0$,

$$t_{3k} = \frac{t_k}{2t_k+1} \in (0, \tfrac{1}{2}) \qquad \text{and} \qquad t_{3k+1} = \frac{2t_k+1}{2t_k+2} \in (\tfrac{1}{2}, 1).$$

Moreover, from Corollary 1, $t_{3k+2} = t_{3(k+1)-1} = 1 + t_k > 1$. □

Remark 1. As $t_0 = 0$, $t_1 = \frac{1}{2}$ and $t_2 = 1$, we can also see that for every $k \in \mathbb{N}$, $t_{3k} \in [0, \frac{1}{2})$, $t_{3k+1} \in [\frac{1}{2}, 1)$ and $t_{3k+2} \in [1, +\infty)$.

Now we prove that the sequence (t_n) satisfies relations similar to (2) and (3).

Proposition 2. *For every $n \in \mathbb{N}^*$, $\lfloor t_{n-1} \rfloor = v_3(n)$.*

Proof. For $n = 1$, clearly $\lfloor t_0 \rfloor = \lfloor 0 \rfloor = 0 = v_3(1)$.

Let $n \geqslant 2$. Assume that, for every positive integer $j \leqslant n-1$, $\lfloor t_{j-1} \rfloor = v_3(j)$ and denote by N_k the parent of N_n ($k \in \mathbb{N}^*$).

If N_n is the left child of N_k, then $n = 3k-1$, whence $v_3(n) = v_3(3k-1) = 0$ and by Remark 1 $\lfloor t_{n-1} \rfloor = \lfloor t_{3(k-1)+1} \rfloor = 0$.

If N_n is the right child of N_k, then $n = 3k+1$, $v_3(n) = v_3(3k+1) = 0$ and $\lfloor t_{n-1} \rfloor = \lfloor t_{3k} \rfloor = 0$.

If N_n is the middle child of N_k then $n = 3k$ and N_{n-1} is the left child of N_k. Hence, by Corollary 1, $t_{n-1} = 1 + t_{k-1}$ and $\lfloor t_{n-1} \rfloor = 1 + \lfloor t_{k-1} \rfloor$. By the induction hypothesis, it follows that $\lfloor t_{n-1} \rfloor = 1 + v_3(k) = v_3(3k) = v_3(n)$.

The proof by induction is now complete. □

From 1 and 2 we get directly

Corollary 3. *Let f be defined as in (4). Then the sequence $(t_n)_{n\in\mathbb{N}}$ satisfies $t_0 = 0$ and, for every $n \in \mathbb{N}^*$,*

$$t_n = \frac{1}{2(1 + 2\lfloor t_{n-1} \rfloor - t_{n-1})} = \frac{f(t_{n-1})}{2}.$$

2.3. The Sequence (t_n) Enumerates $\mathbb{Q}_+$

Theorem 1. *The mapping $n \mapsto t_n$ is a bijection from $\mathbb{N}$ to $\mathbb{Q}_+$.*

Proof. As $t_0 = 0$ and $t_n = \frac{a_n}{b_n}$ is reduced for every $n \in \mathbb{N}^*$, we have to prove that, for every pair of coprime positive integers (α, β), there exists one and only one $n \geqslant 1$ such that $a_n = \alpha$ and $b_n = \beta$.

The proof is by induction on $m = \alpha + \beta$. If $m = 2$ then $\alpha = \beta = 1$ and Corollary 2 implies that $n = 2$ is the only integer such that $a_n = b_n = 1$.

Assume that, for a given integer $m \geqslant 2$, the property is true for every $k \in \{2, \ldots, m\}$. Let (α, β) be a pair of coprime positive integers such that $\alpha + \beta = m + 1$.

1st case: $\beta > 2\alpha$. Then, by Corollary 2, if n exists, there is a $k \in \mathbb{N}^*$ such that $n = 3k$. Hence N_n is the middle child of N_k. Therefore $N_k = (\alpha\,; \beta - 2\alpha\,; c_k)$. Now, $\alpha + (\beta - 2\alpha) = \beta - \alpha \leqslant m$ and α and $\beta - 2\alpha$ are coprime. By the induction hypothesis, there exists one and only one integer k such that $a_k = \alpha$ and $b_k = \beta - 2\alpha$, which proves that $n = 3k$ is the only integer such that $a_n = \alpha$ and $b_n = \beta$.

2nd case: $\beta = 2\alpha$. Then, $(\alpha, \beta) = (1, 2)$ since α and β are coprime. By Corollary 2, $n = 1$ is the sole integer such that $a_n = 1$ and $b_n = 2$.

3rd case: $\beta < 2\alpha < 2\beta$. Then, by Corollary 2, if n exists, there is a $k \in \mathbb{N}^*$ such that $n = 3k+1$. Hence N_n is the right child of N_k. If β is even $N_k = \left(\alpha - \frac{\beta}{2}\,;\beta-\alpha\,;c_k\right)$. Since $\alpha - \frac{\beta}{2} + \beta - \alpha = \frac{\beta}{2} \leqslant m$ and $\alpha - \frac{\beta}{2}$ and $\beta - \alpha$ are coprime, we see, as in the first case, that $n = 3k+1$ is the only integer such that $a_n = \alpha$ and $b_n = \beta$. If β is odd then $N_k = (2\alpha - \beta\,;2\beta - 2\alpha\,;c_k)$. As $2\alpha - \beta + 2\beta - 2\alpha = \beta \leqslant m$ and $2\alpha - \beta$ and $2\beta - 2\alpha$ are coprime (since β is even), we draw the same conclusion.

4th case: $\alpha = \beta$. Then $\alpha = \beta = 1$ since α and β are coprime. But this is impossible because $\alpha + \beta = m + 1 \geqslant 3$.

5th case: $\alpha > \beta$. Then, by Corollary 2, if n exists, there is an integer $k \geqslant 2$ such that $n = 3k-1$. Hence N_n is the left child of N_k. In this case, we cannot argue as before because, for odd b_n, $a_n + b_n$ is not necessarily greater than $a_k + b_k$; this can be seen, for example, when $N_3 = (1\,;4\,;1)$ and $N_8 = (2\,;1\,;0)$. However, by Corollary 1, $t_n = 1 + t_{k-1}$, whence $t_{k-1} = \frac{\alpha-\beta}{\beta}$. As $(\alpha - \beta) + \beta = \alpha \leqslant m$ and $\alpha - \beta$ and α are coprime, by the induction hypothesis there exists one and only one integer $k \geqslant 2$ such that $a_{k-1} = \alpha - \beta$ and $b_{k-1} = \beta$. This shows that $n = 3k$ is the only integer such that $a_n = \alpha$ and $b_n = \beta$.

The proof by induction is now complete. □

Remark 2. Theorem 1 appeared in a slightly different version as a problem in the American Mathematical Monthly [15]. S. Northshield gave a solution to this problem in [16] by constructing an analogue (b_n) of Stern's sequence for $\mathbb{Z}[\sqrt{2}]$. More precisely, by putting $R_n = \sqrt{2}\frac{b_{n+1}}{b_n}$ for every positive integer n, Northshield shows that, on the one hand, the sequence (R_n) is an enumeration of the positive rationals and, on the other hand, it satisfies $R_0 = 2$ and, for every positive integer n, $R_n = 4v_3(n) + 2 - \frac{2}{R_{n-1}}$. Thus, for every positive integer, $R_n = \frac{1}{t_n}$. Moreover, we can see that Northshield's sequence (b_n) is related to the tree $\mathcal{A}_3$ as Stern's sequence is related to the Calkin-Wilf tree.

In summary, the ternary tree $\mathcal{A}_3$ enabled us to construct a sequence (t_n) which enumerates the nonnegative rationals and satisfies recurrence relations similar to (1) and (2). Now we give a similar construction by using a quinary tree.

3. A Quinary Tree

3.1. Definition

We consider the quinary tree $\mathcal{A}_5$ whose vertices are labeled by triples of integers $(a\,;b\,;c)$ such that:

- the top of the tree is $(1\,;3\,;0)$;

- the children of $(a\,;b\,;c)$ are defined by:

  ▶ if 3 does not divide b:

$$(a\,;b\,;c)$$

$$(3(4c+3)a-2b\,;3(6c+5)a-3b\,;0) \qquad\qquad (3a+b\,;6a+3b\,;0)$$

$$((6c+5)a-b\,;6(c+1)a-b\,;0) \qquad (a\,;3a+b\,;c+1)$$

$$(6(c+1)a-b\,;3a\,;0)$$

  ▶ if 3 divides b:

$$(a\,;b\,;c)$$

$$\left((4c+3)a-\frac{2b}{3}\,;(6c+5)a-b\,;0\right) \qquad\qquad \left(a+\frac{b}{3}\,;2a+b\,;0\right)$$

$$((6c+5)a-b\,;6(c+1)a-b\,;0) \qquad (a\,;3a+b\,;c+1)$$

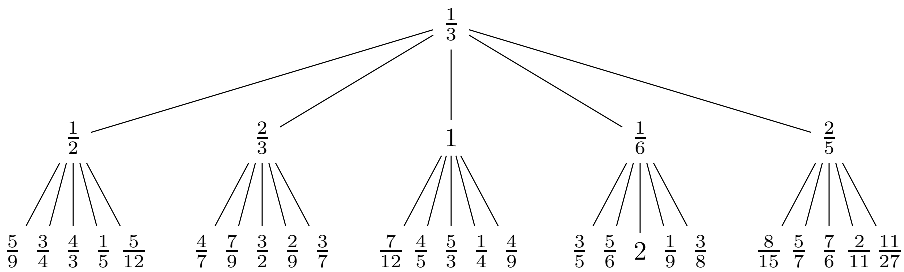
$$\left(2(c+1)a-\frac{b}{3}\,;a\,;0\right)$$

The five children of the vertex $N=(a\,;b\,;c)$ are called from left to right the first, second, third, fourth and fifth child of N, respectively.

For every $n \in \mathbb{N}^*$, we denote by $N_n=(a_n\,;b_n\,;c_n)$ the vertex of index n of the tree $\mathcal{A}_5$ read from the top and, at each level, from left to right. Thus, $N_1=(1\,;3\,;0)$, $N_2=(1\,;2\,;0)$, $N_3=(2\,;3\,;0)$, $N_4=(1\,;1\,;0)$, $N_5=(1\,;6\,;1)$, and so on.

By definition, for every $n \in \mathbb{N}^*$, the i-th child of N_n is $N_{5(n-1)+i+1}$.

It is easy to check, as in Lemmas 1 and 2, that for every $n \in \mathbb{N}^*$, we have $c_n = v_5(n)$, $a_n \in \mathbb{N}^*$, $b_n \in \mathbb{N}^*$ (with, this time, $3a_n \geqslant b_n - 6a_nc_n$) and $\gcd(a_n, b_n) = 1$. Hence, by putting $s_n = \frac{a_n}{b_n}$ for every $n \in \mathbb{N}^*$, we define a sequence $(s_n)_{n\in\mathbb{N}^*}$ of positive reduced rationals, whose first few terms are:



We remark that for every $k \in \mathbb{N}^*$, whether or not 3 divides b_k, we have

$$s_{5k-3} = \frac{3(4c_k+3)a_k - 2b_k}{3(6c_k+5)a_k - 3b_k} = \frac{3(4c_k+3) - \frac{2}{s_k}}{3(6c_k+5) - \frac{3}{s_k}}, \tag{12}$$

$$s_{5k-2} = \frac{(6c_k+5)a_k - b_k}{6(c_k+1)a_k - b_k} = \frac{6c_k+5-\frac{1}{s_k}}{6(c_k+1) - \frac{1}{s_k}}, \tag{13}$$

$$s_{5k-1} = \frac{6(c_k+1)a_k - b_k}{3a_k} = 2(c_k+1) - \frac{1}{3s_k}, \tag{14}$$

$$s_{5k} = \frac{a_k}{3a_k + b_k} = \frac{1}{3 + \frac{1}{s_k}}, \tag{15}$$

$$s_{5k+1} = \frac{3a_k + b_k}{6a_k + 3b_k} = \frac{3s_k + 1}{6s_k + 3} = \frac{3 + \frac{1}{s_k}}{6 + \frac{3}{s_k}}. \tag{16}$$

We extend this sequence to $\mathbb{N}$ by putting $s_0 = 0$. We will now show, as we did for $(t_n)_{n\in\mathbb{N}}$, that $(s_n)_{n\in\mathbb{N}}$ enumerates the elements of $\mathbb{Q}_+$.

3.2. Recurrence Relations

Proposition 3. *For every* $n \in \mathbb{N}^*$, $s_n = \dfrac{1}{3(1 + 2v_5(n) - s_{n-1})}$.

Proof. For $n = 1$ and $n = 2$, the statement is true since

$$\frac{1}{3(1+2v_5(1) - s_0)} = \frac{1}{3} = s_1 \qquad \text{and} \qquad \frac{1}{3(1 + v_5(2) - s_1)} = \frac{1}{2} = s_2.$$

Assume that, for a given $n \geqslant 2$, the property is true for every positive integer $j \leqslant n-1$. Denote by N_k ($k \in \mathbb{N}^*$) the parent of N_n .

1st case. — If N_n is the first child of N_k then $n = 5k-3$ and N_{n-1} is the fifth child of N_{k-1}. By the induction hypothesis,

$$s_k = \frac{1}{3(1+2v_5(k) - s_{k-1})} = \frac{1}{3\,(1 + 2c_k - s_{k-1})}.$$

Hence, by using (12),

$$s_n = \frac{3(4c_k+3)-\frac{2}{s_k}}{3(6c_k+5)-\frac{3}{s_k}} = \frac{3(4c_k+3)-6(2c_k+1-s_{k-1})}{3(6c_k+5)-9(2c_k+1-s_{k-1})} = \frac{1+2s_{k-1}}{2+3s_{k-1}}. \tag{17}$$

As $v_5(n)=0$, (16) yields

$$\frac{1}{3(2v_5(n)+1-s_{n-1})} = \frac{1}{3(1-s_{5(k-1)+1})} = \frac{1}{3\left(1-\frac{3s_{k-1}+1}{6s_{k-1}+3}\right)} = \frac{2s_{k-1}+3}{3s_{k-1}+2} = s_n.$$

2nd case. — If N_n is the second child of N_k then $n=5k-2$ and N_{n-1} is the first child of N_k. As $v_5(n)=0$, (12) and (13) yield

$$\begin{aligned}\frac{1}{3(2v_5(n)+1-s_{n-1})} &= \frac{1}{3(1-s_{5k-3})} = \frac{1}{3\left(1-\frac{3(4c_k+3)a_k-2b_k}{3(6k+5)a_k-3b_k}\right)}\\ &= \frac{(6c_k+5)a_k-b_k}{6(c_k+1)a_k-b_k} = s_{5k-2} = s_n.\end{aligned}$$

3rd case. — If N_n is the third child of N_k then $n=5k-1$ and N_{n-1} is the second child of N_k. As $v_5(n)=0$, (13) and (14) yield

$$\begin{aligned}\frac{1}{3(2v_5(n)+1-s_{n-1})} &= \frac{1}{3(1-s_{5k-2})} = \frac{1}{3\left(1-\frac{(6c_k+5)a_k-b_k}{6(c_k+1)a_k-b_k}\right)}\\ &= \frac{6(c_k+1)a_k-b_k}{3a_k} = s_{5k-1} = s_n.\end{aligned}$$

4th case. — If N_n is the fourth child of N_k then $n=5k$ and N_{n-1} is the third child of N_k. As $v_5(5k)=v_5(k)+1=c_k+1$, (14) and (15) yield

$$\frac{1}{3(2v_5(n)+1-s_{n-1})} = \frac{1}{3\left(2(c_k+1)+1-\left(2(c_k+1)-\frac{1}{3s_k}\right)\right)} = \frac{1}{3+\frac{1}{s_k}} = s_n.$$

5th case. — If N_n is the fifth child of N_k then $n=5k+1$ and N_{n-1} is the fourth child of N_k. As $v_5(n)=0$, (15) and (16) yield

$$\frac{1}{3(2v_5(n)+1-s_{n-1})} = \frac{1}{3\left(1-\frac{a_k}{3a_k+b_k}\right)} = \frac{3a_k+b_k}{6a_k+3b_k} = s_n.$$

This completes the proof by induction. □

Corollary 4. *For every $k\in\mathbb{N}^*$, $s_{5k-1}=1+s_{k-1}$.*

Proof. Let k be a positive integer. By definition, $s_{5k-1}=2(c_k+1)-\frac{1}{3s_k}$ and, by Proposition 3, $s_k=\frac{1}{3(2c_k+1-s_{k-1})}$. Therefore $s_{5k-1}=2(c_k+1)-(2c_k+1-s_{k-1})=1+s_{k-1}$. □

As in Corollary 2, we deduce from Proposition 3 that the rationals s_n belong to one of the five intervals $[\frac{1}{2}, \frac{2}{3})$, $[\frac{2}{3}, 1)$, $[1, +\infty)$, $[0, \frac{1}{3})$ or $[\frac{1}{3}, \frac{1}{2})$ depending on their rank in the tree $\mathcal{A}_5$ as a first, second, third, fourth or fifth child:

Corollary 5. *For every $k \in \mathbb{N}^*$, $s_{5k} \in (0, \frac{1}{3})$, $s_{5k+1} \in (\frac{1}{3}, \frac{1}{2})$, $s_{5k+2} \in (\frac{1}{2}, \frac{2}{3})$, $s_{5k+3} \in (\frac{2}{3}, 1)$ and $s_{5k+4} \in (1, +\infty)$.*

Proof. Let $k \in \mathbb{N}^*$. As $s_k > 0$, (15) and (16) imply that $s_{5k} \in (0, \frac{1}{3})$ and $s_{5k+1} \in (\frac{1}{3}, \frac{1}{2})$. Now (17) yields $s_{5k+2} = \frac{1+2s_k}{2+3s_k}$ whence $s_{5k+2} \in (\frac{1}{2}, \frac{2}{3})$. However, from Proposition 3, $s_{5k+3} = \frac{1}{3(1-s_{5k+2})}$. As $\frac{1}{2} < s_{5k+2} < \frac{2}{3}$, $1 < 3(1 - s_{5k+2}) < \frac{3}{2}$, this yields $s_{5k+3} \in (\frac{2}{3}, 1)$. Finally, $s_{5k+4} > 1$ since $s_{5k+4} = 1 + s_k$ by Corollary 4. $\square$

Remark 3. As $s_0 = 0$, $s_1 = \frac{1}{3}$, $s_2 = 1$, $s_3 = \frac{2}{3}$ and $s_4 = 1$, we see that for every $k \in \mathbb{N}$, $s_{5k} \in [0, \frac{1}{3})$, $s_{5k+1} \in [\frac{1}{3}, \frac{1}{2})$, $s_{5k+2} \in [\frac{1}{2}, \frac{2}{3})$, $s_{5k+3} \in [\frac{2}{3}, 1)$ and $s_{5k+4} \in [1, +\infty)$.

Now we prove that the sequence (s_n) satisfies relations similar to (2) and (3).

Proposition 4. *For every $n \in \mathbb{N}^*$, $\lfloor s_{n-1} \rfloor = v_5(n)$.*

Proof. For $n = 1$, $\lfloor s_0 \rfloor = \lfloor 0 \rfloor = 0 = v_5(1)$. Now assume that, for a given integer $n \geqslant 2$ and every integer $j \leqslant n - 1$, $\lfloor s_{j-1} \rfloor = v_5(j)$. Denote N_k ($k \in \mathbb{N}^*$) the parent of N_n .

If N_n is not the fourth child of N_k then 5 does not divide n. Therefore $v_5(n) = 0$ and $n - 1 \not\equiv 4 \pmod 5$ and, by Corollary 5, $\lfloor s_{n-1} \rfloor = 0$.

If N is the fourth child of N_k then $n = 5k$, whence $n - 1 = 5k - 1$. Now Corollary 4 yields $s_{n-1} = 1 + s_{k-1}$, which implies $\lfloor s_{n-1} \rfloor = 1 + \lfloor s_{k-1} \rfloor$. However, by the induction hypothesis, $\lfloor s_{k-1} \rfloor = v_5(k)$, whence $\lfloor s_{n-1} \rfloor = 1 + v_5(k) = v_5(5k)$, i.e., $\lfloor s_{n-1} \rfloor = v_5(n)$. This completes the proof by induction. $\square$

The following statement is a direct consequence of Propositions 3 and 4.

Corollary 6. *Let f be defined as in (4). Then the sequence $(s_n)_{n\in\mathbb{N}}$ satisfies $s_0 = 0$ and, for every $n \in \mathbb{N}^*$,*

$$s_n = \frac{1}{3(1 + 2\lfloor s_{n-1} \rfloor - s_{n-1})} = \frac{f(s_{n-1})}{3}.$$

3.3. The Sequence (s_n) Enumerates $\mathbb{Q}_+$

Theorem 2. *The mapping $n \mapsto s_n$ is a bijection from $\mathbb{N}$ to $\mathbb{Q}_+$.*

Proof. As in the proof of Theorem 1, we have to prove that, for every pair of coprime positive integers (α, β), there exists one and only one $n \geqslant 1$ such that $a_n = \alpha$ and $b_n = \beta$.

The proof is again by induction on $m = \alpha + \beta$.

If $m = 2$, then $\alpha = \beta = 1$ and Corollary 5 shows that $n = 4$ is the only integer such that $a_n = b_n = 1$.

Assume that, for a given integer $m \geqslant 2$, the property is true for every $k \in \{2, \ldots, m\}$. Let (α, β) be a pair of coprime positive integers such that $\alpha+\beta = m+1$.

As in the proof of Theorem 1, we deduce from Corollary 5 and Remark 2 that $n = 1$ (resp. $n = 2$, $n = 3$ and $n = 4$) if $\beta = 3\alpha$ (resp. $\beta = 2\alpha$, $2\beta = 3\alpha$ and $\beta = \alpha$).

Now we distinguish five cases.

1st case: $\beta > 3\alpha$. Then, by Corollary 5, if n exists, $n = 5k$ with $k \in \mathbb{N}^*$. Hence, by (15), $s_k = \frac{\alpha}{\beta-3\alpha}$. However, $\alpha + (\beta - 3\alpha) = \beta - 2\alpha \leqslant m$ and α and $\beta - 3\alpha$ are coprime, which yields the conclusion by using the induction hypothesis.

2nd case: $2\beta < 6\alpha < 3\beta$. Then, by Corollary 5, if n exists, $n = 5k + 1$ with $k \in \mathbb{N}^*$. Hence, by (15),

$$s_k = \frac{3\alpha - \beta}{3\beta - 6\alpha} \text{ if } 3 \nmid \beta \qquad \text{and} \qquad s_k = \frac{\alpha - \frac{\beta}{3}}{\beta - 2\alpha} \text{ if } 3 \mid \beta$$

which yields the conclusion as in the first case.

3rd case: $3\beta < 6\alpha < 4\beta$. Then, by Corollary 5, if n exists, $n = 5k - 3$ with $k \in \mathbb{N}^*$. Hence, by (15), $s_{k-1} = \frac{2\alpha-\beta}{2\beta-3\alpha}$ which yields the conclusion as in the first case.

4th case: $2\beta < 3\alpha < 3\beta$. By Corollary 5, if n exists, $n = 5k-2$ with $k \in \mathbb{N}^*$. Then, by Proposition 3, $s_{5k-1} = \frac{1}{3(1-s_{5k-2})}$ and therefore $s_{5k-2} = 1-\frac{1}{3s_{5k-1}} = 1-\frac{1}{3(1+s_{k-1})}$ by Corollary 4. Hence,

$$s_{k-1} = \frac{3\alpha - 2\beta}{3\beta - 3\alpha} \text{ if } 3 \nmid \beta \qquad \text{and} \qquad s_{k-1} = \frac{\alpha - 2\frac{\beta}{3}}{\beta - \alpha} \text{ if } 3 \mid \beta$$

which yields the conclusion as in the first case.

5th case: $\alpha > \beta$. By Corollary 5, if n exists, $n = 5k-1$ with $k \in \mathbb{N}$, $k \geqslant 2$. Then, by Corollary 4, $s_n = 1 + s_{k-1}$ and therefore $s_{k-1} = \frac{\alpha-\beta}{\beta}$ and the conclusion holds as in the first case. This completes the proof by induction. □

4. The Relation (5) with $k \geqslant 4$

Newman's result (2) and Propositions 2 and 4 show that the Calkin-Wilf sequence and sequences (t_n) and (s_n) are all defined by a first term $u_0 = 0$ and by a recurrence relation of the form

$$u_n := \frac{f(u_{n-1})}{k} \qquad (n = 1, 2, 3, \ldots)$$

where $k \in \{1, 2, 3\}$ and f is defined by (4). It is natural to ask if such a relation defines an enumeration of $\mathbb{Q}_+$ for every $k \geqslant 1$. We prove now that this is not the case.

Let $k \geqslant 4$ be an integer. Put $f_k = \frac{1}{k} f$ and consider the sequence (u_n) defined by $u_0 = 0$ and, for every $n \in \mathbb{N}^*$, $u_n = f_k(u_{n-1})$. It is easy to check that the only solutions of $f_k(x) = x$ are

$$\gamma_k = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4}{k}} \text{ and } \delta_k = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{k}},$$

and that $0 < \gamma_k \leqslant \delta_k < 1$. Hence, as f_k is increasing on $[0, 1)$ and $f_k(0) = \frac{1}{k} > 0$, $f_k([0, \gamma_k]) \subset [0, \gamma_k]$. Moreover, $u_1 = f_k(0) = \frac{1}{k} > u_0$. Therefore (u_n) is increasing since f_k is increasing, which proves that (u_n) is convergent. As f_k is continuous on $[0, \gamma_k]$, $\lim u_n = \gamma_k$. Hence γ_k is the only accumulation point of (u_n), which proves that (u_n) cannot enumerate $\mathbb{Q}_+$, nor even the rationals of a given interval.

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